Crossing numbers of Sierpiński-like graphs

Sandi Klavžar*
Department of Mathematics, PeF, University of Maribor
Koroška cesta 160, 2000 Maribor, Slovenia
sandi.klavzar@uni-mb.si

Bojan Mohar*
Department of Mathematics, University of Ljubljana
Jadranska 19, 1000 Ljubljana, Slovenia
bojan.mohar@fmf.uni-lj.si

Abstract

The crossing number of Sierpiński graphs $S(n,k)$ and their regularizations $S^+(n,k)$ and $S^{++}(n,k)$ is studied. Explicit drawings of these graphs are presented and proved to be optimal for $S^+(n,k)$ and $S^{++}(n,k)$ for every $n \geq 1$ and $k \geq 1$. These are the first nontrivial families of graphs of "fractal" type whose crossing number is known.

Key words: graph drawing, crossing number, Sierpiński graphs, graph automorphism

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1 Introduction

A *drawing* of a graph G is a pair of mappings $\varphi : V(G) \to \mathbb{R}^2$ and $\psi : E(G) \times [0,1] \to \mathbb{R}^2$ where φ is 1-1 and for each $e = uv \in E(G)$, the induced map $\psi_e : \{e\} \times [0,1] \to \mathbb{R}^2$ is a simple polygonal arc joining $\varphi(u)$ and $\varphi(v)$. It is required that the arc ψ_e is internally disjoint from $\varphi(V(G))$.

The *pair-crossing number*, pair-cr$(\mathcal{D})$, of a drawing $\mathcal{D} = (\varphi, \psi)$ is the number of crossing pairs of $\mathcal{D}$, where a *crossing pair* is an unordered pair $\{e, f\}$ of distinct edges for which there exist $s, t \in (0,1)$ such that $\psi(e,s) =$

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$\psi(f,t)$. The common point $\psi(e,s) = \psi(f,t)$ in $\mathbb{R}^2$ is said to be a *crossing point* of e and f and the pair $\{(e,s),(f,t)\}$ is referred to as a *crossing*. The total number of crossings of $\mathcal{D}$ is called the *crossing number* $\mathrm{cr}(\mathcal{D})$ of $\mathcal{D}$.

The *pair-crossing number*, pair-cr(G), of the graph G is the minimum pair-crossing number of all drawings of G, and the *crossing number*, $\mathrm{cr}(G)$, of G is the minimum crossing number of all drawings of G. It is an open question (see, e.g., [13]) if pair-cr$(G) = \mathrm{cr}(G)$ for every graph G. In this paper we shall restrict ourselves to $\mathrm{cr}(G)$ but all arguments work also for the pair-crossing number.

The exact value of the crossing number is known only for a few specific families of graphs. Such families include generalized Petersen graphs $P(N,3)$ [15], Cartesian products of all 5-vertex graphs with paths [7], and Cartesian products of two specific 5-vertex graphs with the star $K_{1,n}$ [8]. For the Cartesian products of cycles it is conjectured that $\mathrm{cr}(C_m \,\Box\, C_n) = (m-2)n$ for $3 \le m \le n$ and has recently been proved in [2] that for any fixed m the conjecture holds for all $n \ge m(m+1)$. The conjecture has also been verified for $m \le 7$ [1, 14]. Also, the crossing numbers of the complete bipartite graphs $K_{k,n}$ are known for every $k \le 6$ and arbitrary n. We refer to recent surveys [9, 12] for more details.

In this paper we study the crossing number of Sierpiński graphs $S(n,k)$ and their regularizations $S^+(n,k)$ and $S^{++}(n,k)$. They are defined in Section 2. In contrast to all families mentioned above, whose crossing number has been considered in the literature, graphs $S(n,k)$ do not have linear growth. Their number of vertices grows exponentially fast in terms of n, and they exhibit certain "fractal" behavior. Therefore, it seems rather interesting that their crossing number can be determined precisely, see Theorem 4.1.

Let us observe that crossing numbers of extended Sierpiński graphs $S^+(n,k)$ and $S^{++}(n,k)$ in Theorem 4.1 are expressed in terms of the crossing number of the complete graph K_{k+1}. It is known that $\mathrm{cr}(K_r) = 0$ for $r \le 4$, $\mathrm{cr}(K_5) = 1$, $\mathrm{cr}(K_6) = 3$, $\mathrm{cr}(K_7) = 9$, $\mathrm{cr}(K_8) = 18$, $\mathrm{cr}(K_9) = 36$, and $\mathrm{cr}(K_{10}) = 60$. Values of $\mathrm{cr}(K_r)$ for $r \ge 11$ are not known.

2 Sierpiński graphs and their regularizations

Sierpiński graphs $S(n,k)$ were introduced in [5], where it is in particular shown that the graph $S(n,3)$, $n \ge 1$, is isomorphic to the graph of the Tower of Hanoi with n disks. For more results on these graphs see [3, 6]. The definition of the graphs $S(n,k)$ was motivated by topological studies of the Lipscomb's space which generalizes the Sierpiński triangular curve

(Sierpiński gasket), cf. [10, 11].

The *Sierpiński graph* $S(n,k)$ $(n,k \geq 1)$ is defined on the vertex set $\{1,\dots,k\}^n$, two different vertices $u = (u_1,\dots,u_n)$ and $v = (v_1,\dots,v_n)$ being adjacent if and only if there exists an $h \in \{1,\dots,n\}$ such that

(i) $u_t = v_t$, for $t = 1,\dots,h-1$;
(ii) $u_h \neq v_h$; and
(iii) $u_t = v_h$ and $v_t = u_h$ for $t = h+1,\dots,n$.

In the rest we will shortly write $\langle u_1 u_2 \dots u_n \rangle$ for $(u_1, u_2, \dots, u_n)$.

A vertex of the form $\langle ii \dots i \rangle$ of $S(n,k)$ is called an *extreme vertex*. The extreme vertices of $S(n,k)$ are of degree $k-1$ while the degree of any other vertex is k. Note also that in $S(n,k)$ there are k extreme vertices and that $|S(n,k)| = k^n$.

Let $n \geq 2$, then for $i = 1,\dots,k$, let $S^i(n-1,k)$ be the subgraph of $S(n,k)$ induced by the vertices of the form $\langle i v_2 v_3 \dots v_n \rangle$. Note that $S^i(n-1,k)$ is isomorphic to $S(n-1,k)$.

Let

$$\rho_{i,j} = \begin{cases} 1; & i \neq j, \\ 0; & i = j, \end{cases}$$

and set in addition

$$\mathcal{P}^i_{j_1 j_2 \dots j_m} = \rho_{i,j_1} \rho_{i,j_2} \dots \rho_{i,j_m (2)} \,,$$

where the right-hand side term is a binary number, rhos representing its digits. Then we have [5]:

Proposition 2.1 *Let $\langle u_1 u_2 \dots u_n \rangle$ be a vertex of $S(n,k)$. Then its distance in $S(n,k)$ from the extreme vertex $\langle ii \dots i \rangle$ is equal to:*

$$d_{S(n,k)}(\langle u_1 u_2 \dots u_n \rangle, \langle ii \dots i \rangle) = \mathcal{P}^i_{u_1 u_2 \dots u_n} \,.$$

In the rest, in particular when introducing regularizations of the Sierpiński graphs, the following lemma will be useful.

Lemma 2.2 *For any $n \geq 1$ and any $k \geq 1$, $Aut(S(n,k))$ is isomorphic to $Sym(k)$, where $Aut(S(n,k))$ acts as $Sym(k)$ on the extreme vertices of $S(n,k)$.*

Proof. Let $\varphi \in Aut(S(n,k))$. Then the degree condition implies that φ permutes the k extreme vertices of $S(n,k)$. Let $f(\varphi) \in Sym(k)$ be the corresponding permutation. We claim that $f : Aut(S(n,k)) \to Sym(k)$ is a bijection.

We first show that f is surjective. So let $\pi \in Sym(k)$ and define $\varphi : V(S(n,k)) \to V(S(n,k))$ with $\varphi(\langle i_1 i_2 \ldots i_n \rangle) = \langle \pi(i_1)\pi(i_2)\ldots\pi(i_n)\rangle$. Clearly, φ is 1-1. Let $u, v \in V(S(n,k))$. Then u is adjacent to v if and only if $u = \langle i_1 i_2 \ldots i_{k-1} rs \ldots s\rangle$ and $v = \langle i_1 i_2 \ldots i_{k-1} sr \ldots r\rangle$, where $k \in \{1, \ldots, n\}$ and $r \neq s$. But this is if and only if $\varphi(u) = \langle \varphi(i_1) \ldots \varphi(i_{k-1})\varphi(r)\varphi(s) \ldots \varphi(s)\rangle$ is adjacent to $\varphi(v) = \langle \varphi(i_1) \ldots \varphi(i_{k-1})\varphi(s)\varphi(r) \ldots \varphi(r)\rangle$ because $\varphi(r) \neq \varphi(s)$. Hence $\varphi \in Aut(S(n,k))$ and, clearly, maps extreme vertices onto extreme vertices.

To show injectivity we are going to prove that given $\varphi \in Aut(S(n,k))$, φ is the unique automorphism with the image $f(\varphi)$. Let $u = \langle i_1 i_2 \ldots i_n\rangle$ be an arbitrary vertex of $S(n,k)$ and set

$$D(u) = (\, d(u, \langle 11 \ldots 1\rangle), \ldots, d(u, \langle kk \ldots k\rangle)\,)$$

be its vector of distances from the extreme vertices. Since φ is an automorphism that maps extreme vertices onto extreme vertices,

$$D(\varphi(u)) = (\, d(\varphi(u), \varphi(\langle 11 \ldots 1\rangle)), \ldots, d(\varphi(u), \varphi(\langle kk \ldots k\rangle))\,)\,.$$

Moreover, Proposition 2.1 implies that if $u \neq v$ then $D(u) \neq D(v)$. Hence $\varphi(u)$ is uniquely determined, so there is a unique automorphism (namely φ) with the image $f(\varphi)$. $\square$

We now introduce the extended Sierpiński graphs $S^+(n,k)$ and $S^{++}(n,k)$. The graph $S^+(n,k)$, $n \geq 1$, $k \geq 1$, is obtained from $S(n,k)$ by adding a new vertex w, called the *special vertex* of $S^+(n,k)$, and all edges joining w with extreme vertices of $S(n,k)$. The graphs $S^{++}(n,k)$, $n \geq 1$, $k \geq 1$, are defined as follows. For $n = 1$ we set $S^{++}(1,k) = K_{k+1}$. Suppose now that $n \geq 2$. Then $S^{++}(n,k)$ is the graph obtained from the disjoint union of $k+1$ copies of $S(n-1,k)$ in which the extreme vertices in distinct copies of $S(n-1,k)$ are connected as the complete graph K_{k+1}. By Lemma 2.2, this construction defines a unique graph. Fig. 1 shows graphs $S(2,4)$, $S^+(2,4)$, and $S^{++}(2,4)$.

$S^+(n,k)$ is a k-regular graph on k^n+1 vertices; in particular, $S^+(1,k) = K_{k+1}$. Note also that $S^{++}(n,k)$ is a k-regular graph on $k^{n-1}(k+1)$ vertices that can also be described as the graph obtained from the disjoint union of a copy of $S(n,k)$ and a copy of $S(n-1,k)$ such that the extreme vertices of $S(n,k)$ and the extreme vertices of $S(n-1,k)$ are connected by a matching.

For a fixed k we will write S_n, S_n^+, and S_n^{++} for $S(n,k)$, $S^+(n,k)$, and $S^{++}(n,k)$, respectively. Also, $S^i(n-1,k)$ will be denoted by S_{n-1}^i. The

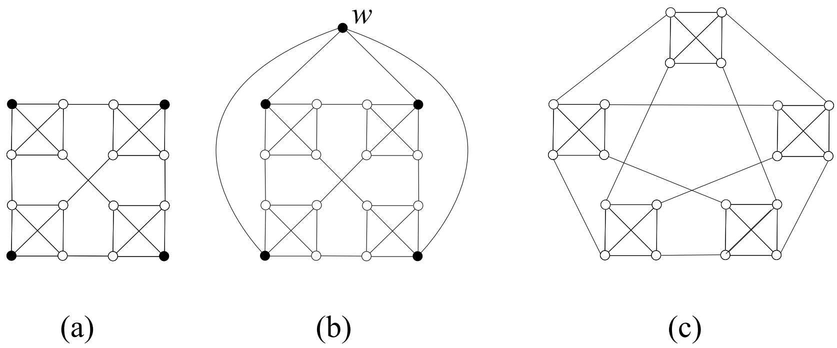

Figure 1: Graphs (a) $S(2,4)$, (b) $S^{+}(2,4)$, and (c) $S^{++}(2,4)$

graph S_n^+ consists of k disjoint copies $S_{n-1}^1, \ldots, S_{n-1}^k$ of S_{n-1} and an additional vertex w. Let $e_{ij} = e_{ji}$ be the edge joining S_{n-1}^i and S_{n-1}^j, where $i \neq j$, and $e_{i0} = e_{0i}$ the edge joining S_{n-1}^i and w, so that

$$V(S_n^+) = \bigcup_{i=1}^{k} V(S_{n-1}^i) \ \cup \ \{w\}, \quad \text{and}$$

$$E(S_n^+) = \bigcup_{i=1}^{k} E(S_{n-1}^i) \ \cup \ \{e_{ij} \mid 0 \leq i < j \leq k\}\,.$$

Similar notation is also used for S_n^{++} where w is replaced by S_{n-1}^0.

Lemma 2.3 *For any fixed $k \geq 3$ and every $n \geq 2$, we have $Aut(S_n^+) \approx Sym(k)$ and $Aut(S_n^{++}) \approx Sym(k+1)$, where $Aut(S_n^+)$ and $Aut(S_n^{++})$ act as $Sym(k)$ on the extreme vertices of the subgraph S_n in S_n^+ and S_n^{++}, respectively.*

Proof. Let $\pi \in Sym(k)$ and let φ be the unique automorphism of S_n that acts on the extreme vertices of $S(n,k)$ as π, cf. Lemma 2.2. Then we can extend φ to an automorphism of S_n^+, resp. S_n^{++} by fixing f on the special vertex w of S_n^+, resp. the special copy S_{n-1}^0 of S_{n-1}.

The blocks of the action of the automorphism group on the vertex set are vertex sets of subgraphs S_{n-1}^i, and any permutation of these blocks defines a unique automorphism of S_n^+ (or S_n^{++}). This easily implies the statement of the lemma. $\square$

3 Drawings of S_n^+, S_n^{++}, and S_n

For $k \leq 3$ and every $n \geq 1$, Sierpiński graphs $S(n, k)$ and their regularizations $S^+(n, k)$ and $S^{++}(n, k)$ are planar, cf. [4]. We now describe explicit drawings of these graphs for any $k \geq 4$. Our results are given in terms of the crossing number of complete graphs, we refer to [16] for more information on $\mathrm{cr}(K_k)$, $k \in \mathbb{N}$.

Lemma 3.1 *For every $k \geq 4$ and $n \geq 1$ we have:*

(i) $\mathrm{cr}(S_n^+) \leq k \cdot \mathrm{cr}(S_{n-1}^+) + \mathrm{cr}(K_{k+1}) \leq \dfrac{k^n - 1}{k - 1}\ \mathrm{cr}(K_{k+1})\,.$

(ii) $\mathrm{cr}(S_n^{++}) \leq \mathrm{cr}(S_n^+) + \mathrm{cr}(S_{n-1}^+) \leq \dfrac{(k+1)k^{n-1} - 2}{k - 1}\ \mathrm{cr}(K_{k+1})\,.$

Proof. (i) For $n = 1$ the graph S_n^+ is K_{k+1}, so it can be (optimally) drawn with $\mathrm{cr}(K_{k+1})$ crossings. For $n \geq 2$ we draw the graph S_n^+ inductively as follows. First take an optimal drawing of S_{n-1}^+. We may assume that the special vertex of S_{n-1}^+ is on the unbounded face of this drawing. It we "erase" a small neighborhood of the special vertex in this drawing, we obtain a drawing $\mathcal{D}'$ of S_{n-1} together with pendant edges incident with all extreme vertices and all sticking out to the infinite face. Clearly, $\mathcal{D}'$ has $\mathrm{cr}(S_{n-1}^+)$ crossings. We now take an optimal drawing of K_{k+1} (with $\mathrm{cr}(K_{k+1})$ crossings). Select an arbitrary vertex w of this drawing to represent the special vertex of S_n^+. Around every remaining vertex v of K_{k+1} select small enough disk Δ_v so that only drawings of edges incident with v intersect Δ_v. For each of such edges uv, follow its drawing from u towards v until Δ_v is reached for the first time, and then erase the rest of the drawing of this edge. Now, add the drawing $\mathcal{D}'$ inside Δ_v and connect its pending edges with the points on $\partial\Delta_v$ where arcs coming from the outside have been stopped. By Lemma 2.3, the resulting drawing is a drawing $\mathcal{D}$ of S_n^+. Clearly, $\mathrm{cr}(\mathcal{D}) = k \cdot \mathrm{cr}(\mathcal{D}') + \mathrm{cr}(K_{k+1}) = k \cdot \mathrm{cr}(S_{n-1}^+) + \mathrm{cr}(K_{k+1})$. This implies the first inequality in (i). The second inequality easily follows by induction.

The same construction in which also the special vertex is replaced by a drawing of S_{n-1}^+ shows (ii). □

In the next section we will prove that the drawings described above are optimal for every $k \geq 4$.

The construction from the proof of Lemma 3.1 can also be used for the graphs S_n with a modification that an optimal drawing of K_k is used instead of K_{k+1}. In this way we obtain, using Lemma 3.1(i),

$$\mathrm{cr}(S_n) \leq k \cdot \mathrm{cr}(S^+_{n-1}) + \mathrm{cr}(K_k) \leq \frac{k(k^{n-1}-1)}{k-1}\ \mathrm{cr}(K_{k+1}) + \mathrm{cr}(K_k)\,. \quad (1)$$

However, in contrast to the optimality of the construction for S^+_n and S^{++}_n, these drawings for the graphs S_n are not always optimal, as shown for $k = 4$ by the following proposition whose upper bound is strictly smaller than the one in (1).

Proposition 3.2 *For any $n \geq 3$,*

$$\frac{3}{16}\,4^n \leq \mathrm{cr}(S(n,4)) \leq \frac{1}{3}\,4^n - \frac{12n-8}{3}\,.$$

Proof. Let $k = 4$ and consider the drawings of $S(2,4)$ and $S(3,4)$ as shown in Fig. 2.

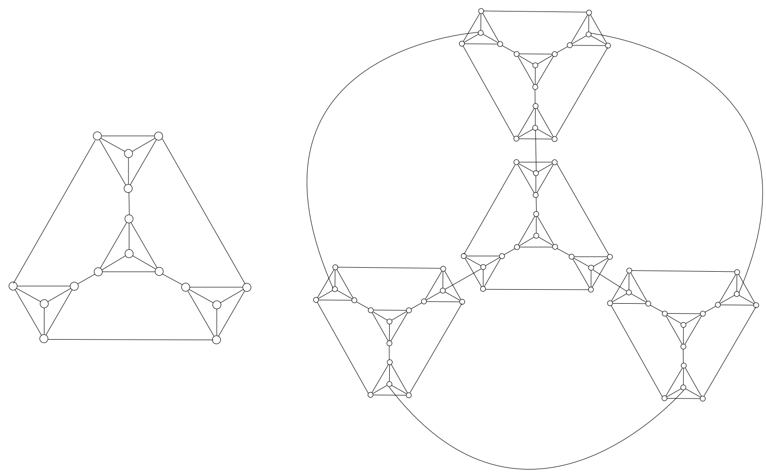

Figure 2: Drawings of $S(2,4)$ and $S(3,4)$

For $n \geq 4$ we inductively construct a drawing of $S(n,4)$ from four copies of $S(n-1,4)$ analogously as the drawing of $S(3,4)$ is obtained from the drawing of $S(2,4)$. Let a_n be the number of crossings of this drawing of $S(n,4)$. Then $a_2 = 0$ and $a_n = 4a_{n-1} + 12(n-2)$, $n \geq 3$ with the solution $a_n = 4^n/3 - 4n + 8/3$.

On the other hand, we are going to show that $\mathrm{cr}(S(3,4)) = 12$. Since $\mathrm{cr}(S(n+1,4)) \geq 4\,\mathrm{cr}(S(n,4))$, this will prove the lower bound.

Let $\mathcal{D}$ be a drawing of $S = S(3,4)$. The graph S contains 16 disjoint copies of K_4, twelve of which do not contain extreme vertices of S. Let $L_1, \ldots, L_{12}$ be these subgraphs. For $i = 1, \ldots, 12$ we define a subgraph L_i^+ such that the following conditions are satisfied:

(a) $L_i \subseteq L_i^+$.

(b) L_i^+ is a nonplanar graph and in the drawing of L_i^+, there is a crossing $C_i = \{(e_i, s_i), (f_i, t_i)\}$ involving an edge $e_i \in E(L_i)$.

(c) If $j < i$, then $e_j \notin E(L_i^+)$.

The graphs L_i^+ can be obtained as follows. Suppose that $L_1^+, \ldots, L_{i-1}^+$ have already been defined and that their edges $e_1, \ldots, e_{i-1}$ have been selected. The graph $S' = S - \{e_1, \ldots, e_{i-1}\}$ is connected. If we contract all edges in $S' - L_i$, the resulting graph is isomorphic to K_5. This implies that S' contains a subgraph L_i^+ that is either homeomorphic to K_5 or to the graph obtained from K_5 by splitting one of its vertices into a pair of adjacent vertices x, y, each of which is adjacent to two vertices of the 4-clique L_i. Clearly, L_i^+ satisfies (a) and (c), and we leave it to the reader to verify (b).

Condition (c) implies that all crossings C_i are distinct. This shows that $\mathrm{cr}(\mathcal{D}) \geq 12$ and completes the proof. □

4 Lower bounds

In this section we prove the main result of this paper. As before, we shall consider $k \geq 2$ as being fixed and will omit it from the notation of S_n, S_n^+ and S_n^{++}.

Theorem 4.1 *For any fixed $k \geq 2$ and every $n \geq 1$ we have:*

(i) $\mathrm{cr}(S_n^+) = \dfrac{k^n - 1}{k-1}\ \mathrm{cr}(K_{k+1})$.

(ii) $\mathrm{cr}(S_n^{++}) = \dfrac{(k+1)k^{n-1} - 2}{k-1}\ \mathrm{cr}(K_{k+1})$.

By Lemma 3.1 we only have to prove that the values on the right hand side of (i) and (ii) are lower bounds for the crossing number. The proof is deferred to the end of this section.

Below we will introduce some notation that applies for all three families of graphs, S_n, S_n^+ and S_n^{++}. Recall that for every i and $j \neq i$, there is precisely one edge, denoted by $e_{ij} = e_{ji}$ connecting S^i_{n-1} with S^j_{n-1}. The edge e_{ij} is incident with an extreme vertex of S^i_{n-1} and this vertex is called the j^{th} extreme vertex of S^i_{n-1} and denoted by z^i_j. With this notation, $e_{ij} = e_{ji} = z^i_j z^j_i$. We also consider S_n as a subgraph of S^+_n, and then the extreme vertices of S_n are precisely the vertices z^i_0, $1 \le i \le k$.

Lemma 4.2 *For every $n \ge 1$, S_n contains a subdivision of the complete graph K_k in which vertices of degree $k-1$ are precisely the extreme vertices of S_n.*

Proof. The proof is by induction on n. The claim is trivially true for $S_1 = K_k$. For $n \ge 2$, the subdivision of K_k in S_n is obtained by taking the union of all edges e_{ij} $(1 \le i < j \le k)$ and all paths in subdivision cliques in S^i_{n-1} joining the extreme vertex z^i_0 with z^i_j, $j \notin \{0, i\}$, $i = 1, \ldots, k$. □

In what follows, we fix a subdivision of K_k in every S^i_{n-1} and denote by $P^i_{j\ell}$ the path in this subdivision joining the extreme vertices z^i_j and z^i_ℓ.

Lemma 4.3 *Let $k \ge 3$ and $n \ge 1$. If $\tau_0, \ldots, \tau_k$ are integers such that $\tau_i \in \{0, \ldots, k\} \setminus \{i\}$ for $i = 0, \ldots, k$, then S^{++}_n contains a subgraph $K(\tau_0, \ldots, \tau_k)$ which is isomorphic to a subdivision of the complete graph K_{k+1} in which vertices of degree k are precisely the vertices $z^i_{\tau_i}$, $i = 0, \ldots, k$.*

There are k^{k+1} distinct choices for parameters $\tau_0, \ldots, \tau_k$; they give rise to k^{k+1} distinct subdivisions of K_{k+1}. Every edge e_{ij} is in every such subdivision. Every edge contained in some path P^i_{jl} is in precisely $2k^k$ of them, while other edges of S^{++}_n are in none.

Proof. The subgraph $K(\tau_0, \ldots, \tau_k)$ consists of all paths $R_{ij} = P^i_{\tau_i j} \cup \{e_{ij}\} \cup P^j_{i\tau_j}$, $0 \le i < j \le k$.

The claims in the second part of the lemma are easy to verify. Let us just observe that the path P^i_{jl} is in $K(\tau_0, \ldots, \tau_k)$ if and only if $\tau_i = j$ or $\tau_i = l$. □

Let $\mathcal{D}$ be a drawing. For subdrawings $\mathcal{K}, \mathcal{L}$ of $\mathcal{D}$, let $\operatorname{cr}(\mathcal{K}, \mathcal{L})$ be the number of crossings involving an edge of $\mathcal{K}$ and an edge of $\mathcal{L}$. We write $\operatorname{cr}(\mathcal{K}) = \operatorname{cr}(\mathcal{K}, \mathcal{K})$. We also allow $\mathcal{K}$ or $\mathcal{L}$ be a subgraph of $S = S_n$, S^+_n or S^{++}_n, in which case $\operatorname{cr}(\mathcal{K}, \mathcal{L})$ refers to their drawings under $\mathcal{D}$.

Two drawings $\mathcal{D}$ and $\mathcal{D}'$ are said to be *isomorphic* if there is a homeomorphism of the extended plane (the plane plus the point at the infinity, which is homeomorphic to the 2-sphere) mapping $\mathcal{D}$ onto $\mathcal{D}'$.

Let $\mathcal{D}$ be a drawing of $S = S_n$, S_n^+ or S_n^{++}. For every $S_{n-1}^i \subseteq S$, let $\mathcal{D}_i$ be the induced drawing of S_{n-1}^i.

Lemma 4.4 *Let $k \geq 4$ be an integer. Let $\mathcal{D}$ be a drawing of S_n^{++}. Then there is a drawing $\mathcal{D}'$ of S_n^{++} such that:*

(a) *For every $i = 0, \ldots, k$, the subdrawings $\mathcal{D}_i$ and $\mathcal{D}'_i$ of S_{n-1}^i in $\mathcal{D}$ and $\mathcal{D}'$, respectively, are isomorphic.*

(b) *For $i \neq j$, $\mathrm{cr}(\mathcal{D}'_i, \mathcal{D}'_j) = 0$ and $\mathcal{D}'_j$ is contained in the unbounded face of $\mathcal{D}'_i$.*

(c) $\mathrm{cr}(\mathcal{D}') \leq \mathrm{cr}(\mathcal{D})$.

Proof. For $i \in \{0, \ldots, k\}$, the graph $B_i = S_n^{++} - S_{n-1}^i$ is isomorphic to S_n. For every extreme vertex z_ℓ^j of S_{n-1}^j in $B_i \approx S_n$, let $Z_{j\ell}^i$ be the subgraph consisting of k (or $k-1$ if $\ell = i$) internally disjoint paths $R_m = P_{\ell m}^j \cup \{e_{jm}\} \cup P_{ji}^m$ ($m \notin \{i, j\}$) and $R_i = P_{\ell i}^j$. Finally, let $W_{j\ell}^i$ be the subgraph of S_n^{++} which is the union of S_{n-1}^i, $Z_{j\ell}^i$, and all edges e_{im} ($m \neq i$) joining S_{n-1}^i with $Z_{j\ell}^i$. Then $W_{j\ell}^i$ is isomorphic to a subdivision of the graph S_{n-1}^+ in which z_ℓ^j plays the role of the special vertex in S_{n-1}^+. Among all such subgraphs $W_{j\ell}^i$ ($j \neq \ell, i \notin \{j, \ell\}$), let S_{n-1}^{i+} be one whose induced drawing has minimum number of crossings in $\mathcal{D}$. Let $\mathcal{D}_i^+$ be a drawing isomorphic to the induced drawing of S_{n-1}^{i+} such that the special vertex is on the outer face of the drawing.

Drawings $\mathcal{D}_0^+, \ldots, \mathcal{D}_k^+$ can be combined (as explained in the proof of Lemma 3.1) so that a drawing $\mathcal{D}'$ of S_n^{++} satisfying (a) and (b) is obtained and such that

$$\mathrm{cr}(\mathcal{D}') = \sum_{i=0}^{k} \mathrm{cr}(\mathcal{D}_i^+) + \mathrm{cr}(K_{k+1})\,. \tag{2}$$

We introduce the following notation, where all crossing numbers are taken with respect to the drawing $\mathcal{D}$:

$$\begin{aligned}
c_i &:= \mathrm{cr}(\mathcal{D}_i), \\
c_{ij} &:= \mathrm{cr}(\mathcal{D}_i, \mathcal{D}_j), \\
f_i &:= \mathrm{cr}(\mathcal{D}_i, F_i), \text{ where } F_i = \{e_{ij} \mid j \neq i\}, \\
\overline{f_i} &:= \mathrm{cr}(\mathcal{D}_i, \overline{F_i}), \text{ where } \overline{F_i} = \{e_{j\ell} \mid j \neq \ell, i \notin \{j, \ell\}\}, \\
f_{ij} &:= \mathrm{cr}(F_i \setminus \{e_{ij}\}, F_j \setminus \{e_{ji}\}).
\end{aligned}$$

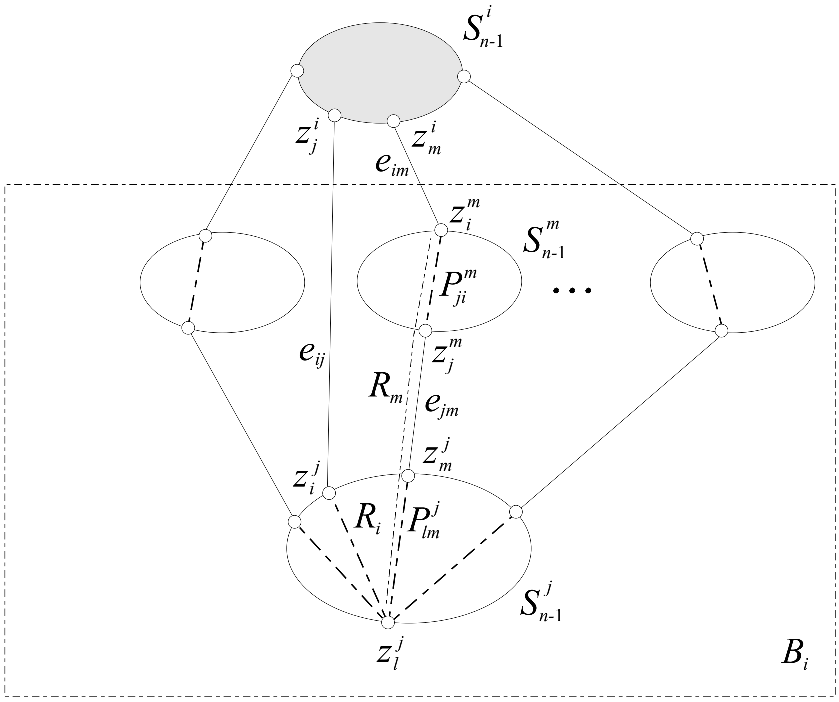

Figure 3: Subgraph $W^i_{j\ell}$

Clearly,

$$\mathrm{cr}(\mathcal{D}) \geq \sum_{i=0}^{k} c_i + \sum_{i=0}^{k} f_i + \sum_{i=0}^{k} \overline{f_i} + \frac{1}{2} \sum_{i=0}^{k} \sum_{j \neq i} (c_{ij} + f_{ij}) \,. \tag{3}$$

Next, let $c_i^+ = \mathrm{cr}(\mathcal{D}_i^+)$ (where the number of crossings is counted with respect to the drawing $\mathcal{D}'$). Then:

$$\begin{aligned} c_i^+ &= c_i + f_i + \min\{\mathrm{cr}(\mathcal{D}_i, Z^i_{j\ell}) \mid j \neq i, \ell \neq j\} \\ &\leq c_i + f_i + \frac{1}{k^2} \sum_{j \neq i} \sum_{\ell \neq j} \mathrm{cr}(\mathcal{D}_i, Z^i_{j\ell}) & (4) \\ &\leq c_i + f_i + \frac{1}{k^2} \Big(\sum_{j \neq i} (k+2) c_{ij} + 2k\overline{f_i} \Big) \,. & (5) \end{aligned}$$

Inequality (4) holds since the minimum is always smaller or equal to the average, while (5) follows from the observation that an edge of S^j_{n-1} $(j \neq i)$

is in at most 2 subgraphs Z^i_{jl} and in at most k subgraphs Z^i_{ml} ($m \notin \{i,j\}$), while an edge e_{jm} ($i \notin \{j,m\}$) belongs to precisely $2k$ subgraphs Z^i_{jl} and Z^i_{ml}.

Combining (2)–(5), we get

$$\begin{aligned}
\operatorname{cr}(\mathcal{D}) - \operatorname{cr}(\mathcal{D}') + \operatorname{cr}(K_{k+1}) &= \operatorname{cr}(\mathcal{D}) - \sum_{i=0}^{k} c_i^+ \\
&\geq \sum_{i=0}^{k} \overline{f_i} + \sum_{i=0}^{k}\sum_{j\neq i}\Big(\frac{1}{2} - \frac{k+2}{k^2}\Big)c_{ij} - \frac{2}{k}\sum_{i=0}^{k}\overline{f_i} + \frac{1}{2}\sum_{i=0}^{k}\sum_{j\neq i} f_{ij} \\
&= \frac{k-2}{k}\sum_{i=0}^{k}\overline{f_i} + \frac{k^2-2k-4}{2k^2}\sum_{i=0}^{k}\sum_{j\neq i} c_{ij} + \frac{1}{2}\sum_{i=0}^{k}\sum_{j\neq i} f_{ij}\,. \qquad (6)
\end{aligned}$$

For $k = 4$ we have

$$\operatorname{cr}(\mathcal{D}) - \operatorname{cr}(\mathcal{D}') \geq \frac{1}{2}\sum_{i=0}^{4}\overline{f_i} + \frac{1}{8}\sum_{i=0}^{4}\sum_{j\neq i} c_{ij} - 1\,. \qquad (7)$$

If $c_{ij} = 0$ for every i and j, and every $\overline{f_i} = 0$, then a drawing isomorphic to $\mathcal{D}$ satisfies (a)–(c). Otherwise, (7) implies that $\operatorname{cr}(\mathcal{D}) - \operatorname{cr}(\mathcal{D}') > -1$. Since the left hand side is an integer, this implies that $\operatorname{cr}(\mathcal{D}') \leq \operatorname{cr}(\mathcal{D})$. This completes the proof for $k = 4$.

Suppose now that $k \geq 5$. By (6), it remains to see that

$$\frac{1}{2}\sum_{i=0}^{k}\sum_{j\neq i} f_{ij} + \frac{k-2}{k}\sum_{i=0}^{k}\overline{f_i} + \frac{k^2-2k-4}{2k^2}\sum_{i=0}^{k}\sum_{j\neq i} c_{ij} \geq r \qquad (8)$$

where $r = \operatorname{cr}(K_{k+1})$.

Let us consider all k^{k+1} subgraphs $K(\tau_0,\ldots,\tau_k)$ of S_n^{++} isomorphic to subdivisions of K_{k+1}; see Lemma 4.3. A crossing (in $\mathcal{D}$) of two edges of $K(\tau_0,\ldots,\tau_k)$ is said to be *pure* if the two edges lie on subdivided edges of K_{k+1} that are not incident in K_{k+1}. Any drawing of $K(\tau_0,\ldots,\tau_k)$ has at least r pure crossings.

Let $C = \{(e,s),(f,t)\}$ be a crossing in $\mathcal{D}$. Let us estimate the maximum number of subgraphs $K(\tau_0,\ldots,\tau_k)$ in which C is a pure crossing.

(i) If $e \in F_i \setminus \{e_{ij}\}$ and $f \in F_j \setminus \{e_{ji}\}$, where $i \neq j$, then C is a pure crossing in at most k^{k+1} subgraphs $K(\tau_0,\ldots,\tau_k)$.

(ii) If $e \in E(S^i_{n-1})$ and $f = e_{jl}$, where $i \notin \{j, l\}$, then by Lemma 4.3, C can be a pure crossing in at most $2k^k$ subgraphs $K(\tau_0, \ldots, \tau_k)$.

(iii) If $e \in E(S^i_{n-1})$ and $f \in E(S^j_{n-1})$, where $i \neq j$, then C can be a pure crossing of $K(\tau_0, \ldots, \tau_k)$ only when $e \in E(P^i_{ab})$, $f \in E(P^j_{cd})$, $\tau_i \in \{a, b\}$, and $\tau_j \in \{c, d\}$. So, $4k^{k-1}$ is an upper bound for the number of such cases.

The bounds derived in (i)–(iii) imply that

$$k^{k+1} r \leq k^{k+1} \frac{1}{2} \sum_{i=0}^{k} \sum_{j \neq i} f_{ij} + 2k^k \sum_{i=0}^{k} \overline{f_i} + 4k^{k-1} \sum_{i=0}^{k} \sum_{j \neq i} c_{ij}. \tag{9}$$

Clearly, $2/k \leq (k-2)/k$ (for $k \geq 4$) and $4/k^2 \leq (k^2 - 2k - 4)/(2k^2)$ (for $k \geq 5$). Therefore (9) implies (8). The proof is complete. □

Inequalities used at the very last step of the above proof are strict for $k \geq 5$. If either some $\overline{f_i} \neq 0$ or some $c_{ij} \neq 0$, this would imply that the lower bound would be strictly greater than the upper bound of Lemma 4.4 (if $\mathcal{D}$ is an optimal drawing). This implies that every optimal drawing of S^{++}_n (for $k \geq 5$) satisfies the condition stated for $\mathcal{D}'$ in Lemma 4.4(b).

Proof of Theorem 4.1. We may assume that $k \geq 4$. By Lemma 3.1 we only have to prove that the values in (i) and (ii) are lower bounds for the crossing number. The proof is by induction on n. The case when $n = 1$ is trivial, so we assume that $n \geq 2$.

By Lemma 4.4, there is an optimal drawing $\mathcal{D}$ of S^{++}_n such that condition (b) of the lemma holds for its subdrawings $\mathcal{D}_0, \ldots, \mathcal{D}_k$. In other words, $\mathrm{cr}(\mathcal{D}_i, \mathcal{D}_j) = 0$ and $\mathcal{D}_j$ is in the unbounded face of $\mathcal{D}_i$ for every $i \neq j$. This implies that

$$\mathrm{cr}(\mathcal{D}) \geq \sum_{i=0}^{k} \mathrm{cr}(\mathcal{D}^+_i) + \mathrm{cr}(K_{k+1}) \geq (k+1)\,\mathrm{cr}(S^+_{n-1}) + \mathrm{cr}(K_{k+1}). \tag{10}$$

By the induction hypothesis for (i), $\mathrm{cr}(S^+_{n-1}) \geq \frac{k^{n-1}-1}{k-1}\,\mathrm{cr}(K_{k+1})$, so (10) implies (ii).

To prove (i), suppose that there is a drawing $\mathcal{D}'$ of S^+_n with $\mathrm{cr}(\mathcal{D}') < \frac{k^n-1}{k-1}\,\mathrm{cr}(K_{k+1})$. As in the proof of Lemma 3.1 we see that $\mathcal{D}'$ and a drawing of S^{+}_{n-1} can be combined in such a way as to get a drawing $\mathcal{D}$ of S^{++}_{n} with

$$\begin{aligned}\mathrm{cr}(\mathcal{D}) &= \mathrm{cr}(\mathcal{D}') + \mathrm{cr}(S^{+}_{n-1}) \\ &< \frac{k^n - 1}{k-1}\,\mathrm{cr}(K_{k+1}) + \frac{k^{n-1}-1}{k-1}\,\mathrm{cr}(K_{k+1}) \\ &= \frac{(k+1)k^{n-1}-2}{k-1}\,\mathrm{cr}(K_{k+1}).\end{aligned}$$

This is a contradiction to the already proved equality in (ii). □

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