2-restricted extensions of partial embeddings of graphs

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Abstract

Let K be a subgraph of G. Suppose that we have a 2-cell embedding of K in some surface and that for each K-bridge in G one or two simple embeddings in faces of K are prescribed. A linear time algorithm is presented that either finds an embedding of G extending the embedding of K in the same surface using only prescribed embeddings of K-bridges, or finds an obstruction which certifies that such an extension does not exist. It is described how the obtained obstructions can be transformed into minimal obstructions in linear time. The geometric and combinatorial structure of minimal obstructions is also analyzed. At the end we apply the above algorithm to solve general embedding extension problems where the embedding of K is a closed 2-cell embedding.

1 Introduction

Let K_0 be a fixed graph together with a fixed 2-cell embedding in some (closed) surface. Let G be a graph containing a subgraph K homeomorphic to K_0. The embedding of K_0 and the homeomorphism $K \to K_0$ determine a 2-cell embedding of K. *Embedding extension problem* asks if it is possible to extend the embedding of K to an embedding of G. A subgraph Ω of $G - E(K)$ is an *obstruction* (for embedding extensions of K to G) if there is no embedding of $K \cup \Omega$ extending the embedding of K. Almost all embedding extension problems can be, roughly speaking, reduced to a number of some special embedding extension subproblems in which for every K-bridge in G, at most two of the possible simple embeddings in faces of K are allowed. (Definitions of the undefined terms are given in the sequel.) Such an embedding extension problem is said to be *2-restricted*. Even though this special problem looks rather simple, possible obstructions may have

Embedding extension problem

obstruction

2-restricted

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quite complicated structure. It is the purpose of this paper to exhibit and analyze this structure. After some computational preliminaries which are discussed in Section 2, embedding extension problems and related notions are introduced in Section 3. In Section 4, a linear time algorithm for solving 2-restricted embedding extension problems (with some additional properties) is described. The algorithm either finds an admissible embedding extension, or returns an obstruction for such embedding extensions. When solving special cases of cylinder and Möbius band embedding extension problems [M2, JM], a special "millipede" structure of obstructions was observed. In Sections 5, 6, and 7 we introduce general *millipedes* and show that every minimal obstruction can be written as the union of a bounded number of K-bridges together with a bounded number of pairwise disjoint millipedes, where the bounds depend only on K_0 . This enables us to transform obtained obstructions into minimal obstructions in linear time. Let us remark that minimal obstructions can be arbitrarily large. In Section 8 we use the millipede structure of obstructions to perform an operation called *compression*. This operation "slightly" changes the subgraph K such that under the embedding extension problem with respect to the new subgraph, an obstruction with a bounded number of branches exists. In our linear time estimates it is crucial that K is homeomorphic to the fixed graph K_0 since the constant factors in these estimates heavily depend on the number of edges of K_0. Section 9 presents a typical application of our results. It is shown how to extend closed 2-cell embeddings of graphs in linear time.

millipedes

compression

Results of this paper play a fundamental role in solving more general embedding extension problems, from the algorithmic and theoretical point of view; see, e.g., [JMM2, JMM3, M4]. In particular, our results are extensively used in the design of a linear time algorithm to construct embeddings of graphs in an arbitrary (fixed) surface [M4], generalizing the well-known Hopcroft-Tarjan algorithm [HT] for testing planarity in linear time.

2 Computational preliminaries

In our algorithms, we consider embeddings of graphs. In case of orientable surfaces, 2-cell embeddings can be described combinatorially [GT] by specifying a *rotation system*: for each vertex v of the graph G we have a cyclic permutation π_v of its incident edges, representing their circular order around v on the surface. To describe embeddings in non-orientable surfaces we need another information, a function $\lambda : E(G) \to \{1, -1\}$, called a *signature* (cf. [GT] for details). However, signatures are really needed only to present a 2-cell embedding of K, while the embeddings of K-bridges in particular faces of K can be encoded by specifying only corresponding rotation systems. In order to make a clear presentation of our algorithm, we have decided to use this description only implicitly. Whenever we say that we have an embedding, we mean such a combinatorial description.

rotation system

signature

Concerning the time complexity of our algorithms, we assume a random-access machine (RAM) model with unit cost for basic operations. This model was introduced by

Cook and Reckhow [CR] and used in [HT] to describe a linear time planarity testing algorithm. More precisely, our model is the *unit-cost* RAM where operations on integers, whose value is $O(n)$, need only constant time (where n is the size of the given graph). unit-cost

We will also need the following simulation of parallelism performed on a unit-cost RAM. At certain steps of our algorithm we will not be able to decide in advance between two possible choices. In such a case we will continue computations simultaneously in both directions. This will enable us to efficiently choose between the two alternatives. During such parallel computations no new parallelism will be introduced.

Denote by $\mathcal{P}_1$ and $\mathcal{P}_2$ both parallel processes. During the parallel computation exactly one of the following three cases will occur:

(i) The process $\mathcal{P}_1$ terminates *successfully*. This means that at the beginning of the parallelism the decision for $\mathcal{P}_1$ would be the right one. In this case, we say that the parallel computation terminates *successfully*. In this case we stop $\mathcal{P}_2$ (if still active) and restore the memory to the state before starting parallelism, choose the alternative $\mathcal{P}_1$ as the proper one and continue with (non-parallel) computation from this point on. successfully successfully

(ii) If $\mathcal{P}_2$ terminates successfully, then we act as in the previous case, except that we stop $\mathcal{P}_1$ and choose the second alternative as the right one.

(iii) If none of $\mathcal{P}_1, \mathcal{P}_2$ terminates successfully, then the parallel computation is said to terminate *non-successfully*. non-successfully

If one of the processes fails, we still continue to run the remaining one. If it succeeds, case (i) or (ii) occurs; if also the other process fails, we have case (iii).

In our application of parallelism, the processes $\mathcal{P}_1$ and $\mathcal{P}_2$ will try to extend a partial embedding of a graph in two different ways. If appropriate embedding extension is found by one of them, this process will be termed as successful. Otherwise an obstruction for a particular type of embedding extension problem will be found. In case (iii) the "union" of both obstructions will give rise to a more general obstruction.

It is explained in [JM] how the memory management and other details are to be handled in order that parallelism increases the overall time complexity only by a constant factor.

3 (2-restricted) embedding extension problems

Our approach to embedding extension problems is based on the concept of a bridge. Let K be a subgraph of a simple graph G. A *K-bridge* B in G is a subgraph of G which is either an edge $uv \in E(G) \setminus E(K)$ (together with its endpoints) such that $u, v \in V(K)$, or it is a connected component B° of $G - V(K)$ together with all edges (and their endpoints) between B° and K. The vertices of $V(B) \cap V(K)$ are the *vertices of attachment* (simply *attachments*) of B. K-bridge vertices of attachment

A vertex of K of degree in K different from 2 is a *main vertex* (or a *branched vertex*) attachments main vertex branched vertex of K. A *closed branch* of K is any path (possibly a closed path) in K whose endpoints are main vertices but no internal vertex on this path is a main vertex. A branch without its endpoints is an (*open*) *branch*. Main vertices and open branches of K are also called *basic pieces* of K. Note that K is a disjoint union of its basic pieces. Every set $T \neq \emptyset$ of basic pieces of K is called a (*combinatorial*) *type*.

closed branch
(open) branch
basic pieces
(combinatorial) type

A K-bridge B is *local* (on e) if there is a closed branch e of K such that all attachments of B are on e. For a K-bridge B, denote by $T = \mathit{type}(B)$ the set of all basic pieces of K that B is attached to. We say that B is of *type* T. If $|T| > 2$, then B is *strongly attached* to K.

local
type
strongly attached

Suppose that K is 2-cell embedded in some surface. Let F be a face of the embedding and T a (combinatorial) type. An *embedding scheme* δ for T in the face $F(\delta) = F$ is a set of appearances of basic pieces from T on ∂F such that each basic piece from T is selected at least once. If each basic piece is selected exactly once, then δ is *simple*.

embedding scheme
simple

Let δ be an embedding scheme for the type T. An embedding of a K-bridge B of type T in $F(\delta)$ is *δ-compatible* if the embedding uses only appearances of basic pieces from δ. We also say that B is *δ-embedded*. An embedding of B in F is *simple* if it is δ-compatible for some simple embedding scheme δ.

δ-compatible
δ-embedded
simple

Embedding schemes δ_1 and δ_2 *overlap* if $F(\delta_1) = F(\delta_2)$ and either they share three or more appearances of basic pieces or they contain pairs of appearances of basic pieces that interlace on $\partial F(\delta_1)$. Schemes δ_1 and δ_2 are *independent* if either $F(\delta_1) \neq F(\delta_2)$ or $F(\delta_1) = F(\delta_2)$ and $\partial F(\delta_1)$ contains appearances x, y of main vertices of K such that δ_1 and δ_2 use distinct segments of $\partial F(\delta_1)$ from x to y (possibly both including x and/or y). Note that there are pairs of embedding schemes that are neither independent nor overlapping.

overlap
independent

For every K-bridge B in G, let us choose a non-empty set of simple embedding schemes $\mathcal{D}(B)$ for $\mathit{type}(B)$. A (*simple*) *embedding extension problem* (shortly EEP) is a quadruple $\Xi = (G, K, \Pi, \mathcal{D})$, where K is a subgraph of G, Π is (a combinatorial description of) a 2-cell embedding of K in some surface, and $\mathcal{D} = \{\mathcal{D}(B) \mid B \text{ a } K\text{-bridge in } G\}$. Embedding schemes from $\mathcal{D}(B)$ are called *admissible embedding schemes* for B. Embeddings of B that are compatible with some admissible embedding scheme are called *admissible embeddings*.

(simple) embedding extension problem
admissible embedding schemes
admissible embeddings

A *solution* for Ξ is an embedding of G extending the embedding Π of K (to an embedding of G in the same surface) such that under this embedding, each K-bridge B is δ-embedded for some $\delta \in \mathcal{D}(B)$. An *obstruction* for Ξ is a set Ω of K-bridges with the property that the EEP $(K \cup \Omega, K, \Pi, \mathcal{D}')$, where $\mathcal{D}' = \{\mathcal{D}(B) \mid B \in \Omega\}$, has no solution. An obstruction Ω is *minimal*, if no bridge from Ω is redundant, i.e., for each $B \in \Omega$, $\Omega \backslash \{B\}$ is not an obstruction.

solution
obstruction
minimal

The EEP Ξ is *2-restricted* (shortly 2-EEP) if for every bridge B in G, we have $|\mathcal{D}(B)| \leq 2$. A bridge that has only one admissible embedding scheme is called *1-embeddable* (or *uniquely embeddable*). A bridge B is *$\frac{3}{2}$-embeddable* if $F(\delta_1) = F(\delta_2)$ (where $\mathcal{D}(B) = \{\delta_1, \delta_2\}$ and $\delta_1 \neq \delta_2$) and there exists a basic piece x of K such that both, δ_1 and δ_2, contain the same appearance of x. In such a case, x is called the *base* of the $\frac{3}{2}$-embeddable bridge B.

2-restricted
1-embeddable
uniquely embeddable
$\frac{3}{2}$-embeddable
base

From now on we will restrict ourselves to 2-EEPs. We will assume that Ξ has some additional properties that are discussed in the sequel. These properties, although primarily of a technical nature, will considerably simplify the description of an efficient algorithm for solving 2-EEPs and the analysis of obstructions.

Firstly, we require that Ξ has the following property:

(P1) For every pair B', B'' of bridges of the same type we have:

$$\mathcal{D}(B') \subseteq \mathcal{D}(B'') \text{ or } \mathcal{D}(B'') \subseteq \mathcal{D}(B') \text{ or } \mathcal{D}(B') \cap \mathcal{D}(B'') = \emptyset\,.$$

Choose a type T and let $\mathcal{B}(T)$ be the set of all K-bridges in G of type T. By (P1), the following defines an equivalence relation in $\mathcal{B}(T)$:

$$B' \sim B'' \overset{\text{def}}{\Longleftrightarrow} \exists B \in \mathcal{B}(T) : \mathcal{D}(B') \cup \mathcal{D}(B'') \subseteq \mathcal{D}(B)\,.$$

The equivalence classes of $\sim$ are called *clusters* of G (with respect to Ξ). By construction, each cluster $\mathcal{C}$ is associated with a type $T = \mathit{type}(\mathcal{C})$ and with (one or) two admissible embedding schemes for $\mathit{type}(\mathcal{C})$. These schemes are denoted by $\delta_1(\mathcal{C})$ and $\delta_2(\mathcal{C})$. If there is only one admissible scheme for $\mathcal{C}$, $\delta_2(\mathcal{C})$ is undefined. For a bridge $B \in \mathcal{C}$ and $\ell \in \{1, 2\}$, we denote by $B[\ell]$ a $\delta_\ell(\mathcal{C})$-embedding of B. clusters

Next, we assume that Ξ satisfies:

(P2) For every K-bridge B and every $\delta \in \mathcal{D}(B)$, B can be δ-embedded.

(P3) There are no local K-bridges in G.

Let us briefly comment on the above properties. Since the admissible embedding schemes are simple, it is easy to check in linear time by planarity testing whether B can be δ-embedded. Therefore (P2) can be guaranteed in linear time by appropriate preprocessing. In applications, (P3) can be achieved in linear time by using the results from [JMM1].

Given a subgraph H of G, the number of branches of H, denoted by $\mathrm{bsize}(H)$, is called the *branch size* of H. In solving 2-EEPs, results of [M3] enable us to perform a linear time preprocessing and henceforth assume that Ξ has also the following property: branch size

(P4) For every K-bridge B, $\mathrm{bsize}(B)$ is bounded by a constant (depending on $|\mathit{type}(B)|$).

Let us remark that it is crucial for this reduction that all admissible embedding schemes are simple.

Additionally, the following two technical assumptions will be helpful:

(P5) For every pair $x, y \in V(K)$ of vertices each cluster contains at most one bridge that is attached to K only at x and y.

(P6) Each cluster contains at most two strongly attached K-bridges.

Note that if there is a cluster that contains more than two strongly attached bridges, then any three of these bridges form an obstruction for Ξ.

4 A linear time algorithm

In this section, a linear time algorithm for solving 2-EEPs is presented.

Theorem 4.1 *There exists an algorithm that, given a* 2-EEP $\Xi = (G, K, \Pi, \mathcal{D})$ *with the properties* (P1)–(P6)*, either finds a solution of* Ξ*, or returns an obstruction for* Ξ*. The time complexity of the algorithm is* $\mathcal{O}(\kappa \cdot |E(G)|)$ *where* κ *depends only on* $\mathrm{bsize}(K)$*.*

The rest of the section is devoted to the proof of Theorem 4.1. In the description of the algorithm we will use *oriented basic pieces* of K. If e is a branch, it is homeomorphic to the open interval $(0, 1)$, and any orientation of $(0, 1)$ determines an *orientation* of e. If x is a main vertex of K, then we associate x with two virtual edges forming an open path P with the middle vertex x. Now, orientations of x correspond to the orientations of P. If K is embedded and we are considering an appearance of x on the boundary of a face F, then the two virtual edges can be identified with the two edges that precede and follow, respectively, the appearance of x in the facial walk of F. oriented basic pieces orientation

Let ε be an oriented basic piece. By *type*(ε) we denote the basic piece whose orientation is given by ε. Suppose that $e = \mathit{type}(\varepsilon)$. Its first edge (with respect to ε) is denoted by $\mathrm{left}(\varepsilon)$. Similarly, the last edge is denoted by $\mathrm{right}(\varepsilon)$. (If e is a main vertex, then $\mathrm{left}(\varepsilon)$ and $\mathrm{right}(\varepsilon)$ are the virtual edges of e.) The orientation of ε gives rise to a linear ordering of vertices (and edges) of e. For $u, v \in V(e)$, we write $u < v$, if u is closer to $\mathrm{left}(\varepsilon)$ than v. We also say that u *precedes* v on ε. Analogously, $u \leq v$, if $u < v$ or $u = v$. The set $(u, v) = \{w \in V(e) \mid u < w < v\}$ is called an *open segment* of e. Segments $(u, v]$, $[u, v)$, and $[u, v]$ are defined similarly. precedes open segment

For an oriented basic piece ε, we denote by $\overline{\varepsilon}$ the orientation of *type*(ε) that is opposite to ε. In particular, $\mathrm{left}(\varepsilon) = \mathrm{right}(\overline{\varepsilon})$, and conversely. Let B be a bridge that is attached to ε. By $\varepsilon(B)$ we denote the first attachment of B on ε. Then $\overline{\varepsilon}(B)$ is the last attachment of B on ε.

Let $\mathcal{C}$ be a cluster and let ε be an oriented basic piece such that $\mathit{type}(\varepsilon) \in \mathit{type}(\mathcal{C})$. In our algorithm, we will have three variables $\alpha(\mathcal{C}, \varepsilon)$, $\beta_1(\mathcal{C}, \varepsilon)$, and $\beta_2(\mathcal{C}, \varepsilon)$. The value of each of them is either a vertex of ε, or one of $\mathrm{left}(\varepsilon)$, $\mathrm{right}(\varepsilon)$. Their interpretation in our algorithm is as follows:

(a) If $u = \alpha(\mathcal{C}, \varepsilon)$, then all bridges B from $\mathcal{C}$ that have an attachment on ε strictly before u have already been considered by the algorithm, one of their admissible embeddings has been chosen, and all restrictions to embeddings of other bridges that are imposed by the chosen embedding have been discovered and used in updating the corresponding values of β_1's and β_2's. Such bridges B will be called *already embedded*. Moreover, no two already embedded bridges (possibly from different clusters) interfere with each other, i.e., all the chosen embeddings can be realized simultaneously. already embedded

(b) $\beta_1(\mathcal{C}, \varepsilon)$ is a vertex of ε satisfying the following requirement. Each bridge of $\mathcal{C}$ that has an attachment on $[\alpha(\mathcal{C}, \varepsilon), \beta_1(\mathcal{C}, \varepsilon))$ is either already embedded, or its $\delta_1(\mathcal{C})$-

embedding is obstructed by some of the already embedded bridges (from the same or from some other clusters). If there is no such vertex, then $\beta_1(\mathcal{C}, \varepsilon) = \text{right}(\varepsilon)$.

(c) $\beta_2(\mathcal{C}, \varepsilon)$ is defined analogously except that $\delta_1(\mathcal{C})$-embeddings are replaced by $\delta_2(\mathcal{C})$-embeddings. If bridges from $\mathcal{C}$ are 1-embeddable, then we set $\beta_2(\mathcal{C}, \varepsilon) = \beta_1(\mathcal{C}, \varepsilon)$.

Let $\mathcal{C}$ be a cluster of weakly attached bridges with $type(\mathcal{C}) = \{e, f\}$. Denote by δ_1, δ_2 the admissible embedding schemes for $\mathcal{C}$. Suppose that ε is an orientation of e. Then ε induces an orientation ϕ_i of f through $F(\delta_i)$ by the following requirement: under every δ_i-embedding of a bridge $B \in \mathcal{C}$ the appearances left(ε) and left(ϕ_i) lie on the boundary of the same subface of $F(\delta_i)$. We say that ϕ_i is *induced* by ε and δ_i. We define two doubly linked lists $S_1 = S(\mathcal{C}, \varepsilon, \delta_1)$ and $S_2 = S(\mathcal{C}, \varepsilon, \delta_2)$. Each list S_i ($i \in \{1, 2\}$) contains all bridges from $\mathcal{C}$. Their order in S_i is consistent with the following requirements: induced

(S1) If $\varepsilon(Q) < \varepsilon(R)$, then Q precedes R in S_i.

(S2) If $\varepsilon(Q) = \varepsilon(R)$ and $\phi_i(Q) < \phi_i(R)$, then Q precedes R in S_i.

(S3) If $\varepsilon(Q) = \varepsilon(R)$, $\phi_i(Q) = \phi_i(R)$, and Q is attached only to $\varepsilon(Q)$ and to $\phi_i(Q)$, while R has at least three vertices of attachment, then Q precedes R in S_i.

If a pair of bridges from $\mathcal{C}$ does not fit any of (S1), (S2), or (S3), then their mutual order in S_i is arbitrary. It is easy to see that there always exists a linear ordering of bridges from $\mathcal{C}$ that is consistent with the above requirements.

It is important to observe that, if a set of bridges from $\mathcal{C}$ is δ_i-embedded, then their order determined by the embedding (and by the choice of ε) is the same as their order in $S(\mathcal{C}, \varepsilon, \delta_i)$ and in $S(\mathcal{C}, \phi_i, \delta_i)$. Note that this property holds only if (P5) is fulfilled.

Let us now describe how to build the lists $S(\mathcal{C}, \varepsilon, \delta_i)$ in linear time. Let $e = type(\varepsilon)$. Denote by $v_1, v_2, \ldots, v_k$ the vertices of e in the order as determined by ε. The list $S(\mathcal{C}, \varepsilon, \delta_i)$ will be obtained as a concatenation of lists $S^j = S^j(\mathcal{C}, \varepsilon, \delta_i)$, $1 \le j \le k$, where each S^j links all bridges $B \in \mathcal{C}$ with $\varepsilon(B) = v_j$ in the order respecting (S2) and (S3). The lists S^j are constructed simultaneously by the following algorithm:

```
S^j := ∅, j = 1, ..., k
for all u ∈ V(f) do
  { The vertices u are taken in order as they appear on
    f according to the orientation φ_i. }
  for all B ∈ C with φ_i(B) = u do
    if B is attached only to two vertices then
      add B at the end of S^j, where v_j = ε(B)
  endfor
  for all B ∈ C with φ_i(B) = u do
    if B is attached to three or more vertices then
      add B at the end of S^j, where v_j = ε(B)
  endfor
endfor
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Concatenate $S^1, \ldots, S^k$ into $S(\mathcal{C}, \varepsilon, \delta_i)$.

It is easy to implement the traversals in the above algorithm so that the overall time spent by the algorithm is linear. Note that the double traversal of bridges with $\phi_i(B) = u$ assures that (S3) will be fulfilled. Condition (S2) is satisfied at the end since the traversal of the "opposite" branch f is performed in the direction imposed by ϕ_i. Clearly, (S1) is guaranteed by the use of sublists S^j and their concatenation at the end. In the main algorithm, these lists are used to efficiently check simultaneous δ_i-embeddability of those bridges from $\mathcal{C}$ that have an attachment in the chosen segment of ε and ϕ_i.

To unify the presentation of the main algorithm, we may assume that the lists $S(\mathcal{C}, \varepsilon, \delta_i)$ are built also for clusters with $|type(\mathcal{C})| > 2$ since such clusters contain at most two bridges.

To be able to remove bridges from the lists in constant time, for every bridge $B \in \mathcal{C}$ a list of pointers to its appearances in the lists $S(\mathcal{C}, \varepsilon, \delta_i)$ is also maintained.

We are now ready to discuss the main part of the algorithm. Denote by $\mathcal{B}$ the set of all K-bridges in G. Roughly speaking, the algorithm Extend is based on the following idea. Suppose that a subset of bridges $\mathcal{B}' \subseteq \mathcal{B}$ is already embedded. Their presence within faces of K blocks some admissible embeddings of the remaining bridges. If there is a bridge that has no admissible embedding left, then we are done. Otherwise, some of the remaining bridges can be embedded in one way only. We say that these bridges are *forced* to have the remaining embedding. By fixing that embedding and forced adding these bridges to $\mathcal{B}'$, we obtain additional bridges with at most one admissible embedding. By repeating this procedure, we either get stuck (which proves that no embedding extension exists with the initial $\mathcal{B}'$ embedded as given), or no more bridges are blocked by the chosen embedding of $\mathcal{B}'$. In the latter case, it is clear that the bridges in $\mathcal{B}'$ can be left embedded as they are without obstructing any admissible embeddings of the remaining bridges. The procedure described above is called Forcing.

It is easy to implement the above idea in quadratic time. To realize it in linear time much more sophisticated approach is needed. Let us now discuss some details needed for efficient implementation of algorithm Extend. At the very beginning, we select an arbitrary non-empty cluster $\mathcal{C}_0$ and a bridge $B_0 \in \mathcal{C}_0$ which is initial in some of the lists $S(\mathcal{C}_0, \varepsilon_0, \delta)$. (The choice of B_0 as the initial bridge in one of the lists is helpful but not necessary.) Then we start two parallel processes: the first one starts with $B_0[1]$, while the second one tries to extend $B_0[2]$. (If B_0 is 1-embeddable, then parallelism is not needed). The details how to perform such parallel computations without increasing the overall time complexity are described in Section 2. Each of the two parallel processes either finds an admissible embedding for a set of bridges which does not interfere with any admissible embedding of the remaining bridges (*successful* termination), or it gets successful stuck (*non-successful* termination). It has been described in Section 2 how the two non-successful parallel processes react if one or the other stops successfully. The parallel computation is *successful* if at least one of the two parallel processes stops successfully. Otherwise, it successful is *non-successful*. non-successful

The lists $S(\mathcal{C}, \varepsilon, \delta)$ are updated during the algorithm by removing the already embedded bridges. We also use bridges $B(\mathcal{C}, \varepsilon, \delta)$. They are needed only for efficient construction of obstructions and their use is described in more details in the next section. We denote by $\mathcal{B}$ the set of bridges that have not yet been embedded. The main part of the algorithm is the following:

algorithm EXTEND
  Determine all lists $S(\mathcal{C}, \varepsilon, \delta)$.
  Initialize auxiliary variables for parallel computations.
  for every pair $(\mathcal{C}, \varepsilon)$ **do**
    Let $\alpha(\mathcal{C}, \varepsilon) = \beta_1(\mathcal{C}, \varepsilon) = \beta_2(\mathcal{C}, \varepsilon)$ be the initial vertex of ε.
  endfor
  while $\mathcal{B} \neq \emptyset$ **do**
    Select $\mathcal{C}_0$ and the initial bridge $B_0 \in \mathcal{C}_0$.
    Split $\mathcal{C}_0$ with respect to B_0.
    { Parallel part follows. }
    for every admissible embedding $\delta_0 \in \mathcal{D}(B_0)$ **do in parallel**
      Embed B_0 compatibly with δ_0.
      for every pair $(\mathcal{C}, \varepsilon)$ **do**
        Update the values of $\beta_i(\mathcal{C}, \varepsilon)$, $B(\mathcal{C}, \varepsilon, \delta_i(\mathcal{C}))$, $i = 1, 2$.
      endfor
      FORCING
    end parallel for
    { Parallel part is finished. }
    if not successful **then** RETURN(obstruction)
  endwhile
  { If we reach this point, all bridges have been embedded. }
  RETURN(solution)

Before the algorithm enters the parallel part, a splitting of the chosen cluster $\mathcal{C}_0$ with respect to the bridge B_0 occurs. The bridge B_0 is chosen so that it is the first bridge in one of the lists $S(\mathcal{C}_0, \varepsilon_0, \delta_i(\mathcal{C}_0))$. If B_0 is strongly attached or if it is also the first bridge in some list $S(\mathcal{C}_0, \varepsilon', \delta_{3-i}(\mathcal{C}_0))$, then the splitting of a cluster is trivial, i.e., the cluster $\mathcal{C}_0$ remains unchanged. Otherwise $\mathcal{C}_0$ is split as follows. We introduce two additional (sub)clusters $\mathcal{C}_0'$ and $\mathcal{C}_0''$. The cluster $\mathcal{C}_0'$ contains those bridges from $\mathcal{C}_0$ that are attached to ε_0 only at $\varepsilon_0(B_0)$, while the cluster $\mathcal{C}_0''$ contains those bridges from $\mathcal{C}_0$ that are attached to ε_0 at $\varepsilon_0(B_0)$ and also in some other vertex. Bridges from $\mathcal{C}_0$ that are not attached to $\varepsilon_0(B_0)$ remain in $\mathcal{C}_0$. Admissible embedding schemes remain unchanged. The distribution of bridges from $\mathcal{C}_0$ among $\mathcal{C}_0$, $\mathcal{C}_0'$, and $\mathcal{C}_0''$ is therefore the same as obtained if we would consider $\varepsilon_0(B_0)$ as a main vertex of K. Observe that if there are more than two bridges in $\mathcal{C}_0''$, then any three of them form a small obstruction for Ξ. By traversing the initial part of $S(\mathcal{C}_0, \varepsilon_0, \delta_i(\mathcal{C}_0))$ it is easy to implement the above splitting (and the initialization of the required auxiliary variables) in time that is proportional to the number of bridges in $\mathcal{C}_0' \cup \mathcal{C}_0''$. It may happen that in the current iteration of the loop **while**, only a part of the bridges from $\mathcal{C}_0' \cup \mathcal{C}_0''$ is embedded. Since a splitting of a new cluster prior to the next iteration can increase the time complexity, we always choose as the initial cluster $\mathcal{C}_0$ one of the subclusters $\mathcal{C}_0'$, $\mathcal{C}_0''$ if they are not empty. This assures that all bridges from $\mathcal{C}_0' \cup \mathcal{C}_0''$ are embedded before another splitting occurs. Hence, the overall time spent by the algorithm on splitting of clusters is proportional to the number of embedded bridges. Splitting of clusters therefore does not increase the overall time complexity.

The splitting of a cluster assures that B_0 is also the first bridge in some of the lists for the embedding scheme δ_{3-i}. This property is used when updating the values of $\beta_i(\mathcal{C}, \varepsilon)$ $(i = 1, 2)$.

The chosen embedding of B_0 is used to start the procedure Forcing that is described below:

procedure Forcing
{ Some bridges are already embedded. They block some embeddings of
  the remaining bridges. A bridge $B \in \mathcal{C}$ that is not yet embedded and is
  attached to a segment $[\alpha(\mathcal{C}, \varepsilon), \beta_i(\mathcal{C}, \varepsilon))$ must be δ_{3-i}-embedded. }
  while $\exists(\mathcal{C}, \varepsilon)$ such that $\alpha(\mathcal{C}, \varepsilon) \neq \beta_1(\mathcal{C}, \varepsilon)$ **or** $\alpha(\mathcal{C}, \varepsilon) \neq \beta_2(\mathcal{C}, \varepsilon)$ **do**
    if $\alpha(\mathcal{C}, \varepsilon) \neq \beta_1(\mathcal{C}, \varepsilon)$ **and** $\alpha(\mathcal{C}, \varepsilon) \neq \beta_2(\mathcal{C}, \varepsilon)$ **then**
      $\beta := \min\{\beta_1(\mathcal{C}, \varepsilon), \beta_2(\mathcal{C}, \varepsilon)\}$ (with respect to orientation ε)
      if $\exists B \in \mathcal{B} \cap \mathcal{C}$ attached to $[\alpha(\mathcal{C}, \varepsilon), \beta)$ **then** Stop(not successful)
      $\alpha(\mathcal{C}, \varepsilon) := \beta$
    endif
    if $\alpha(\mathcal{C}, \varepsilon) \neq \beta_1(\mathcal{C}, \varepsilon)$ **then** $i := 1$ **else** $i := 2$ **endif**
    $\mathcal{B}_i :=$ all bridges in $\mathcal{B} \cap \mathcal{C}$ attached to $[\alpha(\mathcal{C}, \varepsilon), \beta_i(\mathcal{C}, \varepsilon))$
    if $\mathcal{B}_i \neq \emptyset$ **then**
      Embed $\mathcal{B}_i$ compatibly with $\delta_{3-i}(\mathcal{C})$.
      if no embedding exists **then** Stop(not successful)
      Remove bridges $\mathcal{B}_i$ from all lists $S(\mathcal{C}, \varepsilon', \delta)$ in which they occur.
      for every pair $(\mathcal{C}', \varepsilon')$: $(\mathcal{C}', \varepsilon') \neq (\mathcal{C}, \varepsilon)$ **do**
        Update the values of $\beta_j(\mathcal{C}', \varepsilon')$ and $B(\mathcal{C}', \varepsilon', \delta_j(\mathcal{C}'))$, $j = 1, 2$.
      endfor
      $\mathcal{B} := \mathcal{B} \setminus \mathcal{B}_i$
    endif
    $\alpha(\mathcal{C}, \varepsilon) := \beta_i(\mathcal{C}, \varepsilon)$
    $\beta_{3-i}(\mathcal{C}, \varepsilon) := \beta_i(\mathcal{C}, \varepsilon)$
  endwhile
  Stop(successful)
end {Forcing}

As far as the algorithm is concerned, the choice of pairs $(\mathcal{C}, \varepsilon)$ in the main loop of Forcing is arbitrary if there are several candidates. However, to achieve certain additional properties of obtained obstructions, we initially choose a linear ordering of all possible pairs $(\mathcal{C}, \varepsilon)$. At each iteration, we search for candidates $(\mathcal{C}, \varepsilon)$ cyclically in the chosen order from the point where we stopped previously. This assumption will be used in the process of minimizing the obtained obstruction in Sections 5–7. An implementation of Algorithm EXTEND that uses such a selection scheme for choosing pairs $(\mathcal{C}, \varepsilon)$ is called a *BF-implementation* of Algorithm EXTEND. BF-implementation

The search for $B \in \mathcal{B} \cap \mathcal{C}$ that is attached to $[\alpha(\mathcal{C}, \varepsilon), \beta)$ in the procedure FORCING can be easily performed by advancing through the list $S(\mathcal{C}, \varepsilon, \delta_1)$ or $S(\mathcal{C}, \varepsilon, \delta_2)$. Similarly, the construction of the set $\mathcal{B}_i$ and the testing for the simultaneous δ_{3-i}-embeddability of $\mathcal{B}_i$ can be implemented by moving along the list $S(\mathcal{C}, \varepsilon, \delta_i)$ and comparing the extreme vertices of attachment of bridges with the values of $\beta_{3-i}(\mathcal{C}, \tilde{\varepsilon})$ (where $type(\tilde{\varepsilon}) \in type(\mathcal{C})$). Bridges that become embedded are removed from all the lists $S(\mathcal{C}, \varepsilon', \delta)$, from $\mathcal{B}$ and also from the cluster $\mathcal{C}$.

It remains to explain how to update the values of $\beta_i(\mathcal{C}, \varepsilon)$ and $B(\mathcal{C}, \varepsilon, \delta_i)$, $(i = 1, 2)$. Let us first describe how to do that at the very beginning of the parallel part. The parallel part of algorithm EXTEND starts with a δ_0-embedding of the bridge $B_0 \in \mathcal{C}_0$. Splitting of the cluster $\mathcal{C}_0$ guarantees that there is an oriented basic piece ε_0 such that B_0 is the initial bridge in $S(\mathcal{C}_0, \varepsilon_0, \delta_0)$. Given $\mathcal{C}$, ε such that $type(\varepsilon) \in type(\mathcal{C})$, and $i \in \{1, 2\}$, we change the value of $\beta_i(\mathcal{C}, \varepsilon)$ according to one of the following cases:

(a) If δ_0 and $\delta_i(\mathcal{C})$ are independent, then we leave $\beta_i(\mathcal{C}, \varepsilon)$ unchanged.

(b) If δ_0 and $\delta_i(\mathcal{C})$ overlap, then we set $\beta_i(\mathcal{C}, \varepsilon) = \text{right}(\varepsilon)$.

(c) Suppose that $\mathcal{C} \neq \mathcal{C}_0$ and that δ_0, $\delta := \delta_i(\mathcal{C})$ neither overlap nor are independent. Then $F_0 := F(\delta_0) = F(\delta)$ and δ_0 and δ share an appearance of an open branch (or two) on ∂F_0. The embedding of B_0 dissects ∂F_0 in several closed segments. Since δ_0 and δ do not overlap, one of these segments contains (parts of) all the appearances from δ. Denote this segment by S. If $type(\varepsilon)$ is a main vertex or the appearance of ε in δ does not appear in δ_0, we leave $\beta_i(\mathcal{C}, \varepsilon)$ unchanged. If $\text{left}(\varepsilon) \in S$, then we also leave $\beta_i(\mathcal{C}, \varepsilon)$ unchanged. On the other hand, if $\text{left}(\varepsilon) \notin S$, we set $\beta_i(\mathcal{C}, \varepsilon) = \overline{\varepsilon}(B_0)$.

(d) It remains to consider the case when $\mathcal{C} = \mathcal{C}_0$ and cases (a), (b) do not apply. If $|type(\mathcal{C})| > 2$, then we apply exactly the same procedure as in (c). Suppose now that $type(\mathcal{C}) = \{e, f\}$. Excluding (a), at least one of e, f is an open branch. Since the cases when e or f is a main vertex behave like a special case of the possibility when e, f are both open branches, the latter is assumed henceforth. We distinguish two possibilities:

   (d1) Suppose that $\delta_0 = \delta$. Since B_0 is the first bridge in $S(\mathcal{C}_0, \varepsilon_0, \delta_0)$, we set $\beta_i(\mathcal{C}, \varepsilon) = \overline{\varepsilon}(B_0)$ if $\varepsilon \in \{\varepsilon_0, \phi_i\}$ (where ϕ_i is induced by ε_0 and $\delta_i(\mathcal{C})$). Otherwise, we leave the value of $\beta_i(\mathcal{C}, \varepsilon)$ unchanged. This case is illustrated in Figure 1. The changes of $\beta_i(\mathcal{C}, \varepsilon)$ are represented by bold segments which show that δ-embeddings of any bridge from cluster $\mathcal{C}$ attached to a vertex inside these segments is blocked by B_0.

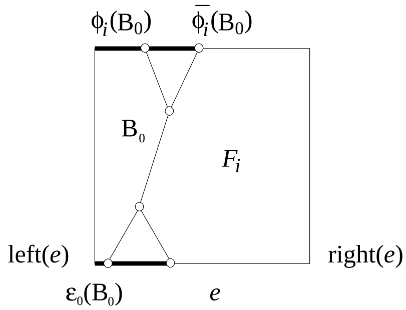

Figure 1: Case (d1)

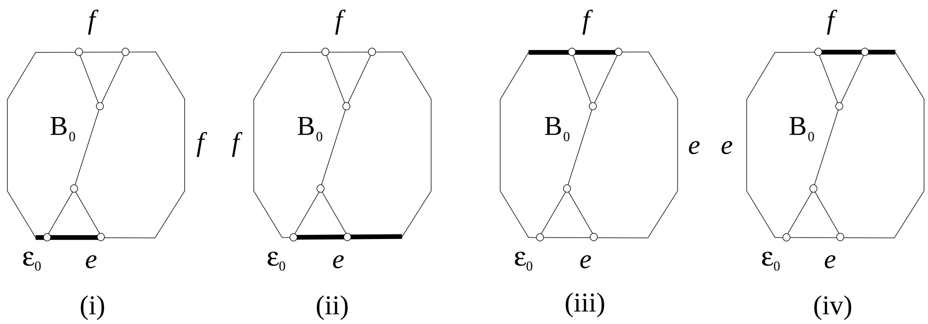

Figure 2: Case (d2)

(d2) The remaining possibility is when $\delta_0 \neq \delta$. Since we have excluded cases (a) and (b), we have $F_0 := F(\delta_0) = F(\delta)$. Moreover, δ_0 and δ use the same appearance of exactly one of e and f on ∂F_0. We may assume that $type(\varepsilon_0) = e$. All four possible positions for the appearances of e, f on ∂F_0 that are used by δ_0 and δ are shown in Figure 2. Bridges from $\mathcal{C}$ that are attached to bold segments have blocked their δ-embeddings by the chosen embedding of B_0. This determines the change of $\beta_i(\mathcal{C}, \varepsilon)$. For example, Figure 2(i) represents the case when $\beta_i(\mathcal{C}, \varepsilon_0)$ is set to $\overline{\varepsilon}_0(B_0)$. In Figure 2(ii) we change $\beta_i(\mathcal{C}, \overline{\varepsilon}_0)$ to $\varepsilon_0(B_0)$. In these two cases, δ_0 and δ share an appearance e. In other two cases, different occurrences of e are used by δ_0 and δ. Note that in none of the above possibilities we need to know that B_0 is the initial bridge in $S(\mathcal{C}_0, \varepsilon_0, \delta_0)$.

In all the cases when we change $\beta_i(\mathcal{C}, \varepsilon)$, we also set $B(\mathcal{C}, \varepsilon, \delta_i(\mathcal{C}))$ to B_0.

To update the values of $\beta_j(\mathcal{C}', \varepsilon')$ and $B(\mathcal{C}', \varepsilon', \delta_j(\mathcal{C}'))$ in procedure FORCING we use the same method as described above. More precisely, if $\mathcal{B}_i$ contains just one bridge, then this bridge takes over the role of B_0. Otherwise, bridges of $\mathcal{B}_i$ are $\delta_{3-i}(\mathcal{C})$-embedded "parallel" to each other and we can speak of the leftmost bridge B_0' and the rightmost bridge B_0'' from $\mathcal{B}_i$. In updating we use the bridge B_0 or each of bridges B_0' and B_0''. The difference from the initial part is that the proposed change of $\beta_j(\mathcal{C}', \varepsilon')$ takes place only if the new value is more restrictive than the current value (and if B_0', B_0'' give different new values, we select the more restrictive one). When we change $\beta_j(\mathcal{C}',\varepsilon')$ we also update $B(\mathcal{C}',\varepsilon',\delta_j(\mathcal{C}'))$ to be the bridge B_0 (B_0' or B_0'') that caused this change. Since $\mathcal{B}_i$ always contains the initial bridges from $S(\mathcal{C},\varepsilon,\delta_1(\mathcal{C}))$ and also from $S(\mathcal{C},\varepsilon,\delta_2(\mathcal{C}))$, a splitting of clusters is not needed in Forcing.

5 Some combinatorial properties of obstructions

Algorithm Extend can be extended in a relatively simple way so that in case when no embedding extension exists, it returns an obstruction. In this section we discuss the structure of resulting obstructions.

Let B be a K-bridge in G. We will write $B[1]$, $B[2]$ and interpret $B[\ell]$ as "B has the ℓth admissible embedding". If for some other bridge B' its ℓ'th embedding ($\ell' \in \{1,2\}$) is obstructed by the ℓth embedding of the bridge B, then we write $B[\ell] \to B'[\neg\ell']$, where $\neg x$ stands for $3-x$. This means that B and B' cannot simultaneously have their ℓth and ℓ'th embedding, respectively. The fact $B[\ell] \to B'[\neg\ell']$ will also be expressed as $B[\ell]$ *overlaps* with $B'[\ell']$ or $B[\ell]$ *forces* $B'[\neg\ell']$.

overlaps

forces

Each application of procedure Forcing starts with a prescribed embedding, say $B[1]$, of a bridge B and builds a directed tree rooted at $B[1]$ and with all edges directed away from $B[1]$. The tree structure is determined by the bridges $B(\mathcal{C},\varepsilon,\delta)$ and describes overlappings of bridges as discovered by the procedure. A path from the root to a vertex $B'[\ell]$ proves that B' must use the ℓth embedding if the initial bridge B is embedded as chosen. Obviously, each bridge appears at most once as a vertex in the tree.

Obstructions produced by Algorithm Extend when no admissible embedding extension is found, are composed of one or two parts, depending on whether the initial bridge B_1 has one or two admissible embeddings. If there are two parts, they start with different embeddings of B_1 and each embedding leads to a chain of overlappings showing that the embedding extension of K to G (using only admissible embeddings) with B_1 embedded as chosen does not exist. For each part there are two possibilities why such a contradiction occurs. The first possibility is that there is a 1-embeddable bridge B_s which is forced to be embedded in a non-admissible way. In such a case, we have a *forcing chain*

forcing chain

$$B_1[\ell_1] \to B_2[\ell_2] \to \ldots \to B_{s-1}[\ell_{s-1}] \to B_s[\ell_s]\,, \tag{FC}$$

where $B_s[\neg\ell_s]$ is the only admissible embedding of B_s. Note that all bridges B_j ($1 \le j \le s$) are distinct.

The second possibility is more complicated. In this case $B_1[\ell_1]$ forces embeddings $B'_{s'}[\ell'_{s'}]$ and $B''_{s''}[\ell''_{s''}]$ (along two forcing chains) that pairwise exclude each other, i.e., $B'_{s'}[\ell'_{s'}] \to B''_{s''}[\neg\ell''_{s''}]$ or equivalently $B''_{s''}[\ell''_{s''}] \to B'_{s'}[\neg\ell'_{s'}]$. In this case, embeddings $B'_{s'}[\ell'_{s'}]$ and $B''_{s''}[\ell''_{s''}]$ are admissible. Since the two forcing chains have a common beginning, we have the following (*branched*) *forcing chain*:

(branched) forcing chain

$$\begin{array}{ccl} & & B'_1[\ell'_1] \to \ldots \to B'_{s'}[\ell'_{s'}] \\ & \nearrow & \\ B_1[\ell_1] \to \ldots \to B_s[\ell_s] & & \\ & \searrow & \\ & & B''_1[\ell''_1] \to \ldots \to B''_{s''}[\ell''_{s''}] \end{array} \tag{BFC}$$

Observe that also in this case all bridges are distinct.

Next we show that it can be achieved that the number of 1-embeddable bridges in an obstruction is at most two.

Lemma 5.1 *Let Ω be an obstruction for a 2-*EEP *Ξ. Then Ω contains a subset Ω' that is also an obstruction for Ξ such that Ω' has at most two* 1*-embeddable bridges. Moreover, Ω' can be found in linear time.*

Proof. Suppose that Ω contains a 1-embeddable bridge B_1. Run (a BF-implementation) of Algorithm Extend starting with B_1 and let $\mathcal{R}$ be the obtained forcing chain. (We may assume that Ω was obtained by Algorithm Extend. Then it is easy to check that $\mathcal{R}$ starts with B_1.) Suppose that $\mathcal{R}$ contains at least two 1-embeddable bridges. If $\mathcal{R}$ is of the form (FC), then the subchain from the last but one to the last 1-embeddable bridge forms an obstruction with two 1-embeddable bridges. If $\mathcal{R}$ is of the form (BFC), let k be the largest index among $1, 2, \ldots, s$ such that B_k is 1-embeddable. Similarly we define indices k' ($1 \le k' \le s'$) and k'' ($1 \le k'' \le s''$), if they exist. Since B_1 is 1-embeddable, B_k always exists. We distinguish three cases. If k' and k'' exist, then

$$B'_{k'}[\ell'_{k'}] \to \ldots \to B'_{s'}[\ell'_{s'}] \to B''_{s''}[\neg\ell''_{s''}] \to \ldots \to B''_{k''}[\neg\ell''_{k''}]$$

forms an obstruction with two 1-embeddable bridges. If none of k', k'' exists, then

$$\begin{array}{ccl} & & B'_1[\ell'_1] \to \ldots \to B'_{s'}[\ell'_{s'}] \\ & \nearrow & \\ B_k[\ell_k] \to \ldots \to B_s[\ell_s] & & \\ & \searrow & \\ & & B''_1[\ell''_1] \to \ldots \to B''_{s''}[\ell''_{s''}] \end{array}$$

is an obstruction with one 1-embeddable bridge. In the remaining case we may assume that k' exists and that k'' does not. Then

$$B_k[\ell_k] \to \ldots \to B_s[\ell_s] \to B''_1[\ell''_1] \to \ldots \to B''_{s''}[\ell''_{s''}] \to B'_{s'}[\neg\ell'_{s'}] \to \ldots \to B'_{k'}[\neg\ell'_{k'}]$$

is the required obstruction. Since the described changes can easily be accomplished in linear time, the lemma is proved. □

The proof also shows that 1-embeddable bridges appear only at the beginning and possibly also at the end of the chains. If they do appear, then Ω is composed of a single (branched) forcing chain. Let us note that the above changes require at most one reversal of the original ordering and that a forcing chain of the form (BFC) may be transformed into a forcing chain of the form (FC).

In Section 4 we assumed that a BF-implementation of procedure Forcing is used. If we omit this assumption, Algorithm Extend (together with a splitting of a cluster at the beginning of each parallel process, if necessary) can be used to traverse in linear time (branched) forcing chains in order that is different than the original one. This observation will be used in the sequel.

We can view a branched forcing chain of the form (BFC) as an ordinary (unbranched) forcing chain

$$B_1[\ell_1] \to \ldots \to B_s[\ell_s] \to B'_1[\ell'_1] \to \ldots \to B'_{s'}[\ell'_{s'}] \to B''_{s''}[\neg\ell''_{s''}] \to \ldots \to B''_1[\neg\ell''_1]$$

with an additional forcing $B_s[\ell_s] \to B''_1[\ell''_1]$ that causes a contradiction. The above observation on the use of Algorithm Extend enables us to perform the above traversal of the chain in linear time. During the traversal, several possibilities arise. It may happen that the required simultaneous embedding of bridges $B_1, \ldots, B_s, B'_1, \ldots, B'_{s'}, B''_{s''}, \ldots, B''_2$ can be realized. In such a case we obtain a forcing chain that is of the form (FC) except that the unique embeddability of the last bridge in the chain is replaced by a "contradicting" forcing from one of the previously embedded bridges. We say that such a forcing chain is of the form (FC′). On the other hand, it may happen that the traversal stops at B''_j, where $j > 1$. (Note that by construction, the bridges $B_1, \ldots, B_s, B'_1, \ldots, B'_{s'}$ can be simultaneously embedded as required.) This can happen only if one of the previous (already embedded) bridges in the chain forces $B''_j[\ell''_j]$. In this case the bridges $B''_1, \ldots, B''_{j-1}$ are redundant and the obtained forcing chain is of the form (FC′) with B''_j being the last bridge in the chain yielding a contradiction with some previous bridge. Similarly, we also traverse forcing chains of the form (FC) and thus achieve that the simultaneous embedding $B_1[\ell_1] \cup \cdots \cup B_{s-1}[\ell_{s-1}]$ exists. Note that during the traversal, the form of the chain may change into (FC′).

We say that a forcing chain $\mathcal{R} = B_1[\ell_1] \to \ldots \to B_s[\ell_s]$ is *forward minimal* if there are no forcings $B_j[\ell_j] \to B_k[\ell_k]$ for $k > j + 1$. To achieve in linear time that $\mathcal{R}$ is forward minimal, we apply the procedure Forcing on bridges B_j ($1 \le j \le s$) starting with $B_1[\ell_1]$. If some bridge forces the required embeddings of more than one bridge, we select as the next bridge the one with the largest index in the chain. For chains of type (FC) such a selection guarantees forward minimality of the resulting subchains. For chains of type (FC′) we must take care not to lose the additional forcing $B_s[\ell_s] \to B''_1[\ell''_1]$. Let i be the smallest index such that $B_i[\ell_i] \to B_s[\neg\ell_s]$. As above, we can achieve that the subchains of $\mathcal{R}$ from B_1 to B_i and from B_i to B_s are both forward minimal. Let B_j ($j \le i$) be the first bridge in the chain that forces some $B_k[\ell_k]$ where $k > i$. Choose the index k as large as possible. If $j < i$, then we replace $\mathcal{R}$ by the chain of type (FC′) (margin note: forward minimal)

$$B_1[\ell_1] \to \ldots \to B_j[\ell_j] \to B_k[\ell_k] \to \ldots \to B_s[\ell_s] \to B_i[\neg\ell_i] \to \ldots \to B_{j+1}[\neg\ell_{j+1}],$$

which is forward minimal. The above procedure for making $\mathcal{R}$ forward minimal can be incorporated in the procedure that changes chains of the form (BFC) into the form (FC$'$).

To summarize, so far we have achieved (in linear time) that our obstruction Ω either consists of a single forcing chain of type (FC) or (FC$'$) or of two forcing chains of type (FC$'$). Furthermore, each forcing chain $\mathcal{R} = B_1[\ell_1] \to B_2[\ell_2] \to \dots \to B_s[\ell_s]$ in Ω has the following properties:

(FC0) All bridges B_j ($1 \leq j \leq s$) are distinct and for $2 \leq j \leq s-1$, B_j has two admissible embeddings. If B_s is 1-embeddable, then so is B_1.

(FC1) There is no simultaneous embedding of $B_1 \cup \dots \cup B_s$ where B_1 is embedded as $B_1[\ell_1]$.

(FC2) The simultaneous embedding $B_1[\ell_1] \cup \dots \cup B_{s-1}[\ell_{s-1}]$ exists.

(FM) $\mathcal{R}$ is forward minimal.

Let $\mathcal{R}$ be a forcing chain of the form (FC$'$) with the properties (FC0)–(FM) and let r be the smallest index such that $B_r[\ell_r] \to B_s[\neg\ell_s]$. Let us traverse $\mathcal{R}$ in the following order:

$$B_1[\ell_1] \to \dots \to B_r[\ell_r] \to B_s[\neg\ell_s] \to B_{s-1}[\neg\ell_{s-1}] \to \dots \to B_{r+1}[\neg\ell_{r+1}]\,.$$

Let B_j be the first bridge in the traversal whose embedding cannot be realized simultaneously with the chosen embeddings of the previous bridges (according to the order from the traversal). Note that such a bridge always exists by (FC1), and that $j > r$ because of (FC2). If $j = r+1$, then $\mathcal{R}$ has the following property:

(FC3) The simultaneous embedding $B_1[\ell_1] \cup \dots \cup B_r[\ell_r] \cup B_s[\neg\ell_s] \cup \dots \cup B_{r+2}[\neg\ell_{r+2}]$ can be realized (where r is the smallest index such that $B_r[\ell_r] \to B_s[\neg\ell_s]$).

Suppose now that $j > r+1$. We claim that the forcing chain $\mathcal{R}'$, defined as

$$B_1[\ell_1] \to \dots \to B_r[\ell_r] \to B_s[\neg\ell_s] \to \dots \to B_j[\neg\ell_j]\,,$$

satisfies (FM), (FC0)–(FC2) and also (FC3). Obviously, $\mathcal{R}'$ fulfils (FC0). By the definition of B_j, it also satisfies (FC1) and (FC2). Since $\mathcal{R}$ satisfies (FM), (FC2), and B_r is the first bridge in $\mathcal{R}$ that forces $B_s[\neg\ell_s]$, $\mathcal{R}'$ is also forward minimal. It remains to check that $\mathcal{R}'$ satisfies (FC3). Let B_k be the first bridge in $\mathcal{R}'$, such that B_k embedded as in $\mathcal{R}'$ forces $B_j[\ell_j]$. Since $\mathcal{R}$ is forward minimal and $j > r+1$, we have $j+1 \leq k \leq s$. It remains to check that the simultaneous embedding

$$B_1[\ell_1] \cup \dots \cup B_r[\ell_r] \cup B_s[\neg\ell_s] \cup \dots \cup B_k[\neg\ell_k] \cup B_j[\ell_j] \cup \dots \cup B_{k-2}[\ell_{k-2}]$$

can be realized whenever $k \geq j+2$. Let $\mathcal{B}_1 = \{B_i \mid 1 \leq i \leq r\}$, $\mathcal{B}_2 = \{B_i \mid j \leq i \leq k-2\}$, and $\mathcal{B}_3 = \{B_i \mid k \leq i \leq s\}$. By (FC2), $\mathcal{B}_1 \cup \mathcal{B}_2$ can be simultaneously embedded as required. Also, $\mathcal{B}_1 \cup \mathcal{B}_3$ can be simultaneously embedded as checked by the traversal.

Since $\mathcal{R}'$ is forward minimal, there are no overlappings among embeddings of bridges from $\mathcal{B}_2$ and $\mathcal{B}_3$. The claim is thus proved.

If $\mathcal{R}$ is of the form (FC), we consider (Fc3) to be equivalent to (Fc2). From now on we thus assume that the forcing chain(s) of Ω also satisfy (Fc3).

Next we traverse (again in linear time using Algorithm Extend) the subchain

$$B_r[\neg\ell_r] \to \ldots \to B_2[\neg\ell_2]$$

of $\mathcal{R}$ (where r is defined in (Fc3) or $r = s$ if $\mathcal{R}$ is of the form (FC)) to check if $\mathcal{R}$ has the following property:

(Fc4) The simultaneous embedding $B_2[\neg\ell_2] \cup \cdots \cup B_r[\neg\ell_r]$ can be realized. (If $\mathcal{R}$ is of the form (FC), we take $r = s$.)

Suppose that the traversal stops at $B_j[\neg\ell_j]$. Consider the forcing chain

$$\mathcal{R}' = B_r[\neg\ell_r] \to \ldots \to B_j[\neg\ell_j]$$

which is of the form (FC$'$). It is easy to see that $\mathcal{R}'$ has the properties (Fm) and (Fc0)–(Fc4). If $\mathcal{R}$ is of the form (FC), then $\mathcal{R}'$ forms an obstruction. Let us remark that this obstruction is in fact minimal. Otherwise, $\mathcal{R}'$ forms an obstruction together with the forcing chain

$$B_r[\ell_r] \to \ldots \to B_s[\ell_s]$$

of the form (FC$'$) that also has the required properties (Fm) and (Fc0)–(Fc4). (Note that for the latter chain, (Fc4) is trivial by the choice of r.)

To simplify the proofs in the rest of the paper we shall add an additional assumption on the EEP Ξ:

(P7) No branch of K appears on the same facial walk of Π twice in the same direction.

This property is always satisfied in orientable embeddings and also in embeddings of K with minimal Euler genus. In particular, it is satisfied in the main applications of our algorithms mentioned in the introduction.

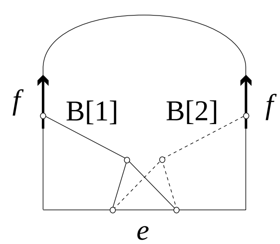

Figure 3: The admissible embeddings of a firmly based $\frac{3}{2}$-embeddable bridge.

Let B be a $\frac{3}{2}$-embeddable bridge with the base e. If B has two or more attachments on e, we say that B is *firmly based* (see Figure 3). firmly based

Lemma 5.2 *The number of firmly based $\frac{3}{2}$-embeddable bridges in the obstruction Ω for the 2-*EEP Ξ *is bounded by a constant which depends only on* $\mathrm{bsize}(K)$*.*

Proof. Let $\mathcal{R}$ be a forward minimal forcing chain in Ω. Suppose that $\mathcal{R}$ contains two firmly based $\frac{3}{2}$-embeddable bridges B and B' from the same cluster $\mathcal{C}$ and that the embedding $B[\ell]$ occurs in $\mathcal{R}$ before $B'[\ell']$. If the bridges from $\mathcal{C}$ are strongly attached to K, then $\mathcal{C} = \{B, B'\}$ by (P6). Otherwise, let $type(\mathcal{C}) = \{e, f\}$ where e is the base of B and B'. Let us first suppose that both, B and B', appear in $\mathcal{R}$ among $B_2, \ldots, B_r$. If $B[\ell]$ and $B'[\ell']$ use distinct occurrences of f, then the embedding $B[\neg\ell] \cup B'[\neg\ell']$ does not exist, a contradiction with (Fc4). So, $B[\ell]$ and $B'[\ell']$ use the same occurrence of f. By (P7) and (Fc4), the union $B \cup B'$ has only one attachment x on f. Let B'' be the bridge in $\mathcal{R}$ that forces B. Since $\mathcal{R}$ is forward minimal, $B'' \in \mathcal{C}$ and it is attached to f only at x, while its attachments on e are between the attachments of B and B'. Consequently, every bridge in $\mathcal{R}$ which forces the embedding of B'' also forces $B[\ell]$. Therefore, B'' must be the first bridge in $\mathcal{R}$. The same arguments, except that (Fc3) is used instead of (Fc4), imply that among the bridges B_i $(r + 2 \leq i < s)$ of $\mathcal{R}$ there is at most one firmly based bridge from $\mathcal{C}$. This completes the proof. □

In the sequel we shall achieve that for each firmly based $\frac{3}{2}$-embeddable bridge B in Ω with oriented base branch ε, at most six bridges that are not $\frac{3}{2}$-embeddable are attached to the segment $S = (\varepsilon(B), \overline{\varepsilon}(B))$.

Let us first assume that Ω is composed of a single forcing chain $\mathcal{R}$. Denote by $\mathcal{B}$ be the set of bridges from $\mathcal{R}$ that are neither 1-embeddable nor $\frac{3}{2}$-embeddable and that are attached to S. Let $B_i \in \mathcal{B}$. Property (Fc2) implies that $B[\ell] \to B_i[\ell_i]$ for $\ell = 1, 2$ if $B \neq B_s$. Suppose first that $\mathcal{R}$ is of the form (FC). Because of (Fc4), B has a simultaneous embedding with $B_i[\neg\ell_i]$, a contradiction. Therefore $\mathcal{B} = \emptyset$. If $\mathcal{R}$ is of the form (FC$'$), the proof is more complicated. The first subcase is when $B \neq B_{r+1}, B_s$. By (Fc2) and (Fc3), $i \notin \{r + 2, r + 3, \ldots, s - 1\}$. Assume that $\mathcal{B} \cap \{B_2, \ldots, B_r\} \neq \emptyset$. Let j $(2 \leq j \leq r)$ be the largest index such that $B_j \in \mathcal{B}$. Since $B[\ell] \to B_j[\ell_j]$, $\ell = 1, 2$, the bridges $B_1, \ldots, B_{j-1}$ in Ω can be replaced by B. After that, only B_j, B_{r+1}, and B_s may be attached to S. (Now we have changed Ω, and we repeat the previous reductions in order to achieve (Fc0)–(Fc4) and (Fm).) The last case to consider is when $B = B_{r+1}$ or $B = B_s$. By symmetry we may assume that $B = B_s$. Since $\mathcal{R}$ satisfies (Fm) and (Fc3), the only bridge B_i $(r + 1 < i < s)$ that might be in the set $\mathcal{B}$ is B_{s-1}. If $s > r + 1$, let j be the largest index such that $B_j \in \mathcal{B}$ and $1 < j \leq r$ (if such an index exists). Then we can remove the bridges $B_1, \ldots, B_{j-1}$ from Ω. If $s = r + 1$, then we select $B_j \in \mathcal{B}$ with the largest index among the bridges B_i $(1 < i < r)$ and proceed as above. In each of the cases, there remain at most four bridges that are attached to S and that are not $\frac{3}{2}$-embeddable. (These may be B_j, B_{r+1}, B_{s-1}, and possibly B_1 since it is 1-embeddable.)

Let us now consider the possibility when Ω is composed of two forcing chains. Select a forcing chain $\mathcal{R}$ in Ω. If $B \in \mathcal{R}$, then we proceed as above in the case when $\mathcal{R}$ is of the form (FC$'$). On the other hand, if $B \notin \mathcal{R}$, let $\mathcal{B}_1$ be the set of those bridges from

$\mathcal{B}$ for which the embedding in $\mathcal{R}$ uses the same appearance of ε as both embeddings of B. Let $\mathcal{B}_2 = \mathcal{B} \backslash \mathcal{B}_1$. If $\mathcal{B}_2 \neq \emptyset$, let B_j be the first bridge from $\mathcal{R}$ which belongs to $\mathcal{B}_2$. If some bridge B_i ($1 \leq i < j$) belongs to $\mathcal{B}_1$, let k be the largest index of such a bridge. Otherwise, let $k = 1$. Then the bridges $B_k, \ldots, B_j$ and B replace the entire chain $\mathcal{R}$. If $B_k \in \mathcal{B}_1$, these bridges form an obstruction for the EEP Ξ. In each case, at most two bridges from $\mathcal{B}$, B_k and B_j, remain attached to S. The last case is when $\mathcal{B}_2 = \emptyset$. If also $\mathcal{B}_1 = \emptyset$, no changes are necessary. Otherwise, let j be the largest index such that $1 \leq j \leq r$ and $B_j \in \mathcal{B}_1$ (or $j = 1$ if such j does not exist). Similarly, let k be the largest index such that $r + 1 \leq k \leq s$ and $B_k \in \mathcal{B}_1$ (or $k = r + 1$). Then the bridges $B_j, \ldots, B_r$ and $B_s, \ldots, B_k$ together with B replace $\mathcal{R}$. Again, at most two bridges are attached to the segment S. If $B_j \in \mathcal{B}_1$, then the selected bridges form an obstruction for Ξ.

After changing Ω and repeating the entire procedure from the beginning, the chain structure of Ω may change. It may even become composed of two forcing chains instead of one. However, no new $\frac{3}{2}$-embeddable bridges occur. By Lemma 5.2, the number of firmly based $\frac{3}{2}$-embeddable bridges is bounded. Therefore the change of Ω is performed only a constant number of times, and the overall time spent is linear. Thus we may assume in the sequel that our obstruction Ω additionally satisfies:

(FB) For each $\frac{3}{2}$-embeddable bridge B with oriented base ε, Ω contains at most six bridges that are attached to the segment $(\varepsilon(B), \overline{\varepsilon}(B))$ and that are not $\frac{3}{2}$-embeddable.

To achieve the above properties, the initial forcing chain $\mathcal{R}_0$, which was obtained by a BF-implementation of Algorithm Extend, has been changed a bounded number of times. After the changes, $\mathcal{R}_0$ is composed of a bounded number of subchains such that each subchain (or its reversal) is a subchain of the original forcing chain. Moreover, we have:

(BF) Each forcing chain in Ω is composed of a bounded number of subchains, each of which (or its reversal) is a (not necessarily contiguous) subchain of an original forcing chain obtained by a BF-implementation of Algorithm Extend.

6 Geometric structure of obstructions

Our next goal is to change the obstruction Ω into a minimal obstruction. It is an easy task to minimize Ω in quadratic time. Just take an arbitrary bridge $B \in \Omega$ and test in linear time (using Algorithm Extend) whether $\Omega \backslash B$ is still an obstruction. If not, the bridge B participates in every minimal obstruction contained in Ω. Otherwise reject B and proceed with $\Omega \backslash B$. If Ω contains a 1-embeddable bridge, it is not too complicated to transform Ω in linear time into a minimal obstruction using only its "combinatorial" properties, i.e., by considering its forcing chain. On the other hand, if Ω is composed of two forcing chains, this task is much more complicated because of the mutual interference between the chains. On the other hand, in Section 8 we will also use the basic "geometric" structure of obstructions to "compress" minimal obstructions.

Therefore we have decided to use the "geometric" approach also to efficiently minimize Ω.

The outline of this approach is as follows. We start with a forcing chain $\mathcal{R}_0$ contained in Ω and step by step subdivide $\mathcal{R}_0$ into a bounded number of smaller and smaller subchains with additional properties that capture the essential properties of the "geometric" structure of $\mathcal{R}_0$. During the process we take care that each of the subchains retains all properties achieved at the previous steps. For example, in Section 5 we showed that we can achieve that $\mathcal{R}_0$ satisfies (FC0)–(FC4) and (FM), (FB), (BF). Initially, we consider the first and the last bridge (and also the bridge B_r if $\mathcal{R}_0$ is of the form (FC′)) and subchains from (BF) as separate subchains.

Subchains consisting of a single bridge are said to be *trivial*, and those that contain at least two bridges are *essential*. Each essential subchain $B_1[\ell_1] \to \ldots \to B_s[\ell_s]$ is forward minimal and satisfies: trivial essential

(C0) All bridges B_i $(1 \leq i \leq s)$ in the chain are distinct and have two admissible embeddings.

(C1) The simultaneous embedding $B_1[\ell_1] \cup \cdots \cup B_s[\ell_s]$ can be realized.

(C2) The simultaneous embedding $B_1[\neg\ell_1] \cup \cdots \cup B_s[\neg\ell_s]$ can be realized.

(C3) For every j $(1 \leq j \leq s)$ the simultaneous embedding $B_1[\ell_1] \cup \cdots \cup B_{j-1}[\ell_{j-1}] \cup B_{j+1}[\neg\ell_{j+1}] \cup \cdots \cup B_s[\neg\ell_s]$ can be realized.

Clearly, (C0) follows from (FC0), and (C1) follows from the property (FC2) of $\mathcal{R}_0$. Properties (C2) and (C3) follow by (FM) and properties (FC2)–(FC4) of $\mathcal{R}_0$. Let us mention that (C3) implies forward minimality of each subchain. To summarize: our obstruction is decomposed into a bounded number of forward minimal subchains with properties (C0)–(C3) (or having length 1). Additionally, each such subchain is contained in one of the subchains from (BF). The "global" properties (FC0)–(FC4) will be used only occasionally in the sequel when comparing distinct subchains.

The forcing $B[\ell] \to B'[\ell']$ is *strong* if the embedding schemes of embeddings $B[\ell]$ and $B'[\neg\ell']$ overlap. Since $\mathcal{R}_0$ is forward minimal, the number of strong forcings in each subchain is bounded by the number of clusters. By (P6), there are at most two strongly attached bridges in each cluster. Hence, a bounded number of subdivisions of subchains assures that each essential subchain $\mathcal{R} = B_1[\ell_1] \to \ldots \to B_s[\ell_s]$ of $\mathcal{R}_0$ also satisfies: strong

(M1) No bridge B_j $(1 \leq j \leq s)$ is strongly attached to K.

(M2) No forcing in $\mathcal{R}$ is strong. (However, forcings between the last member of a subchain and the first member of the next subchain can be strong.)

In the sequel we shall further subdivide the chains in order to assure additional structural properties that will later enable us to efficiently perform minimization and "compression" of obstructions. We shall also use the fact that if $\mathcal{R}$ has properties

(C0)–(C3) and (M1)–(M2), then also every (contiguous) subchain of $\mathcal{R}$ has the same properties.

The subchain $\mathcal{R}$ is obtained by Algorithm Extend and satisfies (M2). This implies that each pair B_i, B_{i+1} of consecutive bridges in $\mathcal{R}$ is attached to a common branch. Moreover, there is an appearance $\tilde{\varepsilon}_i$ of an oriented branch ε_i such that the forcing $B_i[\ell_i] \to B_{i+1}[\ell_{i+1}]$ occurs because of attachments of B_i and B_{i+1} on $\tilde{\varepsilon}_i$. More precisely, if the bridge B_{i+1} was forced because of the value $\beta_\ell(\mathcal{C}, \varepsilon)$, then $B_{i+1} \in \mathcal{C}$, $\ell = \neg \ell_{i+1}$, $\varepsilon = \varepsilon_i$, and $\tilde{\varepsilon}_i$ is the occurrence of ε_i which is used by the embedding $B_{i+1}[\ell]$. For the extreme attachments of B_i and B_{i+1} on ε_i we have $\varepsilon_i(B_{i+1}) < \overline{\varepsilon}_i(B_i)$. In such a case we say that B_i and B_{i+1} *overlap on the oriented branch* ε_i.

overlap on the oriented branch ε_i

Suppose that $\varepsilon_i = \varepsilon_j$ for some $i < j$. If B_{i+1} and B_{j+1} are both in the same cluster, then (Fm) and Algorithm Extend imply that

$$\varepsilon_i(B_{i+1}) < \overline{\varepsilon}_i(B_i) \le \varepsilon_i(B_{j+1}) < \overline{\varepsilon}_i(B_j) . \tag{1}$$

In the sequel we shall prove that for each $\frac{3}{2}$-embeddable cluster $\mathcal{C}$ with the base ε there is only a bounded number of forcings $B_i[\ell_i] \to B_{i+1}[\ell_{i+1}]$ such that at least one of the bridges B_i, B_{i+1} belongs to $\mathcal{C}$ and such that $\varepsilon_i = \varepsilon$. Let $e = \mathit{type}(\varepsilon)$ and $\mathit{type}(\mathcal{C}) = \{e, x\}$. Suppose that for indices $i_1 < i_2 < i_3$, the forcings $B_{i_j-1}[\ell_{i_j-1}] \to B_{i_j}[\ell_{i_j}]$ $(1 \le j \le 3)$ are on the branch ε and such that $B_{i_j} \in \mathcal{C}$ $(1 \le j \le 3)$. Suppose also that $\ell_{i_1} = \ell_{i_2} = \ell_{i_3} =: \ell$. Now, (1) implies that $\varepsilon(B_{i_1}) < \varepsilon(B_{i_2}) < \varepsilon(B_{i_3})$. By (P7) and (C2), the union $B_{i_1} \cup B_{i_2} \cup B_{i_3}$ must be attached to x at one vertex only. Denote this vertex by u. By assumptions, B_{i_1-1} has an attachment on ε after $\varepsilon(B_{i_1})$. Since this bridge does not overlap with B_{i_2} and B_{i_3}, it belongs to the cluster $\mathcal{C}$ and is attached only to the segment $[\overline{\varepsilon}(B_{i_1}), \varepsilon(B_{i_2})] \subseteq \varepsilon$ and to u. Therefore the bridge B_{i_1-2}, which in $\mathcal{R}$ forces the embedding of B_{i_1-1}, also forces the embedding of B_{i_1}. This contradicts (Fm). Hence $i_1 = 2$. Similarly we prove that $\mathcal{R}$ contains at most three forcings $B[\ell] \to B'[\ell']$ on the oriented branch ε where $B \in \mathcal{C}$.

By making an additional bounded number of subdivisions of forcing subchains we can achieve that each essential subchain $\mathcal{R}$ satisfies:

(M3) $\frac{3}{2}$-embeddable bridges in $\mathcal{R}$ do not overlap with other bridges in $\mathcal{R}$ on their base branches.

Our next goal is to achieve one of the following for each subchain $\mathcal{R}$:

(M4a) For all bridges B_i $(1 \le i \le s)$ in $\mathcal{R}$, the embeddings $B_i[1]$ and $B_i[2]$ are in distinct faces of K.

(M4b) For all bridges B_i $(1 \le i \le s)$ in $\mathcal{R}$, the embeddings $B_i[1]$ and $B_i[2]$ are in the same face of K.

Let $B_i \in \mathcal{C}$ and $B_{i+1} \in \mathcal{C}'$ be consecutive bridges in $\mathcal{R}$. It suffices to prove that if $F_1 = F_1(\mathcal{C}) \ne F_2 = F_2(\mathcal{C})$, then also $F_1(\mathcal{C}') \ne F_2(\mathcal{C}')$. Suppose not, say $F_1 = F_1(\mathcal{C}') = F_2(\mathcal{C}')$. Since B_i and B_{i+1} overlap on ε_i, the branch $e = \mathit{type}(\varepsilon_i)$ lies on the boundary of F_1 and

F_2. Therefore e occurs on ∂F_1 just once, and the admissible embeddings of B_{i+1} both use this occurrence of e. In particular, e is the base of B_{i+1}. This contradicts (M3).

If the subchain $\mathcal{R}$ satisfies (M4B), then both admissible embeddings of all bridges in $\mathcal{R}$ are in the same face F. If $\mathcal{R}$ satisfies (M4A), then there are distinct faces F_1 and F_2 of K such that all admissible embeddings of bridges in $\mathcal{R}$ are in F_1 and F_2. This fact is easily proved by induction on the length of $\mathcal{R}$ since the branches ε_i on which consecutive bridges overlap, must all lie on the boundary of F_1 and F_2.

We say that $\mathcal{R}$ satisfies (M4), if it has either property (M4A) or (M4B).

Let B be a bridge that is attached to the oriented branch ε. An embedding $B[\ell]$ of B in the face F splits F into several subfaces. Let $F^{+}_{\varepsilon,B[\ell]}$ be the subface that contains on its boundary the edge right(ε) (from the same occurrence as used by $B[\ell]$). Similarly, let $F^{-}_{\varepsilon,B[\ell]}$ be the subface containing left(ε).

Suppose that bridges $B \in \mathcal{C}$ and $B' \in \mathcal{C}'$ ($\mathcal{C} \neq \mathcal{C}'$) from $\mathcal{R}$ overlap on the oriented branch ε, $B[\ell] \to B'[\ell']$. Let $e = \mathit{type}(\varepsilon)$, $\mathit{type}(\mathcal{C}) = \{e, f\}$, and $\mathit{type}(\mathcal{C}') = \{e, f'\}$. We denote by f_1 the occurrence of f used by $B[\ell]$, and by f_2 the occurrence used by $B[\neg\ell]$. Similarly, let f'_1 and f'_2 be the occurrences of f' used by $B'[\ell']$ and $B'[\neg\ell']$, respectively. Since B and B' belong to distinct clusters and they overlap on ε, f'_2 lies in the subface $F^{+}_{\varepsilon,B[\ell]}$. If also f'_1 lies in the same subface $F^{+}_{\varepsilon,B[\neg\ell]}$, we say that the overlapping of B and B' in $\mathcal{R}$ is *weird* (see Figure 4 for the case (M4A)). weird

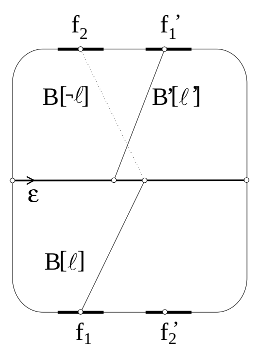

Figure 4: Weird overlapping in two faces.

We claim that in $\mathcal{R}$ for each pair of distinct clusters $\mathcal{C}, \mathcal{C}'$ and each oriented branch ε there are at most two weird overlappings $B[\ell] \to B'[\ell']$ on ε such that $B \in \mathcal{C}$ and $B' \in \mathcal{C}'$. Suppose not. Let $B_{i_j}[\ell_{i_j}] \to B_{i_j+1}[\ell_{i_j+1}]$ ($1 \le j \le 3$) be weird overlappings on ε where $B_{i_j} \in \mathcal{C}$ and $B_{i_j+1} \in \mathcal{C}'$. Suppose that $i_1 < i_2 < i_3$. Then $\varepsilon(B_{i_1+1}) < \overline{\varepsilon}(B_{i_1}) < \overline{\varepsilon}(B_{i_3})$. Weird overlapping implies that $B_{i_1+1}[\ell_{i_1+1}] \to B_{i_3}[\ell_{i_1}]$. This contradicts either (FM) or (C1) (depending on whether $\ell_{i_1} = \ell_{i_3}$ or not). Hence we may assume:

(M5) There are no weird overlappings in $\mathcal{R}$.

A forward minimal essential subchain with properties (C0)–(C3), (FB), and (M1)–(M5) is called a *basic subchain*. With a bounded number of additional splittings of basic subchain
basic subchains we also achieve that each resulting essential subchain $\mathcal{R}$ satisfies: If

$\mathcal{R}$ contains an overlapping $B[\ell] \to B'[\ell']$ on the occurrence $\tilde{\varepsilon}$ of the oriented branch ε, where $B \in \mathcal{C}$ and $B' \in \mathcal{C}'$, then in the basic subchain $\mathcal{R}'$ that contains $\mathcal{R}$ there is a pair of bridges $B^- \in \mathcal{C}$ and $B'^- \in \mathcal{C}'$ that appear in $\mathcal{R}'$ before $\mathcal{R}$ whose embeddings $B^-[\ell]$ and $B'^-[\ell']$ overlap on $\tilde{\varepsilon}$. Similarly, there are bridges $B^+ \in \mathcal{C}$ and $B'^+ \in \mathcal{C}'$ that appear in $\mathcal{R}'$ after $\mathcal{R}$ and whose embeddings overlap on $\tilde{\varepsilon}$.

Suppose that there are indices $i_1 < i_2 < i_3 < i_4$ such that the bridges $R_j := B_{i_j}$ $(1 \le j \le 4)$ belong to the same cluster $\mathcal{C}$, all bridges $R'_j := B_{i_j+1}$ $(1 \le j \le 4)$ belong to the same cluster $\mathcal{C}'$ (possibly $\mathcal{C} = \mathcal{C}'$), and such that all four overlappings of R_j and R'_j $(1 \le j \le 4)$ occur on the same appearance of the oriented branch ε. Then $\ell_{i_1} = \ell_{i_2} = \ell_{i_3} = \ell_{i_4} =: \ell$ and similarly $\ell' := \ell_{i_j+1}$ $(1 \le j \le 4)$. Since $\mathcal{R}$ is forward minimal, the attachments on ε of the considered bridges follow each other in the direction of ε:

$$\varepsilon(R'_1) < \overline{\varepsilon}(R_1) \le \varepsilon(R'_2) < \overline{\varepsilon}(R_2) \le \varepsilon(R'_3) < \overline{\varepsilon}(R_3) \le \varepsilon(R'_4) < \overline{\varepsilon}(R_4)\,. \qquad (2)$$

Let F be the face in which the subchain $\mathcal{R}$ embeds R_3, and let F' be the face of R'_3 (possibly $F = F'$). Let $e = type(\varepsilon)$, $type(\mathcal{C}) = \{e, f\}$ and $type(\mathcal{C}') = \{e, f'\}$. Denote by ε_1 the occurrence of ε used by the embedding $R_3[\ell]$, and by ε_2 the occurrence used by $R'_3[\ell']$. By (2), $\varepsilon_1 \ne \varepsilon_2$. Let f_1 and f_2 be the occurrences of f used by $R_3[\ell]$ and $R_3[\neg\ell]$. Define similarly f'_1 and f'_2 as the occurrences of f' used by $R'_3[\ell']$ and $R'_3[\neg\ell']$, respectively. It may happen that $f_1 = f_2$ or $f'_1 = f'_2$. The bridges R_3 and R'_3 overlap on ε. Therefore ε_1, f'_2, and f_1 occur on F in that order (using the direction of the boundary of F determined by ε_1). By (M5), these basic pieces appear on F' in order ε_2, f_2 and, f'_1 (determined by the orientation of ε_2).

Embeddings $R_3[\ell]$ and $R'_3[\ell']$ split the union $F \cup F'$ into several subfaces. Let F_1 be the subface of F containing right(ε_1), and let F_2 be the subface of F' containing right(ε_2). If $F = F'$, it may happen that $F_1 = F_2$. At most one of the subfaces F_1, F_2 (even when $F = F'$) contains an occurrence of left(ε) and an occurrence of right(ε). Such a subface is said to be *degenerate*. Let S be the segment of ε which is "on both sides" blocked by the embedded bridges R_j and R'_j $(1 \le j \le 4)$, i.e., degenerate

$$S = (\varepsilon(R_1), \overline{\varepsilon}(R_4)) \cap (\varepsilon(R'_1), \overline{\varepsilon}(R'_4)).$$

The choice of F_1 and F_2 is such that the union $F_1 \cup F_2$ contains on its boundary at most one occurrence of left(ε). Properties (C1) and (C3) imply that for $i > i_4$, the embeddings of R_j and R'_j $(1 \le j \le 3)$ determined by $\mathcal{R}$ overlap neither with $B_i[1]$ nor with $B_i[2]$.

Lemma 6.1 *Using the above notation and assumptions, each bridge B_i $(i > i_4)$ from $\mathcal{R}$ satisfies:*

(a) B_i has at least one of its admissible embeddings in $F_1 \cup F_2$. If exactly one of its embeddings is in $F_1 \cup F_2$, then that embedding is in a nondegenerate subface.

(b) If B_i is attached to ε before the segment S, then B_i is $\frac{3}{2}$-embeddable and ε is its base.

Proof. The proof is by induction on i. The case $i = i_4 + 1$ is the base of induction. By (2), $B_{i_4+1} = R'_4$ is attached to ε after R_3 and R'_3. Hence, both its embeddings are in $F_1 \cup F_2$. This proves the claim.

To prove the induction step, suppose that the claim is true for B_i. Let ε' be the oriented branch on which B_i and B_{i+1} overlap and let $e' = type(\varepsilon')$. By (M3), the admissible embeddings of B_{i+1} are attached to distinct occurrences of e'. The same holds for the embeddings of B_i.

We shall first prove (a). Suppose that $e' \neq e, f, f'$. If both embeddings of B_i can be realized in $F_1 \cup F_2$, then both occurrences of e' lie entirely in the boundary of $F_1 \cup F_2$. Therefore also both embeddings of B_{i+1} are in $F_1 \cup F_2$. If only one embedding of B_i can be made in $F_1 \cup F_2$, then by induction hypothesis this embedding lies in a nondegenerate subface $\tilde{F}$. In particular, one of the occurrences of e' is on the boundary of $\tilde{F}$. Hence B_{i+1} has an embedding in the nondegenerate face $\tilde{F}$, and this implies (a).

The case when $e' = f$ or $e' = f'$ is much harder. We shall distinguish several cases.

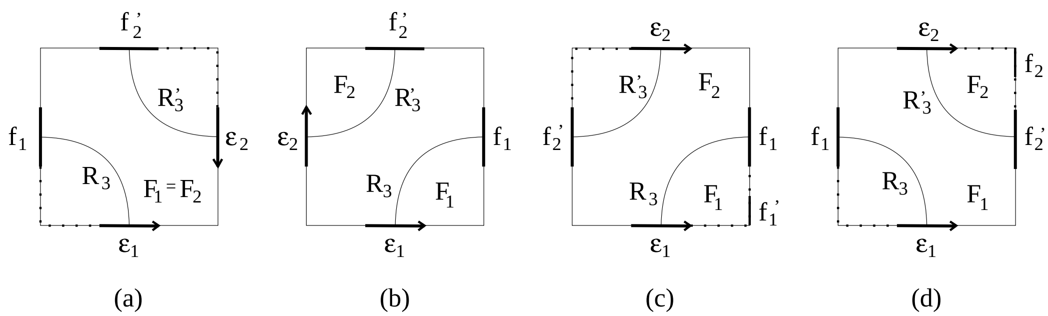

Figure 5: Possible layouts in case (i) when $F = F'$.

(i) Suppose that $\mathcal{C} \neq \mathcal{C}'$ and that the bridges from clusters $\mathcal{C}$ and $\mathcal{C}'$ are not $\frac{3}{2}$-embeddable. The occurrences of f and f' on the boundary of $F_1 \cup F_2$ are such that either both occurrences of left(ε') and right(ε') lie on the boundary of $F_1 \cup F_2$ or at least one of the occurrences of e' entirely lies on the boundary of a nondegenerate subface. If $F \neq F'$, this claim is obvious, while the possibilities in case when $F = F'$ can be verified by using Figure 5. The dotted segments of ∂F show that f and f' do not occur on that segments. Since the admissible embeddings of B_{i+1} use distinct occurrences of e', (a) follows easily.

(ii) Suppose that $\mathcal{C} \neq \mathcal{C}'$ and that the bridges from at least one of the clusters $\mathcal{C}, \mathcal{C}'$ are $\frac{3}{2}$-embeddable. Then $F = F'$. The clusters $\mathcal{C}$ and $\mathcal{C}'$ overlap on e. Hence e is not the base. By (P7), the base is precisely one of f or f', and the union of all bridges from the corresponding cluster is attached to it at a single vertex, call it u. It is easy to see from Figure 6 that if e' is not the base branch, it has an appearance which entirely lies on the boundary of the a nondegenerate subface. Therefore B_{i+1} admits an embedding in that nondegenerate subface. The other case is when e' is the base branch. The occurrence of e' in the admissible embedding schemes

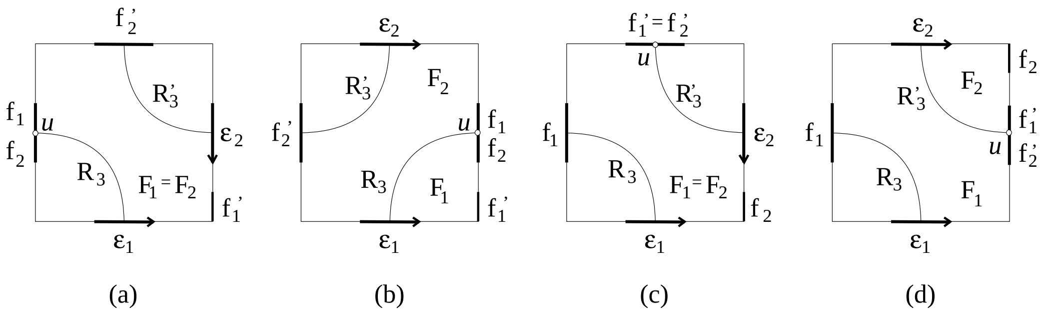

Figure 6: Possible layouts in case (ii).

either lies partially on the boundary of F_1 and partially on F_2, or lies partially on the boundary of the union $F_1 \cup F_2$ and partially out of it (see Figure 6). Let us remark that in the latter case we have $F_1 = F_2$, hence there is no degenerate subface. By (M4), the second occurrence e'_2 of e' also lies on F. If e'_2 appears in a nondegenerate subface, (a) is obviously true. The same is true if e'_2 is in the degenerate subface since then its first occurrence e'_1 lies entirely in $F_1 \cup F_2$. The last possibility is when e'_2 lies out of the boundary of $F_1 \cup F_2$. In that case, B_i has one of the embeddings out of $F_1 \cup F_2$, so the other one must be in a nondegenerate subface. If this is the embedding $B_i[\ell_i]$, then also $B_{i+1}[\neg\ell_{i+1}]$ lies in a nondegenerate subface. So, assume that $B_i[\ell_i]$ is not in $F_1 \cup F_2$. Suppose that B_{i+1} has no admissible embeddings in nondegenerate subfaces. The vertex u splits e' into two closed segments S_1, S_2 with common end u. We may assume that S_1 lies on the occurrence e'_1 in the nondegenerate subface. Clearly, the attachments of B_i on e' are all in S_1, and the attachments of B_{i+1} are in S_2. Since $B_i, B_{i+1} \notin \mathcal{C} \cup \mathcal{C}'$, (C2) implies that also the bridges B_i^- and B_i^+ are entirely attached to S_1, and (C1) implies that B_{i+1}^- and B_{i+1}^+ are attached to S_2. This contradicts (FM).

(iii) Suppose that $\mathcal{C} = \mathcal{C}'$ and $F \neq F'$. Then F_1 and F_2 are both nondegenerate. If (a) does not hold, then we easily see that $B_i, B_{i+1} \notin \mathcal{C}$. If the embedding $B_i[\ell_i]$ is in $F_1 \cup F_2$, then also $B_{i+1}[\neg\ell_{i+1}]$ is in $F_1 \cup F_2$. Hence we may assume that $B_i[\ell_i]$ is not in $F_1 \cup F_2$.

Since B_i and B_{i+1} overlap on e', $e' = f = f'$ is a branch. Let ϕ be the oriented branch f whose orientation is induced by ε and the embedding scheme of $R_3[\ell]$.

Suppose first that the orientation ϕ' of f induced by ε and $R'_3[\ell']$ is equal to ϕ. Suppose also that the embedding $B_i[\ell_i]$ lies in F. Then (C1) and the induction hypothesis imply $\overline{\phi}(B_i) \leq \phi(R_1) \leq \phi(R'_3) \leq \phi(B_i)$. Therefore B_i, R_2, and R'_2 are attached to f only at one vertex, say u. Similarly, if the embedding $B_i[\ell_i]$ is in F', bridges B_i, R'_1, and R_3 are attached to f only at u. In each case one of the bridges B_i^-, B_i^+ is attached to the branch ϕ strictly before u, while the other one is after. But then the union $B_i^-[\ell_i] \cup B_i^+[\ell_i]$ has no common embedding with $R_2 \cup R'_2$, a contradiction with (C1).

On the other hand, if $\phi \neq \phi'$, then (C1) and (C2) imply that the union $R_1 \cup R_2 \cup R_3$ is attached to f only at u. By the same argument, $R'_1 \cup R'_2 \cup R'_3$ has only one attachment, say v. If $u \neq v$, then the segment $[u, v]$ of f lies on the boundary of F_1 and of F_2. Hence, each bridge that is attached to f (e.g., B_{i+1}) has an embedding in $F_1 \cup F_2$. We are left with the case $u = v$. If B_{i+1} cannot be embedded in $F_1 \cup F_2$, then its only attachment to f is u. Similarly as in the case when $\phi = \phi'$, the union $B^-_{i+1}[\ell_{i+1}] \cup B^+_{i+1}[\ell_{i+1}]$ has no common embedding with $R_2 \cup R'_2$, a contradiction.

(iv) Suppose that $\mathcal{C} = \mathcal{C}'$, $F = F'$ and $f_1 \neq f_2$. If at least one of the bridges B_i, B_{i+1} belongs to $\mathcal{C}$, $\varepsilon(B_{i+1}) > \overline{\varepsilon}(R_3)$, hence both embeddings are in $F_1 \cup F_2$. Otherwise, let ϕ be the oriented branch f whose orientation is induced by ε and $R_3[\ell]$. Note that by (P7), this orientation is the same as the one induced by ε and $R'[\ell']$. Let $u_1 = \overline{\phi}(R_3)$ and $u_2 = \overline{\phi}(R'_3)$. Since B_i and B_{i+1} are not in $\mathcal{C}$ and $e' = f$, none of them has an attachment to f strictly between u_1 and u_2. Let u be either u_1 or u_2, whichever appears first on ϕ. Observe that no occurrence of the part of the branch ϕ which is strictly before u does not lie on the boundary of a nondegenerate face and that at most one such occurrence is in the degenerate face (the reader may draw a figure similar to Figures 5, 6 with all four possibilities). By the induction hypothesis, the attachments of B_i to ϕ are all in u or after. If B_i has an attachment distinct from u, then both embeddings of B_i are in $F_1 \cup F_2$. Then also $B_{i+1}[\neg\ell_{i+1}]$ is in $F_1 \cup F_2$. The embedding $B_{i+1}[\ell_{i+1}]$ uses distinct occurrence of f than $B_{i+1}[\neg\ell_{i+1}]$. If $B_{i+1}[\neg\ell_{i+1}]$ is in a degenerate subface, then left(f) and right(f) on the occurrence of f used by $B_{i+1}[\ell_{i+1}]$ are contained in $F_1 \cup F_2$, hence the claim follows. Finally, if the only attachment of B_i on f is u, then the bridges B^-_i and B^+_i cannot have a common embedding with R_3 and R'_3, a contradiction.

(v) The last case is when $\mathcal{C} = \mathcal{C}'$ and $f_1 = f_2$. By (P7) and (C1)–(C2), all bridges R_j and R'_j $(1 \leq j \leq 3)$ are attached to f at a single vertex u. Observe that the subfaces F_1 and F_2 are nondegenerate. Suppose that both embeddings of B_{i+1} are out of $F_1 \cup F_2$. Then the second occurrence of f does not lie on the boundary of $F_1 \cup F_2$, and also the embedding $B_i[\ell_i]$ is out of $F_1 \cup F_2$. There are two possibilities. First, if $F_1 = F_2$, then B_i is attached to f only in u. But then the embeddings $R_3[\ell]$ and $R'_3[\ell']$ obstruct the embedding $B^+_i[\neg\ell_i]$, a contradiction with (C3). Second, if $F_1 \neq F_2$, then B_{i+1} is attached to f only in u. Now, the embedding $B^+_{i+1}[\ell_{i+1}]$ is obstructed by $R_3[\ell] \cup R'_3[\ell']$, a contradiction with (C1).

The last case in the proof of (a) is when $e' = e$. By induction, B_i is attached to ε after the segment S. Hence, both embeddings of B_i are in $F_1 \cup F_2$. If $B_i[\ell_i]$ is in a nondegenerate subface, then also B_{i+1} can be embedded in that subface. Otherwise, $B_i[\ell_i]$ is in the degenerate subface. Since this embedding overlaps on ε with $B_{i+1}[\neg\ell_{i+1}]$, also B_{i+1} must be attached to ε after the segment S. Then $B_{i+1}[\ell_{i+1}]$ lies in a nondegenerate subface. This proves (a).

To prove (b), suppose that B_{i+1} is attached to ε before the segment S. Then B_{i+1} has no embedding in a nondegenerate subface. Hence, both its embeddings are in $F_1 \cup F_2$.

Since this union contains on its border only one occurrence of left(ε), both embeddings of B_{i+1} use the same occurrence. Hence, B_{i+1} is $\frac{3}{2}$-embeddable and ε is its base. This completes the proof. □

Suppose now that the subchain $\mathcal{R}$ contains forcings $B_{i_j}[\ell_{i_j}] \to B_{i_j+1}[\ell_{i_j+1}]$ $(1 \le j \le 5)$, where all bridges B_{i_j} belong to $\mathcal{C}$, all bridges B_{i_j+1} belong to $\mathcal{C}'$, and all overlappings are on the oriented branch ε. Suppose also that $\ell_{i_1} = \ldots = \ell_{i_5}$ and $\ell_{i_1+1} = \ldots = \ell_{i_5+1}$. Moreover, suppose that indices i_j $(1 \le j \le 5)$ have been chosen such that for i_j $(1 \le j \le 4)$, these are the first such overlappings in $\mathcal{R}$, while for i_5, this is the last such overlapping in $\mathcal{R}$. Split $\mathcal{R}$ at the bridges B_{i_j} and B_{i_j+1} $(1 \le j \le 5)$ such that these bridges become subchains of length 1. Let $\mathcal{R}'$ be the part of $\mathcal{R}$ between B_{i_4+1} and B_{i_5}. Among all new subchains, only $\mathcal{R}'$ may contain the same type of overlapping as above.

Let $e = \mathit{type}(\varepsilon)$. Suppose that $\mathcal{R}'$ contains overlappings $R_i[\ell] \to R_i'[\ell']$ $(1 \le i \le 5)$ such that all bridges R_i belong to the cluster $\mathcal{C}_1$, all bridges R_i' belong to $\mathcal{C}_1'$, and the overlappings are on the oriented branch ε', where ε' is one of the orientations of e. Suppose also that $\{\mathcal{C}_1, \mathcal{C}_1'\} \neq \{\mathcal{C}, \mathcal{C}'\}$. By Lemma 6.1, all bridges R_i and R_i' are attached to branch ε after B_{i_4} and B_{i_4+1}. Since they do not belong to the same pair of clusters, they must be attached also after B_{i_5} and B_{i_5+1}. We now distinguish two subcases.

If $\varepsilon' = \varepsilon$, then we get a contradiction with Lemma 6.1, since B_{i_5} and B_{i_5+1} appear in $\mathcal{R}$ after R_i and R_i', while on ε they overlap before.

If $\varepsilon' = \overline{\varepsilon}$, let $\overline{\mathcal{R}}$ be the reverse subchain $\mathcal{R}$. If R_i and R_i' belong to distinct clusters, then they overlap in $\overline{\mathcal{R}}$ on the branch ε' or on $\overline{\varepsilon'}$. Since $\mathcal{R}$ satisfies (M5), they indeed overlap on $\overline{\varepsilon'}$. On the other hand, if they are in the same cluster, let $\mathit{type}(\mathcal{C}_1) = \{e, f\}$. We may assume that f is a branch of K. Let ϕ be the orientation of f induced by ε' and $R_i[\ell]$. For $i > 1$, R_i and R_i' overlap in the chain $\overline{\mathcal{R}'}$ on the branch $\overline{\varepsilon'}$ or on $\overline{\phi}$. Since $\overline{\varepsilon'}(R_i') < \varepsilon'(R_i)$, we may assume that they overlap on $\overline{\varepsilon'} = \varepsilon$. Lemma 6.1 again yields a contradiction since B_{i_j} and B_{i_j+1} $(1 \le j \le 4)$ appear in $\overline{\mathcal{R}}$ after R_i and R_i' $(2 \le i \le 5)$, while they are attached to ε before them.

Consequently, for each orientation ε' of e and for each pair of clusters $\mathcal{C}_1, \mathcal{C}_1'$ distinct from the pair $\mathcal{C}, \mathcal{C}'$, at most five pairs of consecutive bridges that overlap on ε' appear in $\mathcal{R}'$.

Let us repeat the splittings of clusters described above also for other branches f and pairs of clusters whose embeddings overlap on f. The number of subchains obtained by this process from the original subchain is bounded. Therefore, we have achieved that our obstruction Ω is composed of a bounded number of subchains where each essential subchain $\mathcal{R}$ satisfies one of the following:

(M6a) All bridges in $\mathcal{R}$ belong to the same cluster.

(M6b) $\mathcal{R}$ contains bridges from at least two clusters. The embeddings determined by $\mathcal{R}$ use the same embedding scheme for all bridges in the same cluster. For each branch e of K, all overlappings from $\mathcal{R}$ on e occur from pairs of bridges from the same pair of distinct clusters. For each cluster, the order of bridges determined by the embedding of $\mathcal{R}$ is the same as the order of their appearances in $\mathcal{R}$.

The subchain has property (M6), if it either satisfies (M6A) or (M6B).

Suppose that $B \in \mathcal{C}$ and that $B[\ell]$ appears in $\mathcal{R}$ and that bridges of $\mathcal{C}$ overlap in $\mathcal{R}$ on the oriented branch ε with bridges from cluster $\mathcal{C}'$. Suppose that $\mathcal{R}$ satisfies (M6B). Then $\mathcal{C} \neq \mathcal{C}'$. We may assume that all embeddings in $\mathcal{R}$ are the first embeddings. By (M6B), $B[1] \to B'[1]$ for each $B' \in \mathcal{C}'$ with an attachment on ε before $\overline{\varepsilon}(B)$. By (FM), all such bridges, except the bridge that follows $B[1]$ in $\mathcal{R}$, must appear in $\mathcal{R}$ before $B[1]$. By (M5), for each $B' \in \mathcal{C}'$ with $\varepsilon(B) < \overline{\varepsilon}(B')$, we have $B[2] \to B'[2]$. Again, all such bridges, except possibly one, appear in $\mathcal{R}$ after $B[1]$. Suppose now that $\mathcal{R}$ satisfies (M6A). Let $type(\mathcal{C}) = \{e, f\}$ where $e = type(\varepsilon)$. We distinguish several possibilities. If ε and embeddings of B induce the same orientations of f, then we have an overlapping $R[\ell] \to R'[\ell]$ in $\mathcal{R}$ if and only if $R[\neg\ell] \to R'[\neg\ell']$. Hence the embedding $B[i]$ $(i = 1, 2)$ overlaps only with its neighbors in $\mathcal{R}$. If the induced orientations of f are distinct and all bridges of $\mathcal{R}$ are attached to f a single vertex, then the overlappings behave similarly as in the previous case. Otherwise, they behave similarly to the case when $\mathcal{R}$ satisfies (M6B).

Suppose that $\mathcal{R}$ is an essential subchain in the obstruction Ω satisfying (M6). Let $\delta_1, \ldots, \delta_k$ $(k \geq 2)$ be the admissible embedding schemes that appear in $\mathcal{R}$. We define a directed graph $\Re$ with vertex set $V(\Re) = \{1, \ldots, k\}$ where vertex i corresponds to δ_i as follows. If ε is an oriented branch on which embeddings of bridges in $\mathcal{R}$ overlap, then (M6) implies that there are uniquely determined embedding schemes δ_u and δ_v which overlap on ε. If $\mathcal{R}$ contains consecutive bridges where the first is δ_u-embedded and the second is δ_v-embedded, then we have a directed edge from u to v in $\Re$. The edge uv is said to be *associated* with ε. Let us remark that $\Re$ may contain two edges from u to v associated with distinct oriented branches if $\mathcal{R}$ has property (M6A). By construction, $\Re$ contains a directed Hamiltonian path and by (M6), each vertex of $\Re$ is adjacent (irrespective of directions of edges) to at most two other vertices. associated

The sequence of all bridges $B_1, B_2, \ldots, B_s$ in $\mathcal{R}$ determines a directed walk W of length $s - 1$ in $\Re$. For each $uv \in E(\Re)$, mark its first and last appearance in W and split $\mathcal{R}$ at these places. Let L_1', L_1'' and R_1', R_1'' be the bridges that overlap at the first and the last, respectively, appearance of uv. We make our splitting of $\mathcal{R}$ so that L_1', L_1'', R_1', R_1'' become trivial subchains of length 1. We repeat the procedure on all new essential subchains. Let us remark that some edges of $\Re$ no longer appear in subwalks of W corresponding to particular subchains. The splitting process is repeated $2k+8$ times. The entire process assures that for each (remaining) essential subchain M, we call it a *millipede*, and each oriented branch ε_j $(1 \leq j \leq k)$ for which there is an overlapping on ε_j in M, there are bridges $L_i'^{(j)}$, $L_i''^{(j)}$, $R_i'^{(j)}$, $R_i''^{(j)}$ $(1 \leq i \leq 2k + 8,\ 1 \leq j \leq k)$, we call them the (*left* and the *right*) *outer bridges* of M, such that $L_i'^{(j)}$ and $L_i''^{(j)}$ are obtained as L_1' and L_1'' in the ith level of the splitting process and they overlap in $\mathcal{R}$ on ε_j. Similarly, the right outer bridges $R_i'^{(j)}$ and $R_i''^{(j)}$ are obtained on the ith level of the process and they overlap on ε_j. By construction, outer bridges from distinct levels are distinct. However, some bridges from the same level may participate in overlappings on distinct branches. For example, it may happen that $L_i''^{(j)} = L_i'^{(j+1)}$. The *closure* $\tilde{M}$ of millipede outer bridges closure the millipede M is the subchain of $\mathcal{R}$ between the first and the last outer bridge of M. The closure contains all outer bridges of M and all left outer bridges appear before M and all right bridges appear after M. Similarly, if $i < p$ and j, q are arbitrary, then $L_i'^{(j)}$ and $L_i''^{(j)}$ appear before $L_p'^{(q)}$ and $L_p''^{(q)}$.

The branches of K on which there are overlappings in the millipede M are called *inner branches* of M. Appearances of other basic pieces containing attachments of bridges from M are *outer basic pieces* of M. The millipede has at most two outer basic pieces.

inner branches

outer basic pieces

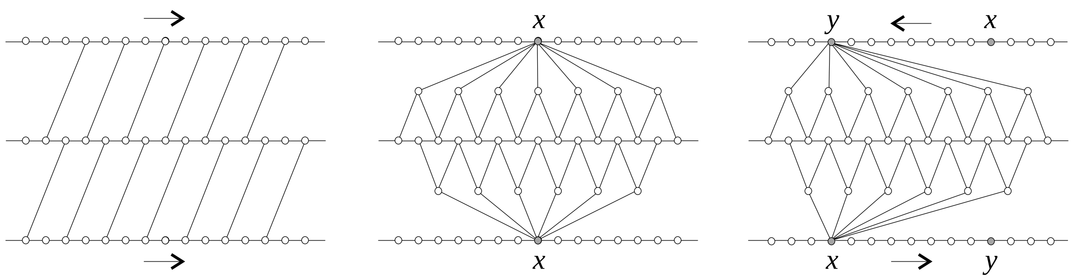

Figure 7: Examples of 2-millipedes: thick, thin, and skew.

Millipedes that contain bridges from one cluster only are called 2-*millipedes*. Such (and only such) millipedes occur, for example, in the EEPs in the cylinder [M2] and the Möbius band [JM]. Results of [M2, JM] show that they can be classified as thick, thin, or skew. Examples are shown in Figure 7. If a millipede contains bridges from more than one cluster, then it is called a *multi-millipede*. An example of a multi-millipede in the torus with $k = 4$ is shown in Figure 8.

2-millipedes

multi-millipede

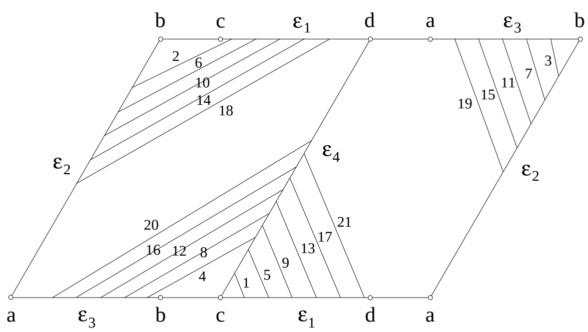

Figure 8: A multi-millipede.

Let M be a multi-millipede. Denote by $\varepsilon_1, \ldots, \varepsilon_l$ the inner branches of M. Clearly, $l \in \{k-1, k\}$. Assume that ε_j $(1 \le j \le l)$ are enumerated in such a way as they appear in a Hamiltonian path in $\Re$, and that under the embedding determined by M, all bridges in M get their first embeddings. Let $\mathcal{C}_j, \mathcal{C}_{j+1}$ $(1 \le j \le l)$ be the clusters that in M overlap on ε_j (with $\mathcal{C}_{k+1} = \mathcal{C}_1$ if $l = k$). From each cluster $\mathcal{C}_j$ $(1 \le j \le k)$

we choose a left outer bridge $L^{(j)}$ and a right outer bridge $R^{(j)}$. By (C3), there exists a common embedding of all outer bridges of M such that the left outer bridges are embedded compatibly with their first, and the right outer bridges with their second admissible embedding scheme. Let F be the face containing $L^{(1)}[1] \cup R^{(2)}[2] \cup \cdots$. The above implies that the appearances of ε_j $(1 \le j \le l)$ used by embeddings $L^{(j)}[1]$ $(j = 1, 3, \ldots)$ and $R_i[2]$ $(j = 2, 4, \ldots)$, follow each other in ∂F as follows: $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_l$ (in the direction determined by ε_1). Any two consecutive oriented branches in this sequence are oriented differently. The same property holds for the appearances of branches used by the embeddings $L^{(j)}[1]$ $(j = 2, 4, \ldots)$ and $R^{(j)}[2]$ $(j = 1, 3, \ldots)$. If $\tilde{M}$ satisfies (M4B), then the face in which the bridges are embedded, can be split into two subfaces such that occurrences of ε_j $(1 \le j \le l)$ used by $L^{(j)}[1]$ $(j = 1, 3, \ldots)$ and $R^{(j)}[2]$ $(j = 2, 4, \ldots)$ are on the boundary of one, all other on the boundary of the second subface.

Let us remark without a proof that millipedes also have the following properties. From (C1)–(C3) it follows that no outer basic piece is an occurrence of an inner branch. Two distinct embedding schemes may use the same outer basic piece. In such a case, if the millipede has embeddings in two faces, then all bridges from the cluster $\mathcal{C}_1$ are attached to the outer basic piece at a single vertex, and similarly for the bridges from $\mathcal{C}_k$. These two vertices are in general distinct. On the other hand, if the bridges of the millipede all lie in one face, then the bridges from exactly one of $\mathcal{C}_1, \mathcal{C}_k$ may have several attachments on the corresponding outer basic piece. The bridges from the other cluster are attached to the outer basic piece at a single vertex and the outer basic piece is their base.

Consider an inner branch $\varepsilon = \varepsilon_j$ of M. The open segment of ε determined by the intersection $(\varepsilon(L_3'^{(j)}), \overline{\varepsilon}(R_3'^{(j)})) \cap (\varepsilon(L_3''^{(j)}), \overline{\varepsilon}(R_3''^{(j)}))$ is called the *inner segment* of M corresponding to ε. Similarly we define the *outer segments* as those segments of outer basic pieces which are covered by embeddings of the corresponding outer bridges from the fifth level.

inner segment

outer segments

With possible further splitting of millipedes we can achieve that for each multi-millipede with a nonempty outer segment on the outer branch ε, the left and the right outer bridge from the sixth level that are attached to ε have attachments in the outer segment.

7 Minimal obstructions

In this section we shall make further analysis of the structure of overlappings among the bridges in millipedes. This will enable us to efficiently minimize obstructions for 2-EEPs. Let us first show how to achieve that to each inner segment of a millipede M only bridges from the closure $\tilde{M}$ are attached.

Lemma 7.1 *Let Ω be an obstruction for the 2-*EEP *Ξ obtained by a BF-implementation of Algorithm* Extend. *Suppose that Ω is composed of (one or) two forcing chains $\mathcal{R}_1$ and $\mathcal{R}_2$. Let M be a millipede in $\mathcal{R}_1$. If there is a bridge $B \in \Omega \backslash \tilde{M}$ which is attached to an inner segment of M, then $B \notin \mathcal{R}_1$ and $\Omega \backslash B$ is also an obstruction for Ξ.*

Proof. Denote by ε the oriented branch containing the inner segment to which B is attached. Let $\mathcal{C}$ and $\mathcal{C}'$ be clusters which in M overlap on ε. Since the inner segments are defined by using the third level outer bridges, there are outer bridges of M attached to ε before B and after B, respectively. By (BF), B has a simultaneous embedding with these outer bridges. Therefore B belongs either to $\mathcal{C}$ or $\mathcal{C}'$. Since $B \notin \tilde{M}$, the same argument shows that $B \notin \mathcal{R}_1$. In particular, Ω is composed of two forcing chains. We may assume that $B \in \mathcal{C}$ and that $B[\ell]$ appears in $\mathcal{R}_2$.

Suppose first that $\mathcal{C} \neq \mathcal{C}'$. By (M6B), we may assume that bridges in M all use their first embeddings. Let $L \in \mathcal{C}$ and $R \in \mathcal{C}$ be the left and the right outer bridge of M, respectively, from the third level. Bridges L and R are attached to ε before and after B, respectively. Suppose that $B'[\ell'] \to B[\ell] \to B''[\ell'']$ in $\mathcal{R}_2$. (If $B[\ell]$ is the last bridge in $\mathcal{R}_2$, $B[\ell] \to B''[\ell'']$ is the additional forcing in $\mathcal{R}_2$.) If $B'[\ell'] \to L[\ell]$ and $B'[\ell'] \to R[\ell]$, then we argue as follows. If $L[\ell] \cup R[\ell]$ forces $B''[\ell'']$, then obviously B can be removed from Ω. Otherwise, $B'' \in \mathcal{C}$ and B'' is attached to ε between L and R. Hence $\ell \neq \ell''$. Because of BF-implementation, B'' has a simultaneous embedding with B and all outer bridges, a contradiction. The remaining possibility is that $B'[\ell'] \to L[\ell]$ or $B'[\ell'] \not\to R[\ell]$. If $B'[\ell'] \to L[1]$ or $B'[\ell'] \to R[2]$, then the subchain $L[1] \to \ldots \to R[1]$ from $\tilde{M}$ proves as above that B is superfluous. If $B'[\ell'] \to R[1]$, then there exists a left outer bridge $L' \in \mathcal{C}'$ of M which is attached to ε after L and such that $R[1] \to L'[1]$ and $L'[1] \to L[1]$. (For L' we can take the left outer bridge from the fifth level.) Similarly in the case when $B'[\ell'] \to L[2]$. Then there is a right outer bridge R' such that $L[2] \to R'[2]$ and $R'[2] \to R[2]$. In both cases we argue as before that B is redundant.

It remains to see what happens in the case when $\mathcal{C} = \mathcal{C}'$. Now, the proof is similar to the above except that we use bridges $L, R \in \mathcal{C}$ from $\tilde{M}$ such that under the embedding of $\tilde{M}$, B lies between L and R and no bridge of $\tilde{M}$ lies between L and R. Since the arguments are similar as above, we leave the details to the reader. □

Lemma 7.1 enables us to achieve in linear time that for each millipede M in $\mathcal{R}_1$ and for each inner segment S of M, only the bridges from $\tilde{M}$ are attached to S. The next lemma will be used to achieve the same property also for the outer segments.

Lemma 7.2 *Let Ω be an obstruction for the 2-EEP Ξ obtained by a BF-implementation of Algorithm* EXTEND. *Suppose that Ω is composed of two forcing chains $\mathcal{R}_1$ and $\mathcal{R}_2$. For $i = 1, 2$, let M_i be a millipede in $\mathcal{R}_i$. Select a cluster $\mathcal{C}$ and an outer branch ε of M_1, and denote by $\mathcal{B}$ the set of those bridges from $\mathcal{C}$ that belong to $M_2 \backslash \tilde{M}_1$ and are attached on ε to the outer segment S_ε of M_1. If the union of bridges from $\mathcal{B}$ is attached to ε in more than one vertex, then $\mathcal{B}$ contains at most two bridges that are not redundant in Ω. Moreover, redundant bridges can be identified in linear time.*

Proof. If both admissible embeddings of bridges from $\mathcal{C}$ use the same occurrence of ε, then ε is an outer branch of M_2. By (P7), the union of all bridges from $\mathcal{C}$ that are in M_2 is attached to ε in a single vertex. Therefore we may assume henceforth that admissible embeddings of bridges from $\mathcal{C}$ use distinct occurrences of ε.

Suppose that in M_1 bridges from $\mathcal{C}_1$ are attached to ε. If both admissible embeddings of bridges from $\mathcal{C}_1$ use the same occurrence of ε, then (P7) implies that the outer segment on ε is empty, hence the claim. Therefore we may as well assume that the admissible embeddings of bridges from $\mathcal{C}_1$ use distinct occurrences of ε. Let $L, R \in \mathcal{C}_1$ be the bridges that determine S_ε. We may assume that in $\tilde{M}_1$ embeddings $L[1]$ and $R[1]$ appear. By possibly reversing the order of bridges in M_2 we also assure that in M_2 the bridges from $\mathcal{C}$ use different occurrence of ε than $L[1]$ and $R[1]$. Let $B_1, B_2 \in \mathcal{C}$ be the first and the last (respectively) bridge from M_2 which are attached to S_ε. Assume also that in M_2, embeddings $B_1[\ell]$ and $B_2[\ell]$ occur.

Suppose that $\mathcal{C} \neq \mathcal{C}_1$. Then either $L[1] \to B_i[\ell]$ $(i = 1, 2)$ or $R[1] \to B_i[\ell]$ $(i = 1, 2)$. Similarly, either $B_1[\ell] \to L[1]$ or $B_1[\ell] \to R[1]$. In the first case, we can substitute the part of M_2 between B_1 and B_2 with forcings $B_1[\ell] \to L[1] \to B_2[\ell]$ or $B_1[\ell] \to L[1] \to \ldots \to R[1] \to B_2[\ell]$. If $B_1[\ell] \to R[1]$, then $L[1]$ does not force $B_1[\ell]$ since the BF-implementation of Algorithm Extend would find $L[1] \to B_1[\ell] \to R[1]$ instead of the part of the forcing chain between L and R in $\tilde{M}_1$. The part of M_2 between B_1 and B_2 can therefore be replaced by $B_1[\ell] \to R[1] \to B_2[\ell]$.

We are left with the case $\mathcal{C} = \mathcal{C}_1$. Then $\ell = 2$. Let ε_1 be the inner branch of M_1 used by bridges from $\mathcal{C}_1$. By Lemma 7.1 and since the definition of outer segments of millipedes uses outer bridges from level five, none of the bridges B_1, B_2 is attached to the segment $S = [\varepsilon_1(L), \overline{\varepsilon}_1(R)]$. If $\varepsilon_1(B_1) < \varepsilon_1(L)$ and $\varepsilon_1(B_2) < \varepsilon_1(L)$, then $B_1[2] \to L[1]$ since $L[2] \cup R[2]$ exists. Now, we replace the part of M_2 between B_1 and B_2 by the chain $B_1[2] \to L[1] \to B_2[2]$. Similarly in the case when both bridges have an attachment on ε_1 after S: we take $B_1[2] \to R[1] \to B_2[2]$. If B_1 is attached before S, while B_2 is attached after, we take $B_1[2] \to L[1] \to \ldots \to R[1] \to B_2[2]$ for replacement. The remaining possibility would be when B_1 is attached after S, and B_2 is attached before S. However, in that case either the embedding $L[2] \cup R[2]$ or $B_1[2] \cup B_2[2]$ does not exist, a contradiction. □

Let us now show how to achieve (in linear time):

(M7) If M is a millipede in Ω, then only the bridges of $\tilde{M}$ are attached to each inner or outer segment of M.

Let us consider a forcing chain $\mathcal{R}_1$ in Ω and let M be a millipede in $\mathcal{R}_1$. Lemma 7.1 enables us to assume (M7) for inner segments. Suppose now that ε is an oriented outer branch of M with nonempty outer segment S. Let x and y be the endvertices of ε. We may assume that the order of bridges in M agrees with the order of their attachments on ε. For $i = 1, 2$, let $\mathcal{B}_i$ be the set of those bridges from $\Omega \backslash \tilde{M}$ that have an attachment in S and belong to the forcing chain $\mathcal{R}_i$. Since $\mathcal{R}_1$ is forward minimal, all bridges in $\mathcal{B}_1$ (except possibly the last bridge of $\mathcal{R}_1$) are $\frac{3}{2}$-embeddable and ε is their base. For each millipede $M' \neq M$ in $\mathcal{R}_1$, (P7) implies that the bridges of $\mathcal{B}_1$ that are in M' are attached to ε at a single vertex. Hence, $\mathcal{B}_1$ has only a bounded number of attachments on ε. By Lemma 7.2, the same holds also for bridges from $\mathcal{B}_2$. Suppose now that $B \in \mathcal{B}_1 \cup \mathcal{B}_2$. If B is attached to ε at a single vertex, then we subdivide the millipede

M at the following bridges: the last bridge with an attachment in the open segment $(x, \varepsilon(B))$ which is entirely attached to $(x, \varepsilon(B)]$, the first bridge attached to $(\varepsilon(B), y)$ which has all its attachments to ε in $[\varepsilon(B), y)$, and the first and the last bridge of M which are attached to ε only at the vertex $\varepsilon(B)$. If M contains a bridge B' which is attached to ε before and after $\varepsilon(B)$, then we take B' instead of bridges attached only to $\varepsilon(B)$. All these bridges become trivial subchains, while for the longer subchains we make sure with further splittings that the required number of outer bridges is obtained. The same method works if B has more than one attachment on ε. In this case, at most six bridges of M are attached to $(\varepsilon(B), \overline{\varepsilon}(B))$: if B is not $\frac{3}{2}$-embeddable, this is assured by (BF); for $\frac{3}{2}$-embeddable B, this follows from $S \neq \emptyset$ and (FB). Therefore we can use the following bridges as the splitting points of the chain: the last bridge in M with an attachment in $(x, \varepsilon(B))$ and the first bridge with an attachment in $(\overline{\varepsilon}(B), y)$ (and possibly the bridges of M that have all attachments in the segment $[\varepsilon(B), \overline{\varepsilon}(B)]$). These splittings of M do not give rise to new bridges attached to the inner or outer segments of the resulting millipedes.

By repeating the described changes on each millipede of $\mathcal{R}_1$, we achieve that the millipedes in $\mathcal{R}_1$ satisfy (M7). Since Lemma 7.2 is used to make the desired changes, the forcing chain $\mathcal{R}_2$ may change and we may lose some of its properties. The easiest solution is to leave $\mathcal{R}_1$ as it is and to perform the algorithm FORCING on the obstruction Ω starting with the embedding $B[\neg\ell]$ where $B[\ell]$ starts $\mathcal{R}_1$. The resulting forcing chain is again denoted by $\mathcal{R}_2$. It may contain some bridges of $\mathcal{R}_1$. By appropriate splittings of $\mathcal{R}_2$ we assure that for each millipede M in $\mathcal{R}_2$, only bridges from the closure of M are attached to each inner or outer segment. Since $\mathcal{R}_2$ "enters" a millipede M_1 in $\mathcal{R}_1$ only through $\tilde{M}_1 \backslash M_1$, there is no need to make changes of the millipedes in $\mathcal{R}_1$ after the change of $\mathcal{R}_2$ has been done. This shows (M7).

To efficiently minimize Ω we also need the following property of millipedes in Ω:

(M8) If M is a millipede in Ω, then the closure $\tilde{M}$ of M contains bridges from precisely the same clusters as M.

This property can be achieved by a bounded number of additional subdivisions as follows. Let $\mathcal{C}_1, \ldots, \mathcal{C}_k$ be the clusters participating in M. First, we repeat on M all $2k+8$ levels of subdivisions to obtain the outer bridges and so that all new millipedes and their closures are contained in M. If one of the new millipedes does not satisfy (M8), its clusters form a proper subset of $\{\mathcal{C}_1, \ldots, \mathcal{C}_k\}$. We repeat the splitting on each such millipede. Clearly, such splittings have to be repeated at most k times in a row. This proves that after a bounded number of splittings the resulting millipedes satisfy (M8).

Let us now prove the property which is essential for an efficient minimization of obstructions. Roughly speaking, it says that millipedes behave like an entity: if a bridge of the millipede M is redundant in the obstruction, then $\Omega \backslash M$ is also an obstruction.

Proposition 7.3 *Let M be a millipede in the obstruction Ω for the 2-EEP Ξ. Let $\mathcal{B} \subseteq \Omega$ be a set of bridges which contains all outer bridges of M and the bridge $B \in M$. If the millipedes in Ω satisfy (M7)–(M8), then the set $\mathcal{B} \backslash M$ forms an obstruction if and only if the set $\mathcal{B} \backslash B$ does.*

Proof. Let $B_1, \ldots, B_s$ be the bridges in M. Without loss of generality we may assume that $\mathcal{B}$ contains M. If $\mathcal{B}\backslash M$ is an obstruction, then also $\mathcal{B}\backslash B$ is. Hence it suffices to show that each embedding of the bridges from $\mathcal{B}\backslash M$ can be transformed and extended to an embedding of $\mathcal{B}\backslash B$.

Suppose that M is a multi-millipede. Choose an arbitrary embedding of $\mathcal{B}\backslash M$. Choosing notation, we may assume that the embeddings of the bridges B_i ($1 \le i \le s$) in M are $B_i[1]$. We will distinguish several cases.

Suppose first that there is a left outer bridge $L_1^{(i)}$ such that under the chosen embedding of $\mathcal{B}\backslash M$ it is embedded as $L_1^{(i)}[2]$. Since M is a multi-millipede, also $L_t^{(i+1)}$ and $L_t^{(i-1)}$ ($t \ge 3$) have embeddings $L_t^{(i+1)}[2]$ and $L_t^{(i-1)}[2]$. The same arguments prove that for each j ($1 \le j \le k$) there is an index $t_j \le 2k-1$ (where k is the number of clusters in M) such that all left outer bridges $L_t^{(j)}$ ($t \ge t_j$) have embeddings $L_t^{(j)}[2]$. By (M8), there is a left outer bridge L at level less than $2k$ such that all bridges from $\mathcal{B} \cap \tilde{M}$ that follow L in $\tilde{M}$ are embedded consistently with their second embedding scheme. Since (M7) holds and the subchain $\tilde{M}$ satisfies (C2), the embedding of $\mathcal{B}\backslash M$ can be extended to an embedding of $\mathcal{B}$ by selecting the second embedding for all bridges in M.

Secondly, if there is a right outer bridge $R_1^{(i)}$ whose embedding is $R_1^{(i)}[1]$, the proof is the same as in the first case except that we embed bridges using their first embedding scheme and apply property (C1) of $\tilde{M}$.

The third possibility is that the left outer bridges $L_1^{(i)}$ ($1 \le i \le k$) have their first embedding and the right outer bridges $R_1^{(i)}$ ($1 \le i \le k$) use their second embedding. We may assume that the same holds for all outer bridges from the second till the eighth level. (Otherwise we proceed as in the first two cases.) Then (M8) implies that all bridges of $\mathcal{B} \cap \tilde{M}$, that appear in $\tilde{M}$ before any left outer bridges of level six, also use their first embedding since their second embedding would force the second embedding of at least one left outer bridge from level eight. Similarly, all bridges that in $\tilde{M}$ follow the right outer bridges of level six, must use their second embedding. Let $\mathcal{B}_L \subseteq \mathcal{B}$ be the set of those bridges that appear in $\tilde{M}$ before B and after some left outer bridge from level six. Similarly, let $\mathcal{B}_R \subseteq \mathcal{B}$ be the set of bridges that appear in $\tilde{M}$ after B and before some right outer bridge from level six. By (C3), there is an embedding of $\mathcal{B}_L \cup \mathcal{B}_R$ such that the bridges from $\mathcal{B}_L$ use their first, and bridges from $\mathcal{B}_R$ their second embeddings. By (M7), this embedding does not interfere with the embeddings of the remaining bridges from $\mathcal{B}\backslash(\mathcal{B}_L \cup \mathcal{B}_R)$.

The proof is similar for 2-millipedes except that forcings in $\tilde{M}$ behave slightly different. We leave details to the reader. □

Proposition 7.3 enables us to test in linear time which bridges from Ω are redundant in the obstruction. Let $M_1, \ldots, M_p$ be the millipedes and $B_1, \ldots, B_q$ the trivial subchains which form the obstruction Ω. First we check for each millipede M_i ($1 \le i \le p$) if it is redundant or not. This task is performed by selecting an arbitrary bridge $B \in M_i$ and applying Algorithm Extend on $\Omega\backslash B$. If this is still an obstruction, then we may remove from Ω all bridges of M_i by Proposition 7.3. Otherwise, Proposition 7.3 guarantees that all these bridges must participate in any minimal obstruction contained in Ω. This way we eliminate (one after another) all redundant millipedes. Then we check which bridges of the trivial subchains are redundant. Since p and q are bounded by a constant depending only on $\mathrm{bsize}(K)$, this procedure can obviously be performed in linear time. Hence we proved:

Theorem 7.4 *There is a linear time algorithm which for a given 2-*EEP $\Xi = (G, K, \Pi, \mathcal{D})$ *satisfying* (P1)–(P7) *either finds a solution for* Ξ *or returns a minimal obstruction* Ω*. In the latter case,* Ω *is composed of a bounded number of bridges and a bounded number of pairwise disjoint millipedes, where the bounds depend only on* $\mathrm{bsize}(K)$*.*

8 Compression of obstructions

Minimal obstructions may have arbitrarily large branch size. In this section we prove that it is possible to replace the subgraph K by a homeomorphic subgraph K' with the same main vertices and their incident edges such that the (minimal) obstruction Ω for the 2-EEP Ξ changes into an obstruction of bounded branch size. Of course, the equivalence classes of (2-cell) embeddings of K' are in natural bijective correspondence with the embeddings of K. Additionally, our change preserves the types of bridges. Therefore, Ξ determines a 2-EEP Ξ' for K', and the bounded branch size obstruction mentioned above is an obstruction for Ξ'. The replacement of K by K' and of Ω by a bounded size obstruction (as described in the sequel) will be referred to as a *compression*. compression

The branches of K will be changed only in inner segments of the millipedes contained in Ω. Let Π' be the embedding of K' which corresponds to Π. Let us consider the 2-EEP $\Xi' = (G, K', \Pi', \mathcal{D}')$ corresponding to the 2-EEP $\Xi = (G, K, \Pi, \mathcal{D})$. The basic pieces of K and K' are in bijective correspondence, $x \mapsto x'$. Let B' be a nonlocal K'-bridge which is not a K-bridge in G. The change of K as described in the sequel will assure that B' consists of parts of bridges and parts of inner branches of K from the same millipede. Such a bridge B' is attached only to two open branches e', f' of K', and in K there is a cluster $\mathcal{C}$ of K-bridges attached to e and f. Then $\mathcal{D}'(B')$ consists of the admissible embeddings for the cluster $\mathcal{C}$. In the rest of this section we describe the compression in more details.

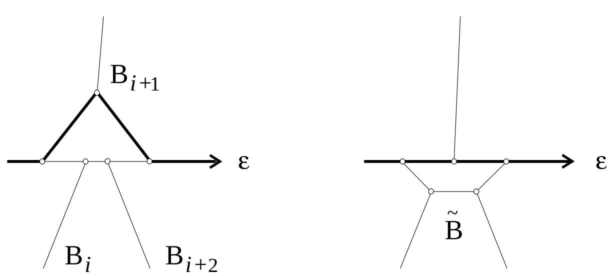

Figure 9: A local change of K.

Let $M = B_1 \cup \cdots \cup B_s$ be the bridges of a millipede in a minimal obstruction Ω. Denote by ε_i the oriented branch of K on which B_i and B_{i+1} overlap. Suppose that M contains three consecutive bridges B_i, B_{i+1}, B_{i+2} such that $\varepsilon := \varepsilon_i = \varepsilon_{i+1}$. Then $\varepsilon(B_{i+1}) < \overline{\varepsilon}(B_i) \leq \varepsilon(B_{i+2}) < \overline{\varepsilon}(B_{i+1})$. Now we replace the segment $(\varepsilon(B_{i+1}), \overline{\varepsilon}(B_{i+1}))$ of ε by a path from $\varepsilon(B_{i+1})$ to $\overline{\varepsilon}(B_{i+1})$ in the bridge B_{i+1} (such that the path is internally disjoint from K). See Figure 9 where $k = 4$ and $p = 1$. (The same millipede is also presented in Figure 8.) Denote the new subgraph by K' and observe that B_i and B_{i+2} merge into a single K'-bridge, say $\tilde{B}$. We may view $\tilde{B}$ as a bridge from the same cluster as B_i and B_{i+2}. (We may also replace $\tilde{B}$ by a subgraph of bounded branch size [M3] to make sure that the bridges remain of bounded size.) The parts of B_{i+1} which do not lie in K' or $\tilde{B}$ are omitted. All other bridges in Ω remain unchanged. This change gives rise to a new millipede $\tilde{M} = B_1 \cup \cdots \cup B_{i-1} \cup \tilde{B} \cup B_{i+3} \cup \cdots \cup B_s$ which is shorter than M. It is easy to see that $(\Omega \backslash (B_i \cup B_{i+1} \cup B_{i+2})) \cup \tilde{B}$ is a minimal obstruction. By repeated application of the above changes we either achieve that M changes into a millipede without consecutive forcings on the same (oriented) branch, or it becomes a trivial subchain consisting of a single bridge.

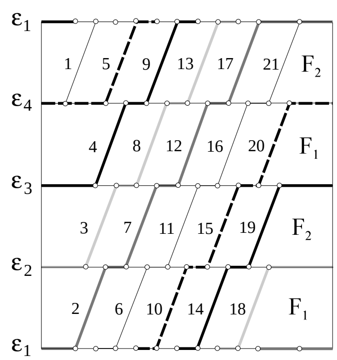

Figure 10: A global change of K.

Let M be the millipede resulting after the above changes. Consider the corresponding graph $\Re$ of M which was introduced in Section 6. Observe that M is closed and that $\Re$ is a directed cycle. Let $k = |V(\Re)|$. Then $\varepsilon_i = \varepsilon_{i+k}$ for each index i. Let $\kappa = k(k+1)$. Suppose that the length s of M is at least $\kappa + 1$ and that $s - 1$ is a multiple of κ. Let $p = (s-1)/\kappa$. For $i = 1, \ldots, k$, let P_i be a path in $K \cup M$ joining vertices $\varepsilon_i(B_{i+1})$ and $\overline{\varepsilon}_i(B_{p\kappa-k+i})$ on ε_i composed of the paths between the vertices $\varepsilon_{i+j}(B_{i+1+j(k+1)})$ and $\overline{\varepsilon}_{i+j+1}(B_{i+1+j(k+1)})$ in $B_{i+1+j(k+1)}$ $(0 \leq j < pk)$ and segments of branches ε_{i+j+1} between $\overline{\varepsilon}_{i+j+1}(B_{i+1+j(k+1)})$ and $\varepsilon_{i+j+1}(B_{i+1+(j+1)(k+1)})$ $(0 \leq j \leq pk-2)$. See Figure 10. The paths P_i $(1 \leq i \leq k)$ are pairwise disjoint and do not intersect any of the bridges $B_{1+j(k+1)}$ $(0 \leq j \leq pk)$. After replacing segments of the branches ε_i from $\varepsilon_i(B_{i+1})$ to $\overline{\varepsilon}_i(B_{p\kappa-k+i})$ by the paths P_i $(1 \leq i \leq k)$, the bridges $B_{1+j(k+1)}$ $(0 \leq j \leq pk)$ merge together into a single K'-bridge $\tilde{B}$ where K' is the new subgraph. Since only bridges from $\tilde{M}$ are attached to the inner branches of M where the change has been done, the other K-bridges do not change. Therefore, we have replaced M in Ω by a single bridge $\tilde{B}$. Since $\tilde{B}$ retains the forcing properties of M in the obstruction Ω, the new set of K'-bridges forms a minimal obstruction. The segments of ε_i $(1 \leq i < k)$ between $\varepsilon_i(B_{i+1})$ and $\overline{\varepsilon}_i(B_i)$ and those parts of bridges from M that belong neither to K' nor to $\tilde{B}$ can be omitted. We also replace $\tilde{B}$ by a subgraph of bounded branch size [M3]. By repeating the compression described above for all millipedes in Ω, we end up with an obstruction of bounded branch size.

In applications we need another improvement of the results proved so far. In solving more general EEPs, we may need to solve several 2-EEPs, and by solving each new case, we do not want to change the subgraph K in such a way that previous obstructions are changed (or even destroyed). Performing compression in previous steps, we may assume that obstructions of those steps are of bounded branch size. Now, we allow compression to change only those segments of branches of K that do not contain any vertex of previous obstructions. More precisely, let $U \subseteq V(K)$ be a set of vertices of K of bounded size. (In mentioned applications, U may be taken as the set of all main vertices of K and all vertices of attachment of the bridges in previous obstructions.) If we interpret all vertices in U as being main vertices of K, the inner segments of the millipedes will be automatically disjoint from U and hence our goal is achieved. The time complexity of the entire procedure is still linear (with bigger constant). We say that the *compression* is performed *with respect to U*. Such a compression finds a subgraph K' homeomorphic to K. The homeomorphism is the identity on U and all edges of K incident with U.

compression with respect to

Theorem 8.1 *Let $\Xi = (G, K, \Pi, \mathcal{D})$ be a 2-EEP with properties (P1)–(P7), and let $U \subseteq V(K)$ be a set of vertices such that each main vertex of K is contained in U. There is an algorithm that either finds a solution for Ξ or returns a subgraph K' homeomorphic to K and an obstruction Ω obtained by a compression with respect to U. There is an upper bound on the number of bridges in Ω that depends only on $\mathrm{bsize}(K)$ and $|U|$. The time complexity of the algorithm is $\mathcal{O}(\kappa \cdot |E(G)|)$ where κ depends only on $\mathrm{bsize}(K)$ and $|U|$.*

Let us conclude this section with some remarks how one could simplify the overall procedure yielding Theorem 8.1 if we would not insist on finding minimal obstructions for Ξ, and would only care of getting an obstruction of bounded branch size, allowing compressions. In that case, we take one of the forcing chains obtained by Algorithm EXTEND, make it forward minimal and assure by splitting that each subchain satisfies (C0)–(C3) and (M1)–(M6). Then we could split the chain into millipedes and perform the compression. This would make one of the forcing chains consist of a bounded number of bridges. The same procedure can then be repeated on the second forcing chain, and because of the bounded number of bridges in the first one, no troubles arise from the mutual interference of the bridges from both chains.

9 Extending closed 2-cell embeddings

The algorithm of Theorem 8.1 is a powerful tool in solving more general EEPs [M1, JMM3, M4]. Let us show some of its strength by presenting a linear time solution for general EEPs in which the subgraph K is closed 2-cell embedded.

Suppose that K is a 2-connected subgraph of G. Let G_K^3 be the graph obtained from G by adding three vertices each of which is adjacent to all main vertices of K. We assume that G_K^3 is 3-connected and that there are no local K-bridges in G. Denote by $\mathcal{B}$ the set of all K-bridges in G. Suppose that an EEP $\Xi = (G, K, \Pi, \mathcal{D})$ is given where Π is a closed 2-cell embedding of K.

Proposition 9.1 *Let G, K, and Ξ be as above. There is an algorithm whose input is Ξ and a set $U_0 \subseteq V(K)$ (containing all main vertices of K) that either finds a solution for Ξ or returns an obstruction Ω obtained by a compression with respect to U_0. The algorithm spends $\mathcal{O}(\kappa \cdot |E(G)|)$ time where κ depends only on* bsize(K) *and $|U_0|$. If Ξ has no solution, then* bsize$(K' \cup \Omega)$ *is bounded with the bound depending only on* bsize(K) *and $|U_0|$ (where K' is the graph obtained from K by the compression).*

Proof. Let $\mathcal{B}_0$ be the set of K-bridges that have an attachment in an open branch of K. Denote by $\mathcal{D}_0$ the restriction of $\mathcal{D}$ to $\mathcal{B}_0$. Then the EEP $\Xi_0 = (K \cup \mathcal{B}_0, K, \Pi, \mathcal{D}_0)$ is a 2-EEP. By construction, Ξ_0 satisfies (P1), and since Π is a closed 2-cell embedding, it also fulfils (P7). By using the auxiliary results from [JMM1, M1], also (P2)–(P6) can be achieved in linear time. Moreover, results of [M1] show that (P4) can be achieved also for each bridge in $\mathcal{B} \backslash \mathcal{B}_0$. Each bridge from $\mathcal{B} \backslash \mathcal{B}_0$ is attached to main vertices of K only. We may assume that to each pair of main vertices of K at most one bridge that is not strongly attached to K is attached. Since in each face of K at most one strongly attached bridge of each type can be embedded, the number of bridges in $\mathcal{B} \backslash \mathcal{B}_0$ is bounded in terms of bsize(K) (or we have an obstruction of bounded size).

We start with $\Omega = \mathcal{B} \backslash \mathcal{B}_0$ and with the set $U \subseteq V(K)$ consisting of U_0 and all vertices of attachment of bridges in $\mathcal{B} \backslash \mathcal{B}_0$. For each $\mathcal{D}$-compatible embedding of bridges $\mathcal{B} \backslash \mathcal{B}_0$, the EEP for the remaining bridge set $\mathcal{B}_0$ is still 2-restricted. Applying Theorem 8.1 (and performing the compression with respect to the current set U), we either get a solution and stop, or we get an obstruction composed of a bounded number of bridges. We add these bridges in Ω and extend U with all vertices of attachment of these bridges. This guarantees that after each step, Ω is an obstruction for the already treated embeddings of $\mathcal{B} \backslash \mathcal{B}_0$. The compression may give rise to local bridges which can in turn be eliminated since G_K^3 is 3-connected [JMM1].

Each $B \in \mathcal{B} \backslash \mathcal{B}_0$ may be embedded in one or more faces of K (whose number is bounded in terms of bsize(K)). Since Π is a closed 2-cell embedding, all embeddings of B in the same face use the same appearances of their vertices of attachment. Hence the number of distinct embeddings of $\mathcal{B} \backslash \mathcal{B}_0$ that has to be treated is also bounded in terms of bsize(K). This implies that the time complexity and the branch size of the obstruction are as claimed. □

If the embedding Π of K in the EEP Ξ is a closed 2-cell embedding, then each bridge admits at most one embedding scheme in each face of K, and each such embedding is simple. The following corollary of Proposition 9.1 solves general EEPs whose subgraph K is closed 2-cell embedded. Note that the assumption on Π implies that for each K-bridge B, all its embeddings that extend Π are simple.

Corollary 9.2 *Let $\Xi = (G, K, \Pi, \mathcal{D})$ be an* EEP *where Π is a closed 2-cell embedding of K. Suppose that for each bridge B, $\mathcal{D}(B)$ contains all possible embedding schemes for B. There is a linear time algorithm that either finds a solution for Ξ, or returns a subgraph K' of G obtained from K by a sequence of compressions and an obstruction Ω for the corresponding* EEP *Ξ'. The branch size of $K' \cup \Omega$ is bounded by a constant depending on* $\mathrm{bsize}(K)$ *only.*

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