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# THE CIRCULAR CHROMATIC NUMBER OF A DIGRAPH

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# The circular chromatic number of a digraph

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**Abstract**

We introduce the circular chromatic number $\chi_c$ of a digraph and establish various basic results. They show that the coloring theory for digraphs is similar to the coloring theory for undirected graphs when independent sets of vertices are replaced by acyclic sets. Since the directed $k$-cycle has circular chromatic number $k/(k-1)$, for $k \geq 2$, values of $\chi_c$ between 1 and 2 are possible. We show that in fact, $\chi_c$ takes on all rational values greater than 1. Furthermore, there exist digraphs of arbitrarily large digirth and circular chromatic number. It is NP-complete to decide if a given digraph has $\chi_c$ at most 2.

*Keywords:* circular chromatic number, chromatic number, digraph, acyclic homomorphism, NP-completeness, digirth.

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## 1 Introduction

For graphs, the circular chromatic number is a refinement of the usual chromatic number; see e.g. [10]. This paper introduces the chromatic and the circular chromatic numbers for digraphs. We define the first of these invariants by replacing the requirement that color classes are independent sets by the weaker condition that they are acyclic. Our results show that this is a natural way to define the chromatic number of a digraph.

We adopt standard graph theory notation and terminology. We shall consistently use $D$ to denote digraphs and $G$ to denote (simple, undirected) graphs. An edge of $G$ denoted by $uv$ represents the edge joining the vertices $u$ and $v$. In a digraph, the arc $uv$ has initial vertex $u$ and terminal vertex $v$. All graphs and digraphs are simple, but we allow oppositely oriented arcs $uv$ and $vu$ to belong to the arc set of a digraph.

For a positive real number $p$, denote by $S_p \subset \mathbb{R}^2$ the circle with perimeter $p$ (hence with radius $p/2\pi$) centered at the origin of $\mathbb{R}^2$. We can identify $S_p$ with the set $\mathbb{R}/p\mathbb{Z}$ in the obvious way. For $x, y \in S_p$, let us denote by $S_p(x, y)$ the arc on $S_p$ from $x$ to $y$ in the clockwise direction, and let $\mathrm{d}(x, y)$ denote the length of this arc. The set $\mathbb{R}/p\mathbb{Z}$ can also be identified with the real interval $[0, p)$, where the "distance" function $\mathrm{d}(x, y)$ can be expressed as

$$\mathrm{d}(x,y) = \begin{cases} y - x, & \text{if } x \text{ precedes } y \text{ on } [0,p) \\ p + y - x, & \text{otherwise.} \end{cases}$$

A *circular $p$-coloring* of a digraph $D$ is a function $c\colon V(D) \to S_p$ such that every arc $uv \in E(D)$ satisfies $\mathrm{d}(c(u), c(v)) \geq 1$. If $D$ has at least one edge, then the *circular chromatic number* $\chi_c(D)$ of $D$ is the infimum of all real numbers $p$ for which there exists a circular $p$-coloring of $D$. If $D$ has no edges, then we define $\chi_c(D) = 1$. It is possible that $D$ admits no circular $\chi_c(D)$-coloring; i.e., the infimum may not be attained.

An alternative definition of circular colorings overcomes this trouble. A subset $U \subseteq V(D)$ is *acyclic* if it induces an acyclic subdigraph of $D$. Let $p \geq 1$. We call $c\colon V(D) \to S_p$ a *weak circular $p$-coloring* of $D$ if, for every arc $uv \in E(D)$, either $c(u) = c(v)$ or $\mathrm{d}(c(u), c(v)) \geq 1$, and for every $x \in S_p$, the *color class* $c^{-1}(x)$ is an acyclic vertex set of $D$. It is easy to see that $\chi_c(D)$ is equal to the infimum of all real numbers $p \geq 1$ for which there exists a weak circular $p$-coloring of $D$. The results of [7] show that this infimum is always attained; i.e., every digraph $D$ admits a weak circular $\chi_c(D)$-coloring. Moreover, $\chi_c(D)$ is a rational number for every $D$.

Finally, we define the *chromatic number* $\chi(D)$ of $D$ to be the minimum integer $k$ such that $V(D)$ can be partitioned into $k$ acyclic subsets. We shall call such a partition a *k-coloring* of $D$. It is easy to see that $\chi(D)$ coincides with the minimum number $p$ for which there exists a weak $p$-coloring $c\colon V(D) \to [1,p]$ into the real interval $[1,p]$ such that every arc $uv$ satisfies $c(v) \notin (c(u), c(u)+1]$, i.e., $\mathrm{d}(c(u), c(v)) \geq 1$ or $c(v) = c(u)$.

The notion of the digraph chromatic number gives rise to a new coloring parameter for undirected graphs which may be of independent interest. For an (undirected) graph $G$, set

$$\vec{\chi}(G) = \max\{\chi(D) \mid D \text{ is an orientation of } G\}$$

and

$$\vec{\chi}_c(G) = \max\{\chi_c(D) \mid D \text{ is an orientation of } G\}.$$

Clearly $\vec{\chi}(G) \leq \chi(G)$ and $\vec{\chi}_c(G) \leq \chi_c(G)$. It was proved by Fountoulakis et al. [4] that $\vec{\chi}(G) \geq \chi(G)/\log(\chi(G))$ and that there are examples for which $\vec{\chi}(G) = \chi(G)$.

## Preliminary results

We shall assume a general familiarity with the basic theory of the circular chromatic number for undirected graphs, as surveyed, e.g., in [10]. This theory and the present paper are connected by the following observation. If a digraph $D$ is obtained from an undirected graph $G$ by replacing each edge of $G$ by a pair of oppositely oriented arcs between the same pair of vertices, then $\chi_c(D)$ agrees with the (undirected) circular chromatic number $\chi_c(G)$.

Observe that $\chi_c(D) \geq 1$, with equality if and only if $\chi(D) = 1$, and this holds if and only if $D$ is acyclic. In general, $\chi$ and $\chi_c$ are related as follows:

**Proposition 1.1** $\chi(D) - 1 < \chi_c(D) \leq \chi(D)$.

**Proof.** Clearly, a $k$-coloring of $D$ determines a weak circular $k$-coloring of $D$. This yields the second inequality.

Let $p = \chi_c(D)$, $k = \lceil p \rceil$, and $\varepsilon = p/2n$, where $n$ is the order of $D$, and let $c$ be a circular $(p+\varepsilon)$-coloring. Then $S_{p+\varepsilon}$ can be written as the union of $k+1$ (disjoint) arcs $A_0, A_1, \ldots, A_k$, each of length less than 1, and such that $c^{-1}(A_0) = \emptyset$. For $i = 1, \ldots, k$, let $V_i = c^{-1}(A_i)$. Clearly, each $V_i$ is acyclic, and the partition of $V(D)$ into these acyclic sets is a $k$-coloring of $D$. This verifies the first inequality. □

Let us recall that the relations in Proposition 1.1 also hold between the chromatic and circular chromatic numbers of undirected graphs; cf. [10].

This result, together with the fact that $\chi_c(C_{2k+1}) = 2 + 1/k$, was one motivation for introducing the circular chromatic number. The fact that $\chi_c$ for odd cycles monotonically decreases towards 2 has an even more natural digraph counterpart. Namely, for directed cycles $\vec{C}_n$, the circular chromatic number monotonically approaches 1 as the length $n$ increases:

$$\chi_c(\vec{C}_n) = 1 + \frac{1}{n-1}.$$

This result is a special case of Proposition 1.2 below.

Let $C(k,d)$ be the undirected graph with vertex set $\{0,\ldots,k-1\}$ in which distinct vertices $i,j$ are adjacent if and only if $d \le |i-j| \le k-d$. If $k \ge 2d$, then this graph has circular chromatic number $k/d$; see [2] or [9]. Here we define a directed analogue of $C(k,d)$: let $\vec{C}(k,d)$ be the digraph with vertex set $V(\vec{C}(k,d)) = \{0,\ldots,k-1\}$ whose arcs emanate from a given vertex $i \in V(\vec{C}(k,d))$ to the vertices $i+d$, $i+d+1$, $\ldots$, $i+k-1$, with arithmetic modulo $k$. We display $\vec{C}(7,3)$ in Figure 1. Notice that $\vec{C}(n,n-1) \cong \vec{C}_n$.

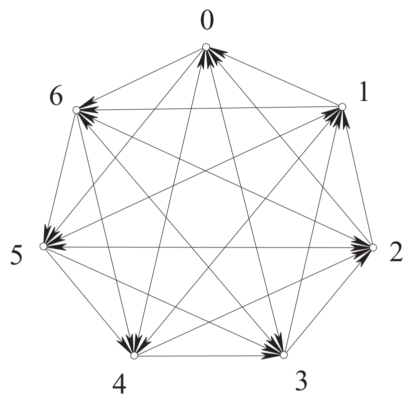


Figure 1: The digraph $\vec{C}(7,3)$

As noted above, $\chi_c(D)$ is always rational and at least 1. The next result shows that every such rational number is the circular chromatic number of some digraph.

**Proposition 1.2** *If $k$ and $d$ are positive integers with $k \ge d$, then*

$$\chi_c(\vec{C}(k,d)) = \frac{k}{d}.$$

*In particular, for every rational number $p \geq 1$, there exists a digraph with circular chromatic number $p$.*

**Proof.** Let $p = k/d$. It is easy to see that $c\colon V(\vec{C}(k,d)) \to \mathbb{R}/p\mathbb{Z}$ defined by $c(i) := i/d$ is a circular $p$-coloring. Therefore $\chi_c(\vec{C}(k,d)) \leq k/d$.

If $k \geq 2d$, then $\vec{C}(k,d)$ contains $C(k,d)$ as a subdigraph (where each edge of $C(k,d)$ is replaced by a pair of oppositely oriented arcs). Thus, in this case $\chi_c(\vec{C}(k,d)) \geq \chi_c(C(k,d))$, which, by [9], is $k/d$.

It remains to consider the case $d < k < 2d$. Suppose, for a contradiction, that $\vec{C}(k,d)$ admits a circular $q$-coloring $c$ with $q < k/d$. We may assume that $c(0) = 0$. Let $d_{ij} = \mathrm{d}(c(i), c(j))$. Then $\sum_{i=0}^{k-1} d_{i,i+1} = \ell q$ for some positive integer $\ell$. Since $\vec{C}(k,d)$ contains each arc $i0$, for $1 \leq i \leq k-d$, we have $0 < c(i) \leq q-1$ for each such $i$. Since $q - 1 < 1$, it follows that $c(i) < c(j)$ whenever $i < j$ and $i, j \in \{1, \ldots, k-d\}$. Thus $\sum_{i=0}^{k-d-1} d_{i,i+1} \leq q-1$, and by symmetry, the same bound holds for the sum of any $k-d$ consecutive values $d_{i,i+1}$. Summing the resulting $k$ inequalities, we obtain $(k-d)\ell q = (k-d)\sum_{i=0}^{k-1} d_{i,i+1} \leq k(q-1)$, which shows that $\ell \leq k(q-1)/(k-d)q < 1$, contradicting $\ell \in \mathbb{Z}^+$. Therefore $\chi_c(\vec{C}(k,d)) \geq k/d$. □

Recall (cf. [2]) that the graphs $C(k,d)$ provide an important connection between graph homomorphisms and circular chromatic numbers. Namely, a graph $G$ has circular chromatic number at most $k/d$ if and only if there exists a graph homomorphism $G \to C(k,d)$. We shall see that the digraphs $\vec{C}(k,d)$ play an analogous role in the theory of digraphs.

An *acyclic homomorphism* of a digraph $D$ into a digraph $D'$ is a mapping $\phi\colon V(D) \to V(D')$ such that:

(i) for every arc $uv \in E(D)$, either $\phi(u) = \phi(v)$ or $\phi(u)\phi(v)$ is an arc of $D'$;

(ii) for every vertex $v \in V(D')$, the subgraph of $D$ induced on $\phi^{-1}(v)$ is acyclic.

**Proposition 1.3** *A digraph $D$ has circular chromatic number at most $k/d$ if and only if there exists an acyclic homomorphism $D \to \vec{C}(k,d)$.*

**Proof.** Let $p = k/d$. Suppose that $\chi_c(D) \leq p$ and let $c\colon V(D) \to \mathbb{R}/p\mathbb{Z}$ be a weak circular $p$-coloring of $D$. Then $\phi\colon V(D) \to V(\vec{C}(k,d))$ defined by $\phi(u) := i$, where $c(u) \in [i/d, (i+1)/d)$, is an acyclic homomorphism from $D$ into $\vec{C}(k,d)$.

Conversely, if $\phi\colon V(D) \to V(\vec{C}(k,d))$ is an acyclic homomorphism, then $c(u) := \phi(u)/d$ defines a weak circular $p$-coloring of $D$. □

The following result is an immediate consequence of Proposition 1.3.

**Corollary 1.4** *Suppose that $k, k', d, d'$ are positive such that $d \leq k$ and $\frac{k}{d} \leq \frac{k'}{d'}$. If a digraph $D$ admits an acyclic homomorphism into $\vec{C}(k, d)$, then $D$ also admits an acyclic homomorphism into $\vec{C}(k', d')$.*

It is easy to see that the composition of acyclic homomorphisms is again an acyclic homomorphism. In particular, if there is an acyclic homomorphism $D \to D'$, then $\chi_c(D) \leq \chi_c(D')$.

# 2 Tight cycles and degeneracy

Let $c$ be a weak circular $p$-coloring of a digraph $D$. A cycle $C = v_1 v_2 \dots v_k v_1$ in the underlying graph of $D$ is *tight* (with respect to $c$) if for every edge $v_i v_{i+1}$ of $C$ ($i = 1, \dots, k$, with indices modulo $k$) we have $\mathrm{d}(c(v_i), c(v_{i+1})) = 1$ whenever $v_i v_{i+1}$ is an arc of $D$, and $c(v_i) = c(v_{i+1})$, otherwise. If $C$ is a tight cycle, then its *weight* $a(C)$ is the number of edges $v_i v_{i+1}$ that are also arcs of $D$. Clearly, the weight of a tight cycle is an integral multiple of $p$; we call the value $w(C) = a(C)/p$ the *winding number* of $C$. In Section 5 we shall need the following result from [7].

**Proposition 2.1** *If $p = \chi_c(D)$, then there is a weak circular $p$-coloring of $D$ which has a tight cycle.*

Let $k$ be a nonnegative integer. Recall that a graph $G$ is *$k$-degenerate* if every subgraph of $G$ contains a vertex of degree at most $k$. It is easy to see that every $k$-degenerate graph is $(k+1)$-colorable. A digraph $D$ is *weakly $k$-degenerate* if every subdigraph of $D$ contains a vertex with indegree or outdegree at most $k$. For example, $D$ is weakly 0-degenerate if and only if $D$ is acyclic.

**Proposition 2.2** *If $D$ is weakly $k$-degenerate, then $\chi(D) \leq k + 1$.*

**Proof.** Let $v_1, \dots, v_n$ be the vertices of $D$ enumerated so that for $i = 1, \dots, n$, the vertex $v_i$ has either indegree or outdegree at most $k$ in the induced subdigraph $D_i = D[\{v_1, \dots, v_i\}]$. Define $A_0, \dots, A_k$ as follows. Start with empty sets. For $i = 1, \dots, n$, there is a set $A_j$, with $j = j(i)$, such that $A_j$ contains either no out-neighbors or no in-neighbors of $v_i$ in $D_i$. Now, put $v_i$ in $A_j$.

Suppose that one of the resulting sets $A_j$ contains a cycle $C$. If $v_i$ is the vertex on $C$ with largest index $i$, then $v_i$ has an in- and an out-neighbor among the other vertices on $C$, which is impossible by the construction of the sets $A_0, \dots, A_k$. Therefore, the partition into these sets determines a $(k+1)$-coloring of $D$. □

The next result shows that the upper bound in Proposition 2.2 is sharp.

**Proposition 2.3** *For every nonnegative integer $k$, there exists a $2k$-degenerate graph $G_k$ and a weakly $k$-degenerate orientation $D_k$ of $G_k$ with $\chi(D_k) = k+1$.*

**Proof.** The digraphs $D_k$ are constructed inductively for $k \geq 0$. We let $D_0$ be the graph $K_1$, and, having constructed $D_{k-1}$, obtain $D_k$ as follows. If $V(D_{k-1}) = V_1 \cup \cdots \cup V_k$ is a $k$-coloring of $D_{k-1}$, let $r$ be the number of color classes $V_i$ whose induced subdigraph has at least one arc. We say that this $k$-coloring has *strength* $r$. Next, we define weakly $k$-degenerate digraphs $D_k^r$ for $r = 0, 1, \dots, k+1$ such that every $k$-coloring of $D_k^r$ has strength at least $r$. In particular, $D_k^{k+1}$ has no $k$-colorings, and we shall take this graph as $D_k$.

Let $D_k^0 = D_{k-1}$. Inductively, having constructed $D_k^r$, where $0 \leq r \leq k$, we consider all $k$-colorings of $D_k^r$ of strength $r$. For every such $k$-coloring with color classes $V_1, \dots, V_k$, we add a new vertex $v$ to $D_k^r$ which has precisely one outgoing arc to each color class and at most one incoming arc from each color class. If $V_i$ has no arcs, then $v$ is joined to an arbitrary vertex in $V_i$. If $V_i$ has an arc $v_i u_i$, then we add arcs $vv_i$ and $u_i v$. It is clear that the given $k$-coloring of $D_k^r$ cannot be extended to a $k$-coloring of $D_k^r + v$ of strength at most $r$. The digraph obtained by adding vertices for all $k$-colorings of $D_k^r$ of strength $r$ is the new digraph $D_k^{r+1}$.

It is clear by construction that all digraphs $D_k^r$ and hence also $D_k$ are weakly $k$-degenerate and also that the underlying graph $G_k$ of $D_k$ is $2k$-degenerate. □

Proposition 2.2 implies that every planar digraph $D$ without 2-cycles satisfies $\chi(D) \leq 3$, but we believe that there is room for improvement in this bound:

**Conjecture 2.4** *If $D$ is a planar digraph without 2-cycles, then $\chi(D) \leq 2$.*

This conjecture, which was suggested by Škrekovski, is supported by the fact that all small planar graphs have vertex arboricity at most 2. Clearly, if $G$ has vertex arboricity $r$, then the chromatic number of any orientation of $G$ is at most $r$ (use an arboricity decomposition as a coloring).

Another immediate consequence of Proposition 2.2 is:

**Corollary 2.5** *Let $G$ be a connected graph with maximum degree at most 4. If $D$ is an orientation of $G$ which is not Eulerian, then $\chi(D) \le 2$.*

There are digraphs of maximum degree at most 3 with circular chromatic number 2. Clearly, if $D$ contains a 2-cycle, then $\chi_c(D) = 2$.

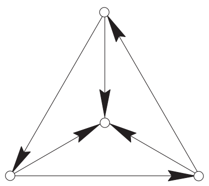

Figure 2: A 2-critical digraph for $\chi_c$

Another example is the cubic digraph in Figure 2. To see this, let 1, 2, 3 be the consecutive vertices of the oriented 3-cycle, and let 4 be the central vertex (which has indegree 3). Assume that $p < 2$ and let $c\colon V \to \mathbb{R}/p\mathbb{Z}$ be a circular coloring with $c(1) = 0$. Then $c(2) \in [1, p)$ and $c(3) \in [c(2) + 1 - p, c(2)) \cap (0, p-1]$. Hence $c(1), c(2), c(3)$ partition $[0, p]$ into 3 intervals, each of length less than 1. This leaves no room for $c(4)$.

It would be interesting to classify all subcubic digraphs which are 2-critical for $\chi_c$. Is their number finite or infinite? Are there examples with digirth at least 4?

# 3 Computational issues

For the usual chromatic number, the threshold between the "easy" and "hard" computable chromatic numbers is between 2 and 3. For digraphs, already deciding whether the chromatic number is $\le 2$ is NP-complete.

**Theorem 3.1** *The decision problem whether the chromatic number of a digraph is at most 2 is NP-complete.*

**Proof.** It is clear that the problem is in NP. To show its completeness, we shall describe a polynomial-time reduction of 2-colorability of 3-uniform hypergraphs, a well-known NP-complete problem [5, 6], to digraph 2-colorability.

Suppose that $H$ is a given 3-uniform hypergraph with vertex set $V(H) = \{v_1, \ldots, v_n\}$. Let $D$ be the digraph obtained as follows. We start with the vertex set $V(H)$ and add, for each hyperedge $v_iv_jv_k \in E(H)$ $(i < j < k)$, a copy of the digraph $F$ shown in Figure 3 and identify its vertices $a, b, c$ with $v_i, v_j, v_k$, respectively. Observe that all arcs between pairs of vertices of $V(H)$ are directed from a vertex with smaller index to a vertex with larger index. Therefore, every directed cycle in the resulting digraph $D$ contains at least one vertex which is not in $V(H)$.

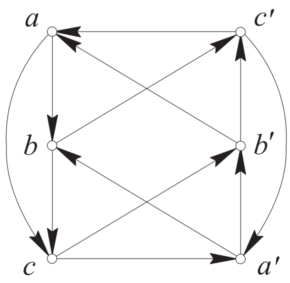


Figure 3: The digraph $F$ in the proof of Theorem 3.1

It is easy to see that in any partition of $V(F)$ into two acyclic subsets, the vertices $a, b, c$ cannot all be in the same class. This shows that every 2-coloring of $D$ determines a 2-coloring of the hypergraph $H$.

On the other hand, suppose that $H$ has a 2-coloring. The 2-coloring of a hyperedge $v_iv_jv_k \in E(H)$ $(i < j < k)$ can be extended "locally" to a 2-coloring of the corresponding copy of $F$ by giving $a$ and $a'$ the color of $v_i$, $b$ and $b'$ the color of $v_j$, and $c$ and $c'$ the color of $v_k$. Doing this for all hyperedges of $H$, we get a partition of $V(D)$ into two classes $A \cup B$ such that the induced partition in every copy of $F$ is acyclic. We claim that $A$ and $B$ are both acyclic. If not, let $C$ be a shortest directed cycle contained in one of them, say in $A$. As noted above, $C$ has a vertex $y$ that is not contained in $V(H)$. Let $F'$ be the corresponding copy of $F$. Since $C$ is not contained in $F'$, there is a segment $S = x \cdots y \cdots z$ of $C$ contained in $F'$ where $x, z \in \{a, b, c\}$. Since $C$ is shortest possible, this segment cannot be replaced by the arc $xz$. Thus the ordered pair $(x, z)$ belongs to $\{(b, a), (c, a), (c, b)\}$, and a simple case analysis gives a contradiction. This shows that a 2-coloring of $H$ yields a 2-coloring of $D$.

Therefore, the digraph $D$ is 2-colorable if and only if the hypergraph $H$ has a 2-coloring. □

Theorem 3.1 implies that for every fixed integer $p \geq 2$, the decision whether $\chi_c(D) \leq p$ is NP-complete. However, it is not clear for which rational numbers the same conclusion holds.

**Problem 3.2** *Let p be a fixed rational number. How difficult is it to verify if a given digraph has circular chromatic number at most p?*

It is possible that the decision task in Problem 3.2 is polynomially solvable for values of $p$ that are smaller than 2 and NP-complete for $p \geq 2$.

**Problem 3.3** *For a fixed rational number p between 1 and 2, how difficult is it to decide for a (sub)cubic digraph D whether* $\chi_c(D) \leq p$*?*

# 4 Chromatic number and girth

It may not be immediately clear that triangle-free (i.e. digirth at least 4) digraphs can have arbitrarily large chromatic number. In fact, we shall prove that there are digraphs with arbitrarily large digirth and circular chromatic number.

For the probabilistic proof, we need the notion of a *random digraph* $\vec{G}(n,p)$, chosen by picking each pair among $n$ vertices as an (unoriented) edge randomly and independently with probability $p$, and then flipping an independent fair coin to determine the orientation of each edge. We also need a directed analogue of the usual independence number: $\alpha = \alpha(D)$ is the maximum size of a set of vertices of $D$ inducing an acyclic subdigraph of $D$. Clearly, every digraph $D$ satisfies the following basic inequality:

$$\chi(D) \geq \frac{|V(D)|}{\alpha(D)}. \tag{1}$$

The next result is analogous to a famous theorem of Erdős [3] on the chromatic number and girth of undirected graphs. Our proof is a refinement of the corresponding proof in [1].

**Theorem 4.1** *For all* $k, \ell \in \mathbb{N}$*, there exists a digraph D with* $\chi_c(D) > k$ *and* $\mathrm{digirth}(D) > \ell$*.*

**Proof.** Fix $\vartheta < 1/\ell$, let $p = 2n^{\vartheta-1}$, and choose $\vec{G} = \vec{G}(n,p)$. Let $X$ be the number of directed cycles in $\vec{G}$ of length at most $\ell$. Then

$$E[X] = \sum_{i=3}^{\ell} \frac{(n)_i \, p^i}{i2^i} \leq \sum_{i=3}^{\ell} \frac{n^{\vartheta i}}{i}$$

(where $(n)_i$ denotes falling factorial). Since $\vartheta\ell < 1$, we have $E[X] = o(n)$. Now Markov's inequality shows that

$$\Pr\left(X \geq \frac{n}{2}\right) = o(1). \tag{2}$$

For $t \in \{1, \ldots, n\}$, we have

$$\begin{aligned}
\Pr(\alpha \geq t) &\leq \sum_{A \in \binom{V}{t}} \Pr(A \text{ induces an acyclic subdigraph of } \vec{G}) \\
&\leq \binom{n}{t} t! \sum_{m=0}^{\binom{t}{2}} \binom{\binom{t}{2}}{m} \left(\frac{p}{2}\right)^m (1-p)^{\binom{t}{2}-m} \\
&< n^t \left(1 - p + \frac{p}{2}\right)^{\binom{t}{2}} \\
&= \left[n \left(1 - \frac{p}{2}\right)^{(t-1)/2}\right]^t \\
&\leq \left(ne^{-p(t-1)/4}\right)^t (1 + o(1)).
\end{aligned}$$

Thus, if we set $t = \lceil (5/p) \ln n \rceil + 1$, so that $t - 1 \geq (5/p) \ln n$, then

$$\Pr(\alpha \geq t) < n^{-t/4}(1 + o(1)) = o(1). \tag{3}$$

If $n$ is chosen large enough to ensure that both of the events in (2) and (3) have probability less than 1/2, then *some* $\vec{G}$ has fewer than $n/2$ directed cycles of length at most $\ell$ and satisfies $\alpha(\vec{G}) < 5n^{1-\vartheta} \ln n/2 + 1$. Let $D$ be obtained from $\vec{G}$ by removing a vertex from each directed cycle of length at most $\ell$. Then $D$ has at least $n/2$ vertices, has digirth greater than $\ell$, and satisfies $\alpha(D) \leq \alpha(\vec{G})$.

Thus, from (1), we see that

$$\chi(D) \geq \frac{|V(D)|}{\alpha(D)} \geq \frac{n/2}{5n^{1-\vartheta} \ln n/2 + 1} = \frac{n^{\vartheta}}{5 \ln n + 2n^{\vartheta - 1}} \to \infty.$$

If $n$ is so large that $\chi(D) > k + 1$, then Proposition 1.1 yields $\chi_c(D) > k$. □

## 5 Examples

We define the *antioriented prism* $R_n$, for $n \geq 3$, to be the digraph with vertex set $\mathbb{Z}_n \times \mathbb{Z}_2$ and arc set $\{(i,0)(i+1,0), (i,1)(i-1,1), (i,0)(i,1) \mid i \in \mathbb{Z}_n\}$; $R_5$ is shown in Figure 4. To simplify notation, we write $i \equiv (i,0)$ and $i' \equiv (i,1)$.

Let us first show that

$$\chi_c(R_3) = \frac{5}{3}.$$

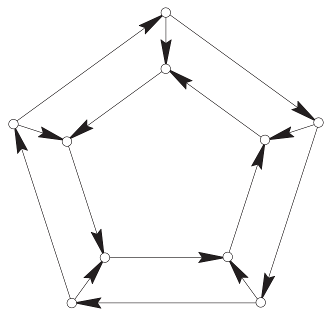

Figure 4: The antioriented prism $R_5$

It is easy to check that the mapping defined by $0, 0' \mapsto 0$, $1 \mapsto 3$, $2 \mapsto 1$, $1' \mapsto 2$, $2' \mapsto 4$ is an acyclic homomorphism from $R_3$ to $\vec{C}(5, 3)$. Therefore $\chi_c(R_3) \le 5/3$. On the other hand, $\chi_c(R_3) \ge 3/2$ since $R_3$ contains a directed 3-cycle, and by Proposition 2.1 this is the only possible value for $\chi_c(R_3)$ smaller than $5/3$. But the value $3/2$ cannot be attained since $R_3$ admits no acyclic homomorphism into $\vec{C}(3, 2) \cong \vec{C}_3$.

When $n \ge 4$, we claim that

$$\chi_c(R_n) = \frac{3}{2}\,.$$

This is slightly surprising since presumably one would expect that $\chi_c(R_n) \to 1$ as $n \to \infty$. A weak circular $\frac{3}{2}$-coloring is easy to find: an acyclic homomorphism into $\vec{C}_3$ is given by mapping $0, 0', (n-1)'$ to 0, sending $1', 2', \ldots, (n-3)', n-1$ to 1, and carrying $1, 2, \ldots, n-3, (n-2)'$ to 2. On the other hand, suppose that there exists a circular $p$-coloring $c$ with $p < 3/2$. Choose $i \in \mathbb{Z}_n$ so that $\mathrm{d}(c(i), c(i'))$ is maximum. Then it is easy to see that the 4-cycle $i$, $i+1$, $(i+1)'$, $i'$, $i$ "winds" twice around $S_p$, while

$$\begin{aligned} &\mathrm{d}(c(i), c(i+1)) + \mathrm{d}(c(i+1), c((i+1)')) \\ &\quad + \mathrm{d}(c((i+1)'), c(i')) + \mathrm{d}(c(i'), c(i)) \quad > \quad 1 + 1 + 1 + 0 = 3\,. \end{aligned}$$

This implies that $p > 3/2$, which is a contradiction.

As a less trivial example, we exhibit another family of digraphs (which we call *daisies*) whose circular chromatic number can be determined. For integers $k, \ell \ge 2$, let $D(k, \ell)$ be the digraph obtained from $\vec{C}_k$ with consecutive vertices $v_1, \ldots, v_k$ by adding, for each $i = 1, \ldots, k$, a directed path of length $\ell - 1$ from $v_{i+1}$ to $v_i$ (indices modulo $k$), completing an $\ell$-cycle.

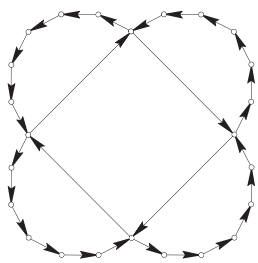

Figure 5: The daisy $D(4,7)$

Figure 5 depicts $D(4,7)$. We can show that

$$\chi_c(D(k,\ell)) = \begin{cases} \dfrac{k}{k-1} & \text{if } k \le \ell \\ \dfrac{(\ell-1)k}{(\ell-2)k + \lfloor k/\ell \rfloor} & \text{if } k > \ell. \end{cases} \tag{4}$$

To prove (4), we will need the following

**Lemma 5.1** *Suppose that $c_0, c_1$ are distinct points on $S_p$, and let $n \ge 2$. Then there is a circular $p$-coloring of $\vec{C}_n$ with winding number $w$ $(1 \le w \le n-1)$ such that two consecutive vertices of $\vec{C}_n$ have respective colors $c_0$ and $c_1$ (in this order) if and only if*

$$1 \le \mathrm{d}(c_0, c_1) \le wp - n + 1. \tag{5}$$

**Proof.** Let $v_0, v_1, \ldots, v_{n-1}$ be the consecutive vertices of $\vec{C}_n$. Having a circular $p$-coloring $c$ of $\vec{C}_n$, let $c_i := c(v_i)$ for $i = 0, \ldots, n-1$. By averaging the values $\mathrm{d}(c_i, c_{i+1})$, we may assume that $\mathrm{d}(c_i, c_{i+1}) =: \alpha$ is constant for $i = 1, 2, \ldots, n-1$. Clearly, $1 \le \alpha < p$ and $\mathrm{d}(c_0, c_1) + (n-1)\alpha = wp$. This yields (5).

For the converse, let $\alpha := (wp - \mathrm{d}(c_0, c_1))/(n-1)$. By (5), we have $\alpha \ge 1$. Since $w \le n-1$, we also have $\alpha < p$. Thus $c(v_i) := c_1 + (i-1)\alpha \pmod p$, for $i = 2, \ldots, n-1$, defines a circular $p$-coloring of $\vec{C}_n$ with winding number $w$. □

This yields an immediate

**Corollary 5.2** *There is a circular $p$-coloring of $\vec{C}_n$ with winding number $w$ such that two consecutive vertices have respective colors $c_0$ and $c_1$ (in this* *order) if and only if*

$$\frac{\mathrm{d}(c_0, c_1) + n - 1}{p} \leq w \leq n - 1.$$

Let us now prove (4). First, let $k \leq \ell$. Observe that $\chi_c(D(k,\ell)) \geq k/(k-1)$, since $D(k,\ell)$ contains $\vec{C}_k$ as a subdigraph. As $(\ell-1)\frac{k}{k-1} - \ell = \frac{\ell-1}{k-1} \geq 1$, Lemma 5.1 shows that the circular $\frac{k}{k-1}$-coloring of the $k$-cycle can be extended to each of the $\ell$-cycles.

Suppose now that $k > \ell$. Let $p$ be the minimum value for which some circular $p$-coloring of the $k$-cycle can be extended to each of the $\ell$-cycles of $D(k,\ell)$. Averaging, if necessary, we may by Lemma 5.1 assume that $\mathrm{d}(c(v_i), c(v_{i+1}))$ is constant for $i = 0, \ldots, k-1$, and denote the common value by $\alpha$. Now

$$\alpha k = w_0 p, \tag{6}$$

for some integer $w_0 \in [1, k-1]$. By Lemma 5.1, we deduce that

$$p \geq \frac{\alpha}{\ell - 1} + 1 \geq \frac{\ell}{\ell - 1}.$$

By our choice of $p$, we may assume that

$$p = \frac{\alpha}{\ell - 1} + 1. \tag{7}$$

Substituting (6) into (7) gives

$$(k(\ell - 1) - w_0)p = k(\ell - 1). \tag{8}$$

Since $\alpha \geq 1$, the relation (6) implies that $p \geq k/w_0$, which, with (8) gives

$$w_0 \geq \frac{k(\ell - 1)}{\ell}. \tag{9}$$

It follows from (8) that $p$ and $w_0$ attain their minima simultaneously. Determining $p$ is thus equivalent to setting

$$w_0 = \left\lceil \frac{k(\ell - 1)}{\ell} \right\rceil. \tag{10}$$

Combining (8) and (10) finally yields the second case in (4).

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