Approximating the List-Chromatic Number and the Chromatic Number in Minor-Closed and Odd-Minor-Closed Classes of Graphs

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ABSTRACT

It is well-known (Feige and Kilian [24], Håstad [39]) that approximating the chromatic number within a factor of $n^{1-\varepsilon}$ cannot be done in polynomial time for $\varepsilon > 0$, unless coRP $=$ NP. Computing the list-chromatic number is much harder than determining the chromatic number. It is known that the problem of deciding if the list-chromatic number is k, where $k \geq 3$, is Π_2^p-complete [37].

In this paper, we focus on minor-closed and odd-minor-closed families of graphs. In doing that, we may as well consider only graphs without K_k-minors and graphs without odd K_k-minors for a fixed value of k, respectively. Our main results are that there is a polynomial time approximation algorithm for the list-chromatic number of graphs without K_k-minors and there is a polynomial time approximation algorithm for the chromatic number of graphs without odd-K_k-minors. Their time complexity is $O(n^3)$ and $O(n^4)$, respectively. The algorithms have multiplicative error $O(\sqrt{\log k})$ and additive error $O(k)$, and the multiplicative error occurs only for graphs whose list-chromatic number and chromatic number are $\Theta(k)$, respectively.

Let us recall that H has an odd complete minor of order l if there are l vertex disjoint trees in H such that every two of them are joined by an edge, and in addition, all the vertices of trees are two-colored in such a way that the edges within the trees are bichromatic, but the edges between trees are monochromatic. Let us observe that the complete bipartite graph $K_{n/2,n/2}$ contains a K_k-minor for $k \leq n/2$, but on the other hand, it does not contain an odd K_k-minor for any $k \geq 3$. Odd K_5-minor-free graphs are closely related to one field of discrete optimization which is finding conditions under which a given polyhedron has integer vertices, so that integer optimization problems can be solved as linear programs. See [33, 34, 64]. Also, the odd version of the well-known Hadwiger's conjecture has been considered, see [28].

Our main idea involves precoloring extension. This idea is used in many results; one example is Thomassen's proof on his celebrated theorem on planar graphs [69].

The best previously known approximation for the first result is a simple $O(k\sqrt{\log k})$-approximation following algorithm that guarantees a list-coloring with $O(k\sqrt{\log k})$ colors for K_k-minor-free graphs. This follows from results of Kostochka [54, 53] and Thomason [67, 68].

The best previous approximation for the second result comes from the recent result of Geelen et al. [28] who gave an $O(k\sqrt{\log k})$-approximation algorithm.

We also relate our algorithm to the well-known conjecture of Hadwiger [38] and its odd version. In fact, we give an $O(n^3)$ algorithm to decide whether or not a weaker version of Hadwiger's conjecture is true. Here, by a weaker version of Hadwiger's conjecture, we mean a conjecture which says that any $27k$-chromatic graph contains a K_k-minor. Also, we shall give an $O(n^{2500k})$ algorithm for deciding whether or not any $2500k$-chromatic graph contains an odd-K_k-minor.

Let us mention that this presentation consists of two papers which are merged into this one. The first one consists of results concerning minor-closed classes of graphs by two current authors, and the other consists of results concerning odd-minor-closed classes of graphs by the first author.

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1. INTRODUCTION

1.1 History and background

Graph coloring is one of the central subjects in both Discrete Mathematics and Theoretical Computer Science. Graph coloring is a mapping from $V(G)$ to a set of colors C so that any two adjacent vertices of G receive different colors. The chromatic number $\chi(G)$ is the minimum number of colors for which there exists a coloring of G. The book by Jensen and Toft [41] is an excellent reference about the current research related to colorings of graphs.

Also, it is one of the central problems in combinatorial optimization, and it is one of the hardest problems to approximate. In general, the chromatic number is inapproximable in polynomial time within factor $n^{1-\epsilon}$ for any $\epsilon > 0$, unless $coRP = NP$, cf. Feige and Kilian [24] and Håstad [39]. Even for 3-colorable graphs, the best known polynomial approximation algorithm achieves a factor of $O(n^{3/14} \log^{O(1)} n)$ in [6]. An interesting variant of the classical problem of properly coloring the vertices of a graph with the minimum possible number of colors arises when one imposes some restrictions on the colors or the number of colors available to particular vertices. This variant received a considerable amount of attention by many researchers, and that led to several beautiful conjectures and results. This subject, known as *list-coloring*, was first introduced in the second half of the 1970s, in two papers by Vizing [72] and independently by Erdős, Rubin and Taylor [23].

If $G = (V, E)$ is a graph, and f is a function that assigns to each vertex of v in G a positive integer $f(v)$, we say that G is f*-choosable* (or f*-list-colorable*) if for every assignment of sets of integers $S(v) \subseteq \mathbb{Z}$, where $|S(v)| = f(v)$ for all $v \in V(G)$, there is a proper vertex coloring $c : V \to \mathbb{Z}$ so that $c(v) \in S(v)$ for all $v \in V(G)$.

The smallest integer k such that G is f-choosable for $f(v) = k$ $(v \in V(G))$ is the *list-chromatic number* $\chi_l(G)$. Clearly, $\chi(G) \leq \chi_l(G)$, and there are many graphs for which $\chi(G) < \chi_l(G)$. A simple example is the complete bipartite graph $K_{2,4}$, which is not 2-choosable. Another well-known example is the complete bipartite graph $K_{3,3}$. In fact, it is easy to show that for every k, there exist bipartite graphs whose list-chromatic number is bigger than k. It is actually NP-hard to determine the list-chromatic number of bipartite graphs, see [71]. Let us mention that list coloring is well motivated in a sense of providing natural interpretations for various kinds of scheduling problems, see [4, 5, 8]. Further applications include issues in VLSI theory.

In this paper, we are interested in approximating the list chromatic number in minor-closed class of graphs, and approximating the chromatic number of odd-minor-closed class of graphs.

1.2 Approximating the list chromatic number in a minor-closed class of graphs

Although there are many negative results as stated above, there are some positive results, which are mainly related to the Four Color Theorem and coloring planar graphs. One celebrated example is Thomassen's result on planar graphs [69]. It says that every planar graph is 5-choosable, and its proof is within 20 lines and gives rise to a linear time algorithm to 5-list-color planar graphs. In contrast with the Four Color Theorem, there are planar graphs that are not 4-choosable [73]. It is well-known that planar graphs are closed under taking minor operations; that is, deleting edges, deleting vertices and contracting edges. So one natural question is: can we extend the result of Thomassen to more general minor-closed families of graphs? This is our main motivation. Our main theorem is the following.

Theorem 1.1. *Let $\mathcal{M}$ be a minor-closed family of graphs and suppose that some graph of order k is not a member of $\mathcal{M}$. Then there is a polynomial time algorithm for list-coloring graphs in $\mathcal{M}$ with $O(k)$ colors. If a coloring is not found, the algorithm finds a small certificate that the graph is not $\Theta(k)$-choosable. The time complexity is $O(n^3)$.*

Previously, the best known result needed $O(k\sqrt{\log k})$ colors was using a theorem of Thomason and Kostochka [67, 68, 54, 53]. Their result says that any graph without K_k-minors is $O(k\sqrt{\log k})$*-degenerate*, i.e., any induced subgraph contains a vertex of degree $O(k\sqrt{\log k})$. Hence it implies a linear time algorithm to list-color graphs without K_k-minors with at most $O(k\sqrt{\log k})$ colors.

The above discussion also shows that the algorithm of Theorem 1.1 can also be viewed as an approximation algorithm for the list-chromatic number of graphs without K_k-minors, whose additive error is $O(k)$ for graphs whose list-chromatic number is $O(k)$, and whose multiplicative error is $O(\sqrt{\log k})$ if the list-chromatic number is $\Theta(k)$.

Our main idea involves precoloring extension. This idea is used in many results; one example is Thomassen's proof on his celebrated theorem on planar graphs [69]. Actually, the starting point of the investigation on precoloring extension was the observation that, on interval graphs, it provides an equivalent formulation of a practical problem on scheduling. For instance, this occurs if some of flights have to be assigned to a given number of airplanes according to the schedule of a timetable under an additional condition that the fixed maintenance schedule assigned to each airplane must not be obeyed [5].

Let us state our main result more precisely. We say that a number $f(k)$ which depends on the parameter k is *computable* if $f(k)$ can be expressed as a specific value, depending on k. In the next result we meet such values. The reader interested in specific expressions yielding these constants should consult [9]. Here is our main result.

Theorem 1.2. *Let k be an integer. There is a computable constant $f(k)$ and there is an $O(n^3)$ algorithm whose input is a graph G of order n, a $15.5k$-list-assignment $L : V(G) \to 2^{\mathbb{N}}$, a set $Z \subseteq V(G)$ with $|Z| \leq 6(k+1)$ and an L-coloring of $G[Z]$. The algorithm either*

(1) finds an L-coloring of G extending the precoloring of Z, or

(2) concludes that G contains K_k as a minor, or

(3) *finds a subset $Z' \subseteq V(G)$ with $|Z'| \leq 6(k+1)$, an L-coloring of Z', and a K_k-minor-free subgraph H of G of bounded size such that the precoloring of Z' cannot be extended to an L-coloring of H.*

If (3) holds in Theorem 1.2, then H is a counterexample to the list-coloring version of Hadwiger's conjecture. In fact, this would be a counterexample to the weaker conjecture that every graph whose list-chromatic number is at least $9.5k-6$ has K_k as a minor. If (2) holds, then we can actually detect a K_k-minor by the result of Robertson and Seymour [59]. The conclusion of Theorem 1.2(3) that such a subgraph has bounded number of vertices is an interesting theoretical outcome of the algorithm.

Corollary 1.3. *Let k be an integer. There is a computable constant $f(k)$ such that every graph without K_k minors is either $15.5k$-choosable, or it contains a subgraph of order at most $f(k)$ that is not $(9.5k-6)$-choosable.*

Another corollary given below is also of interest since there are no efficient algorithms for checking (nontrivial cases of) choosability questions for large classes of graphs.

Corollary 1.4. *For every fixed k, there is a computable constant $f(k)$ and an algorithm with running time $O(n^3)$ for deciding either that*

(1) *G is $15.5k$-choosable, or*

(2) *G contains a K_k-minor, or*

(3) *G contains a subgraph H of bounded size which does not contain a K_k-minor and is not $(9.5k-6)$-choosable.*

We also consider algorithms on the arboricity of graphs in minor-closed families. An *arboreal k-coloring* of G is a partition of the vertices of G into at most k classes, each of which induces an acyclic subgraph of G (a forest). The *arboricity* of G, denoted by $a(G)$, is the the minimum number k for which G has an arboreal k-coloring. The problem of computing $a(G)$ for a given graph G is known to be NP-hard. However, a good upper bound on $a(G)$ is also known in the literature. Suppose G is d-degenerate. Then it is easy to see that $a(G)$ is at most $1+d/2$. Let us remark that the problem of finding arboreal colorings of graphs has applications in the domain of design for testability in VLSI circuits, see [30, 63] for details.

Arboricity has been extensively examined for planar graphs. There is a linear-time algorithm for finding an arboreal 3-coloring of planar graphs. This was also extended to K_5-minor-free graphs and $K_{3,3}$-minor-free graphs. See [16, 18, 17] for details. These algorithms produce arboreal colorings which use at most $a(G)+1$ colors. We are interested in extending these results to general minor-closed families of graphs, for which we may assume that they are without K_k-minors.

Theorem 1.5. *Let $\mathcal{M}$ be a minor-closed family of graphs such that K_k is not its member. There is an $O(n^3)$ time algorithm which either finds an arboreal coloring of a given graph $G \in \mathcal{M}$ using $O(k)$ colors, or concludes that some minor of G has arboricity $\Theta(k)$.*

Previously, the best known algorithm would use $O(k\sqrt{\log k})$ colors, being based on the afore-mentioned result of Thomason and Kostochka.

To prove Theorems 1.15 and 1.5, we need the following result from [9].

Theorem 1.6 **([9]).** *For every k, there exists a constant $N(k)$ such that every $2k$-connected graph with minimum degree at least $\frac{31k}{2}$ and with at least $N(k)$ vertices contains K_k as a minor.*

Actually, in the proof of Theorem 1.2, we need a slightly stronger corollary of the main result in [9]. See Section 2.

1.3 Approximating the chromatic number in odd-minor-closed classes of graphs

Odd-minor-closed classes of graphs seem to be a much weaker concept than minor-closed families. Let us remind that we say that H has an odd complete minor of order l if there are l vertex disjoint trees in H such that any two of them are joined by an edge, and in addition, all the vertices of the trees are two-colored in such a way that the edges within the trees are bichromatic, but the edges between trees are monochromatic. Clearly, an odd minor is a special case of a minor. Let us observe that the complete bipartite graph $K_{n/2,n/2}$ contains a K_k-minor for $k \leq n/2$, but does not contain any odd K_k-minor when $k \geq 3$. In fact, any graph G without K_k-minors is $O(k\sqrt{\log k})$-degenerate, i.e, every induced subgraph has a vertex of degree at most $O(k\sqrt{\log k})$, see [67, 68, 54, 53]. So G has at most $O(k\sqrt{\log k}\,n)$ edges. On the other hand, graphs without odd-K_k-minors may have $\Theta(n^2)$ edges. So a given graph may be dense. This seems to make huge difference. On the other hand, as we shall see later, odd minors are actually motivated by graph minor theory and graph structural theory, and many researchers believe that there would be some analogue of graph minor theory, and some connection to the well-known conjecture of Hadwiger [38].

There is another motivation for odd minors. A long-standing area of interest in the field of discrete optimization is finding conditions under which a given polyhedron has integer vertices, so that integer optimization problems can be solved as linear programs. In the case of a particular set covering formulation for the maximum cut problem, there is a nice structure theorem, which has something to do with odd-minors. Let us give the notation. A *signed graph* is a pair (G, Σ), where $G = (V, E)$ is an undirected graph and $\Sigma \subseteq E$. Call a set of edges, or a path or a circuit *odd* (*even*, respectively) if it contains an odd (even) number of edges in Σ. An *odd circuit cover* is a set of edges intersecting all odd circuits. Following Grötschel and Pullyblank [32], a signed graph (G, Σ) is called *weakly bipartite* if each vertex of the polyhedron determined by

1. $x(e) \geq 0$ for each edge e,
2. $\sum_{e \in C} x(e) \geq 1$ for each odd circuit C,

is integral, that is, the incidence vector of an odd circuit cover. Weakly bipartite graphs are important since a maximum capacity cut in such graphs can be found in polynomial time (by using the ellipsoid method [31]). Let us observe that the problem of solving the related integer program contains the maximum cut problem, which is NP-hard.

Guenin [33, 34] gave a characterization of weakly bipartite graphs in terms of forbidden minors, and thus proving a special case of the well-known conjecture of Seymour [65]. The theorem says that a signed graph $(G, \sum)$ is weakly bipartite if and only if G does not contain odd-K_5-minors. Also, this generalizes the result of Seymour [65] who proved that a signed graph $(G, \sum)$ is strongly bipartite if and only if G does not contain odd-K_4-minors. For the definition of strongly bipartite graphs, we refer to [33, 34, 64]. Guenin's result [33, 34] has motivated several remarkable subsequent papers, see [29, 64].

As mentioned abovee, odd minors have been used to prove structure theorems in discrete optimization. For usual minors, there are important graph minor results, due to Robertson and Seymour. Since odd-minors generalize the usual ones, for which there exist powerful algorithmic results, one may want to extend these to odd-minors. This is motivation of our work. Let us first present some results concerning minor-closed families of graphs using the Graph Minor Theory of Robertson and Seymour.

The seminal Graph Minor Theory of Robertson and Seymour gives a powerful structural theorem. At a high level, the theorem says that every graph without K_k-minor can be decomposed into a collection of graphs each of which can "almost" be embedded into a bounded-genus surface, combined in a tree structure. The algorithmic part of the Graph Minor Theory is a polynomial-time algorithm for testing the existence of fixed minors [59] which, combined with the proof of Wagner's Conjecture, implies the existence of a polynomial-time algorithm for deciding any minor-monotone graph property.

This consequence has been used to prove existence of polynomial-time algorithms for several graph problems, some of which were not previously known to be decidable [25]. Algorithms for H-minor-free graphs for a fixed graph H have been studied extensively; see e.g. [12, 36, 15, 52, 57, 20]. In particular, it is generally believed that several algorithms for planar graphs can be generalized to H-minor-free graphs for any fixed H [36, 52, 57, 20]. The decomposition theorem provides the key insight into why this might be possible: first extend an algorithm for planar graphs to handle bounded-genus graphs, then extend it to graphs "almost-embeddable" into bounded-genus surfaces, and finally to tree decompositions of such graphs. The highlight of this approach was done in [20] to lead a 2-approximation algorithm for the graph coloring problem and a constant factor approximation for tree-width and the size of grid-minor.

But this approach may not work for odd-minor-closed graphs. First, "almost-embeddable" graphs into bounded-genus surfaces are rather sparse. For example, planar graphs can have at most $3n - 6$ edges, and bounded-genus surfaces have at most $3n - 6 + 6g$ edges with Euler genus g. On the other hand, dense(complete) bipartite graphs cannot contain any odd K_k-minor for $k \geq 3$. It seems that the argument of a a constant factor approximation for tree-width and the size of grid-minor in [20] should fail completely, and the 2-approximation algorithm for the graph coloring problem fails unless there is a polynomial time algorithm to get a feasible structure for graphs with no odd-K_k-minor. Second, it is not easy to figure out which edge is possible to contract because we need to keep the "parity" in odd-minor-closed graphs. Deleting edges and vertices certainly keep the parity, but it is not clear when we can contract edges. This causes a lot of trouble for the algorithm since we may need a reduction process.

Recently, however, Geelen et al. [28] proved that any graph G without odd K_k-minors is $O(k\sqrt{logk})$-colorable. Their result implies an $O(k\sqrt{\log k})$-approximation algorithm for the graph coloring problem of graphs with no odd-K_k-minor. (Geelen et al. [28] did not explicitly state the algorithm for the coloring, but we shall give a sketch of an algorithm for the completeness.) Motivated by this work, we obtain our second main result.

Theorem 1.7. *There is an $O(k)$-approximation graph coloring algorithm for graphs with no odd K_k-minors.*

Actually, the main result is the following.

Theorem 1.8. *Let k be an integer. There is a computable constant $f(k)$ and there is an $O(n^4)$ algorithm whose input is a graph G of order n, a set $Z \subseteq V(G)$ with $|Z| \leq 192k$ and a precoloring of $G[Z]$. The algorithm either*

(1) finds a $496k$-coloring of G extending the precoloring of Z, or

(2) concludes that G contains an odd-K_k-minor (actually detects an odd-K_k-minor), or

(3) finds a subset $Z' \subseteq V(G)$ with $|Z'| \leq 192k$, a coloring of Z', and an odd-K_k-minor-free subgraph H of G of bounded size such that the precoloring of Z' cannot be extended to a $496k$-coloring of H.

If (3) holds in Theorem 1.8, then H is a counterexample to the odd version of Hadwiger's conjecture. In fact, this would be a counterexample to the weaker conjecture that every graph whose chromatic number is at least $304k$ has an odd K_k-minor. The conclusion of Theorem 1.8(3) that such a subgraph has bounded number of vertices is a theoretical outcome of the algorithm worth to be mentioned.

Corollary 1.9. *Let k be an integer. There is a computable constant $f(k)$ such that every graph without odd-K_k minors is either $496k$-colorable, or it contains a subgraph of order at most $f(k)$ that is not $304k$-colorable.*

Another corollary given below is also of interest since this corollary gives rise to an algorithm for Theorem 1.8.

Corollary 1.10. *For every fixed k, there is a computable constant $f(k)$ and an algorithm with running time $O(n^4)$ for deciding either that*

(1) G is $496k$-colorable, or

(2) G contains an odd-K_k-minor, or

(3) G contains a subgraph H of bounded size which does not contain an odd-K_k-minor and is not $304k$-colorable.

The approximation algorithm follows from the following theorem together with Theorem 1.8.

Theorem 1.11 (Geelen et al. [28]). *There exists a constant c such that any graph with no odd K_k-minor is $ck\sqrt{\log k}$-colorable.*

In the conclusion, we shall explain the algorithm to color graphs without odd-K_k-minors with at most $ck\sqrt{\log k}$-colors.

1.4 Hadwiger's Conjecture and the Odd Hadwiger Conjecture

Hadwiger's Conjecture from 1943 suggests a far-reaching generalization of the Four Color Theorem [1, 2, 58] and is considered by many as the deepest open problems in graph theory. It claims the following.

Conjecture 1.12. *For every $k \geq 1$, every k-chromatic graph has a K_k-minor.*

Conjecture 1.12 is trivially true for $k \leq 3$, and reasonably easy for $k = 4$, as shown by Dirac [22] and Hadwiger himself [38]. However, for $k \geq 5$, Conjecture 1.12 implies the Four Color Theorem. In 1937, Wagner [74] proved that the case $k = 5$ of Conjecture 1.12 is, in fact, equivalent to the Four Color Theorem. In 1993, Robertson, Seymour and Thomas [62] proved that a minimal counterexample to the case $k = 6$ is a graph G which has a vertex v such that $G - v$ is planar. By the Four Color Theorem, this implies Conjecture 1.12 for $k = 6$. Hence the cases $k = 5, 6$ are each equivalent to the Four Color Theorem [1, 2, 58]. Conjecture 1.12 is open for $k \geq 7$. For the case $k = 7$, Toft and the first author [50] proved that any 7-chromatic graph has K_7 or $K_{4,4}$ as a minor. Recently, the first author [43] proved that any 7-chromatic graph has K_7 or $K_{3,5}$ as a minor.

It is not known if there exists an absolute constant c such that any ck-chromatic graph has a K_k minor. So far, it is known that there exists a constant c such that any $ck\sqrt{\log k}$-chromatic graph has a K_k-minor [53, 54, 67, 68].

So it would be of great interest to prove that a linear function of the chromatic number is sufficient to force K_k as a minor. We refer to [70] for further information on the Hadwiger Conjecture.

When relaxing the Hadwiger Conjecture to allow ck colors, the following conjecture from [47] involving list colorings may also be true:

Conjecture 1.13. *There is a constant c such that every graph without K_k minors is ck-choosable.*

Conjecture 1.12 does not hold for list colorings. For example, there exist planar graphs (without K_5 minors) which are not 4-choosable. However, Conjecture 1.13 is formulated in such a way that it may also be true for $c = 1$. We believe that Conjecture 1.13 holds at least with $c = \frac{3}{2}$.

Gerards and Seymour (see [41], page 115) conjectured the following.

Conjecture 1.14. *For all $l \geq 1$, every graph with no odd K_{l+1} minor is l-colorable.*

This is an analogue of Conjecture 1.12. Clearly, Conjecture 1.14 implies Hadwiger's conjecture. Again, Conjecture 1.14 is trivially true when $l = 1, 2$. In fact, when $l = 2$, this means that if a graph has no odd cycles, then it is 2-colorable. The $l = 3$ case was proved by Catlin [11]. Recently, Guenin [35] announced a solution of the $l = 4$ case. This result would imply the Four Color Theorem because a graph having no K_5-minors certainly contains no odd-K_5-minors. Conjecture 1.14 is open for $l \geq 5$.

It is not known if there exists a constant c such that any ck-chromatic graph contains K_k as a minor. We show in this paper that from an algorithmic point of view, we can "decide" this problem in $O(n^3)$ time. We also prove the following.

Theorem 1.15. *For every fixed k, there is an algorithm with running time $O(n^3)$ for deciding either that*

(1) a given graph G of order n is $27k$-colorable, or

(2) G contains K_k-minor, or

(3) G contains a minor H of bounded size which does not contain a K_k-minor and has no $27k$-colorings.

Let us remark the following:

(a) If (3) holds, then H is a counterexample to Hadwiger's conjecture. In fact, this would be a counterexample to the weaker conjecture that any $27k$-chromatic graph has K_k as a minor. The conclusion of Theorem 1.15(3) that such a minor has bounded number of vertices is an interesting theoretical outcome of the algorithm.

(b) If (1) holds, we can actually color the graph using at most $27k$ colors. If (3) holds, we can exhibit the minor H by means of a subgraph $\tilde{H}$ of G whose contraction yields H. If (2) holds, then we can actually detect a K_k-minor.

(c) We need the result of [9] (Theorem 1.6) in order to prove correctness of the algorithm, but we do not need complicated algorithmic results from the Graph Minor series. However, the proof of Theorem 1.6 given in [9] depends on Robertson and Seymour's deep results, see [9, 61, 60].

Related to Hadwiger's Conjecture, Chartrand, Geller, Hedetniemi [13], and Woodall [75] proposed the following (m, n)-Contraction-Conjecture:

Conjecture 1.16. *For integers $1 \leq n \leq m$, every graph G without a K_{m+1}-minor and without a $K_{\lfloor \frac{m+2}{2} \rfloor, \lceil \frac{m+2}{2} \rceil}$-minor, has a partition of G into $m-n+1$ parts, each part inducing a subgraph without a K_{n+1}-minor and without a $K_{\lfloor \frac{n+2}{2} \rfloor, \lceil \frac{n+2}{2} \rceil}$-minor.*

This conjecture is true for $m \leq 4$ as proved by Chartrand, Geller and Hedetniemi [13] except for the case $(m, n) = (4, 1)$, which is equivalent to the Four Colour Theorem. In these cases, the conjecture is best possible in the sense that there are graphs whose vertex set cannot be partitioned into fewer sets with the desired property. That there exist planar graphs of arboricity 3 was first shown by Chartrand and Kronk [14]. Several interesting applications are obtained in [21].

It is not known if there exists a constant c such that any graph without K_k as a minor has arboricity at most ck. We show in this paper that from an algorithmic point of view, we can "decide" this problem in polynomial time.

Theorem 1.17. *Let k be an integer. There is a computable constant $f(k)$ and there is an $O(n^3)$ algorithm whose input is a graph G of order n, a set $Z \subseteq V(G)$ with $|Z| \leq 6(k+1)$ and a precoloring of $G[Z]$. The algorithm either*

(1) finds an $8k$-coloring of G extending the precoloring of Z such that each color class is a forest, or

(2) concludes that G contains K_k as a minor, or

(3) finds a minor H of G, a subset $Z' \subseteq V(H)$ with $|Z'| \leq 6(k+1)$, and an arboreal coloring of Z' such that H has bounded size, has no K_k-minor, and the precoloring of Z' cannot be extended to an arboreal $8k$-coloring of H.

Let us observe that if the output is (3), then H is a counterexample to Conjecture 1.16. In fact, this would be a counterexample to the weaker conjecture as we shall see below. The above theorem has the following corollary.

Corollary 1.18. *For every fixed k, there is a computable constant $f(k)$ and an algorithm with running time $O(n^3)$ for deciding either that*

(1) G has arboricity at most $8k$,

(2) G contains a K_k-minor, or

(3) G contains a minor H of bounded size which does not contain a K_k-minor and has arboricity more than $2k$.

It is not known if there exists a constant c such that any ck-chromatic graph contains an odd-K_k-minor. We show in this paper that from an algorithmic point of view, we can "decide" this problem in polynomial time. Actually, the time complexity is $O(n^{2496k})$.

Theorem 1.19. *For every fixed k, there is an algorithm with running time $O(n^{2496k})$ for deciding either that*

(1) a given graph G of order n is $2496k$-colorable, or

(2) G contains an odd-K_k-minor, or

(3) G contains an odd-minor H of bounded tree-width which does not contain an odd-K_k-minor and has no $2496k$-colorings.

Let us remark the following:

(a) If (3) holds, then H is a counterexample to the odd Hadwiger's conjecture. In fact, this would be a counterexample to the weaker conjecture that any $2496k$-chromatic graph has an odd-K_k-minor.

(b) If (1) holds, we can actually color the graph using at most $2496k$ colors. If (2) holds, then we can actually detect the odd-K_k-minor.

(c) We need the result of [9], Theorem 1.6, but we do not need any result in Graph Minor series. However, the proof of Theorem 1.6 given in [9] depends on Robertson and Seymour's deep results, see [9, 61, 60].

Consider the following problem: *Does every $2497k$-chromatic graph contain an odd-K_k-minor?*

Our proof of Theorem 1.19 has the following remarkable corollary.

Corollary 1.20. *There are only finitely many minimal counterexamples to the above problem.*

Let us point out that the best previous known ratio for the chromatic number in the above problem is $O(k\sqrt{logk})$, which follows from Theorem 1.11. Although we do not know whether or not chromatic number being linear in k is enough to force an odd K_k-minor, the above corollary shows that there are only finitely minimal counterexamples, so the problem can be answered for every graph in polynomial time.

Another corollary of our proof of Theorem 1.19 is that every minimal counterexample to the odd Hadwiger's conjecture for fixed k is $k/2497$-connected. This is the first result showing that minimal counterexamples have linear connectivity. This strengthens Guenin's result [35] which says that minimal counterexamples are 4-connected.

The detailed proof will be given in the full paper. Let us point out that the idea is similar to that of the weaker Hadwiger's conjecuture case, but it is more involved and quite lengthly.

2. COLORING EXTENSION AND LIST COLORINGS

In this section, we provide details about the proof of Theorem 1.2. We will use a corollary of the following result from [9].

Theorem 2.1. *For any integers k, s and t, there exists a computable constant $N_0(k,s,t)$ such that every $(3k+2)$-connected graph of minimum degree at least $15.5k$ and with at least $N_0(k,s,t)$ vertices either contains $K_{k,st}$ as a topological minor or a minor isomorphic to s disjoint copies of $K_{k,t}$.*

Let A and B be induced subgraphs of G such that $G = A \cup B$. If $V(A) \setminus V(B) \neq \emptyset$ and $V(B) \setminus V(A) \neq \emptyset$, then we say that the pair (A,B) is a *separation* of G. The *order* of this separation is equal to $|V(A \cap B)|$. Let $Z \subseteq V(G)$ be a vertex set. We say that the separation (A,B) of G is *Z-essential* if $(A-Z, B-Z)$ is a separation of $G-Z$. If l is a positive integer, we say that G is *l-connected relative to Z* if it has no Z-essential separations of order less than l.

We will need the following corollary of Theorem 2.1:

Theorem 2.2. *For any integers k and z, there exists a constant $N_1(k,z)$ such that for every graph G and every vertex set $Z \subseteq V(G)$ of cardinality at most z, if G is $(3k+2)$-connected relative to Z, the degree of every vertex in $V(G) \setminus Z$ is at least $15.5k$, and G has at least $N_1(k,z)$ vertices, then G contains the complete graph K_k as a minor.*

Proof. Let G and Z be as assumed in the statement of the theorem. Let Z' be the set of all vertices in Z whose degree is at most $3k+1+z$. Let D be a set of vertices in $G-Z$ of cardinality $3k+2$ such that no vertex in Z' is adjacent to D. If $|V(G)| \geq (3k+2+z)z+3k+2$ (which we may assume), then D exists. Let G' be the graph obtained from G by adding all edges between Z' and D.

In G', every vertex in Z has at least $3k+2$ neighbors that are not in Z. Since G is $(3k+2)$-connected relative to Z and is a spanning subgraph of G', this implies that G' is $(3k+2)$-connected. Suppose that $|V(G)| \geq N_0(k,s,t)$. By Theorem 2.1, G' either contains a subdivision of $K_{k,st}$ or a minor isomorphic to s disjoint copies of $K_{k,t}$. Let us take $s = z+1$ and $t = 3k+2+z$. If G' has s copies of $K_{k,t}$ as a minor, then G' contains a $K_{k,k}$-minor (and hence also a K_k-minor) that is disjoint from Z. As for the other alternative, when G' contains a subgraph K which is a subdivision of $K_{k,st}$, none of the vertices of degree st in K belong to Z' since the vertices in Z' have degree less than $(3k+2+z)z+3k+2 < (3k+2+z)(z+1) = st$. Therefore, $G'-Z'$ contains a subgraph which is a subdivision of $K_{k,st-z}$. Since $st-z \geq k$, G has $K_{k,k}$ and hence also K_k as a minor. So, the theorem holds for the value $N_1(k,z) = N_0(k, z+1, 3k+2+z)$. □

Let L be a list-assignment for a graph G, and let Z be a set of vertices of G. By *precoloring of Z*, we mean that we L-color the subgraph $G[Z]$ induced by Z.

We will need the following facts.

Fact 2.3. *Let Z be a vertex set of G with $|Z| \leq 6k+4$. Suppose G has a separation (A,B) of order at most $3k+2$ such that both $B-A-Z$ and $A-B-Z$ are nonempty. Then either $|(Z \cap A) \cup (A \cap B)| \leq 6k+4$ or $|(Z \cap B) \cup (A \cap B)| \leq 6k+4$.*

Fact 2.4. *For every fixed k, every $15.5k$-list-assignment L, every $Z \subseteq V(G)$ with $|Z| \le 6k+6$, and every precoloring of Z, if G has a separation (A, B) of order at most $3k+2$ such that both $B-A-Z$ and $A-B-Z$ are nonempty and $|(Z \cap B) \cup (A \cap B)| \le 6k+4$, and if the precoloring of Z can be extended to an L-coloring of $A \cup Z$, then either the resulting coloring of $(Z \cap B) \cup (A \cap B)$ can be extended to B (and hence the precoloring of Z can be extended to the whole graph G), or B has a K_k-minor, or B has a subgraph H which does not contain a K_k-minor and contains a vertex set Z' with $|Z'| \le 6k+4$ such that some precoloring of Z' cannot be extended to an L-coloring of H.*

Fact 2.5. *Let Z be a vertex set of G with $|Z| \le 192k$. Suppose G has a separation (A, B) of order at most $64k$ such that both $B-A-Z$ and $A-B-Z$ are nonempty. Then either $|(Z \cap A) \cup (A \cap B)| \le 192k$ or $|(Z \cap B) \cup (A \cap B)| \le 192k$.*

Fact 2.6. *For every fixed k, every $Z \subseteq V(G)$ with $|Z| \le 192k$, and every precoloring of Z, if G has a separation (A, B) of order at most $96k$ such that both $B-A-Z$ and $A-B-Z$ are nonempty and $|(Z \cap B) \cup (A \cap B)| \le 192k$, and if the precoloring of Z can be extended to a coloring of $A \cup Z$, then either the resulting coloring of $(Z \cap B) \cup (A \cap B)$ can be extended to B (and hence the precoloring of Z can be extended to the whole graph G), or B has an odd-K_k-minor, or B has a subgraph H which does not contain an odd-K_k-minor and contains a vertex set Z' with $|Z'| \le 192k$ such that some precoloring of Z' cannot be extended to a coloring of H.*

A K_l- minor in a graph G can be thought as a subgraph of G consisting of l vertex disjoint trees $T_1, \dots, T_l$, called the *nodes* of the minor, together with $\binom{l}{2}$ edges joining all pairs of distinct nodes. We call such a minor *even* if the union of the nodes and connecting edges is bipartite. We call such a minor *odd* if its vertices can be two-colored so that the edges in the nodes are bichromatic but the edges between the nodes are monochromatic.

Recall that a *block* of a graph H is a maximal 2-connected subgraph of H. We need the following result.

Theorem 2.7. *Let N be a $K_{32k,\,(16k-1)\binom{32k}{16k}+1}$-minor in a graph G. Then either G has an odd-K_k-minor or G has a vertex set X of order at most $8k$ such that $G-X$ has a bipartite block F containing all but at most $8k$ nodes of N.*

This was proved in [49], but for the completeness, we shall include the proof. To prove this theorem, we need the following.

Geelen et al. [28] proved the following result.

Theorem 2.8. *If G has an even complete minor N of order at least $16k$, then either G has an odd complete minor of order k or G has a vertex set X with $|X| < 8k$ such that $G-X$ has a bipartite block F containing more than $8k$ nodes of N.*

We can think of a $K_{32k,\,(16k-1)\binom{32k}{16k}+1}$-minor as follows:

> There are $32k + (16k-1)\binom{32k}{16k} + 1$ disjoint trees $T_1, \dots, T_{32k}, T'_1, \dots, T'_{(16k-1)\binom{32k}{16k}+1}$ such that there is an edge between T_i and T'_j for any i, j with $1 \le i \le 32k$ and $1 \le j \le (16k-1)\binom{32k}{16k} + 1$.

We first two-color the trees $T_1, \dots, T_{32k}$ (using colors 1 and 2) such that each T_i is bichromatic. Then for each j, we two-color T'_j in such a way that T'_j is bichromatic and there are at least $16k$ bichromatic edges between T'_j and $\bigcup_{i=1}^{32k} T_i$, for $1 \le j \le (16k-1)\binom{32k}{16k} + 1$. This is possible since we have two choices for two-coloring of T'_j. Then by the Pigeonhole Principle, there are $16k$ disjoint trees in $\{T_1, \dots, T_{32k}\}$, say trees $T_1, \dots, T_{16k}$, and there are $16k$ disjoint trees in $\{T'_1, \dots, T'_{(16k-1)\binom{32k}{16k}+1}\}$, say trees $T'_1, \dots, T'_{16k}$, such that the edge of the minor joining T_i and T'_j is bichromatic for $i = 1, \dots, 16k$ and $j = 1, \dots, 16k$.

Now let $T^*_i = T_i \cup T'_i$, where $i = 1, \dots, 16k$. Clearly $\bigcup_{i=1}^{16k} T^*_i$ is bipartite and forms an even complete minor of order $16k$ in G. By Theorem 2.8, either G has an odd complete minor of order k or G has a vertex set X of order at most $8k$ such that $G-X$ has a bipartite subgraph F and each odd cycle is contained in either components of $G-X$ that do not intersect F or components after deleting a cut vertex to F.

This completes the proof of Theorem 2.7. □

Let us remark that in the above proof, once we detect a $K_{32k,\,(16k-1)\binom{32k}{16k}+1}$-minor, then we can find an even complete minor of order $16k$ in linear time. By the result of Robertson and Seymour [59], we can detect the minor in $O(n^3)$ steps. It remains to find a polynomial time algorithm for Theorem 2.8. The proof of Theorem 2.8 in [28] certainly implies the polynomial time algorithm to find the desired conclusion. Actually, it detects either a desired odd minor or a vertex set X of bounded size in G such that $G-X$ has a bipartite subgraph F and each odd cycle is contained in either components of $G-X$ that do not intersect F or components after deleting a cut vertex to F. Geelen et al. [28] reduced the problem to the problem of finding a maximum matching that can be solved in $O(n^3)$ time, see [26, 55, 19, 27].

3. ALGORITHM FOR APPROXIMATING LIST-CHROMATIC NUMBER IN MINOR-CLOSED CLASSES

Algorithm for Theorem 1.2

Input: A graph G, $15.5k$-list-assignment L, a set $Z \subseteq V(G)$ with $|Z| \le 6k+6$, and a precoloring of $G[Z]$.

Output: As described in Theorem 1.2.

Running time: $O(f(k)n^3)$ for some function $f : \mathbb{N} \to \mathbb{N}$.

Description:

Step 1. If G has a vertex of degree at most $15.5k-1$ in $V(G) \setminus Z$, then we delete it. We continue this procedure until there are no vertices of degree at most $15.5k-1$ apart from those in Z. This can be done in linear time. Let G' be the resulting graph. Proceed to Step 2.

Step 2. Test whether G' has a separation (A, B) of order at most $3k+1$ such that both $B-A-Z$ and $A-B-Z$ are nonempty. Suppose first that G' has no such separation. If $|G'| \ge g(k) := \max\{N_1(k, z) \mid 0 \le z \le 6k+6\}$, then it follows from Theorem 2.2 that G' contains K_k as a minor. So, we output that G has K_k as a minor. If $|G'| < g(k)$, then we can check (1), (2) and (3) of Theorem 1.2 in constant time and output the outcome.

If G' has such a separation, then go to Step 3.

Step 3. Detect a separation (A, B) of G' of minimum order ($\leq 3k+1$) such that both $B-A-Z$ and $A-B-Z$ are nonempty. This can be done in polynomial time by standard methods. The best known algorithm is that of Henzinger, Rao, and Gabow [40] which needs $O(n^2)$ time for this task.

By Fact 2.3, either $|(Z \cap A) \cup (A \cap B)| \leq 6k+4$ or $|(Z \cap B) \cup (A \cap B)| \leq 6k+4$. Suppose $|(Z \cap B) \cup (A \cap B)| \leq 6k+4$.

Then, we test $A \cup Z$ (recursively), starting from Step 1. If $A \cup Z$ has an L-coloring extending the precoloring of Z, then we precolor $A \cap B$ by this coloring. Then we test B (recursively) with $Z' = (Z \cap B) \cup (A \cap B)$ and using the corresponding precoloring of $A \cap B$ and $Z \cap B$. If B has an L-coloring extending the precoloring of $A \cap B$ and $Z \cap B$, this give rise to an L-coloring of their union, and this coloring gives rise to an L-coloring of G'. Of course, we can extend this coloring of G' to G.

If either $A \cup Z$ or B contains a K_k-minor, then we obtain a K_k-minor in G'. Similarly, if outcome (3) appears for $A \cup Z$, we get the same outcome for G'. Also, if outcome (3) appears for B, we get the same outcome for G'.

This algorithm stops when either the current graph is small or when it has no separation (A, B) of order at most $3k+1$ such that both $B-A-Z$ and $A-B-Z$ are nonempty, and minimum degree is at least $15.5k$. By Fact 2.4, it is easy to see that this algorithm correctly gives one of outcomes in Theorem 1.2 for G'. Then it is easy to extend the coloring of G' to G.

Finally, let us estimate time complexity of the algorithm. All steps used in the algorithm can be done in quadratic time. Another factor of n pops up because of applying the recursion in Step 3. This completes the analysis of the correctness and of the stated time complexity of the algorithm. $\square$

4. ALGORITHM FOR ARBORICITY

In this section, we shall give a proof of Theorem 1.17.

The following fact is easy to prove.

Fact 4.1. *If G is $(16k-1)$-degenerate, then G has arboricity at most $8k$ and an arboreal $8k$-coloring can be found in linear time.*

The algorithm below is quite close to the one in Theorem 1.2, except that we need to take care of one issue.

Algorithm for Theorem 1.17

Input: A graph G, a set $Z \subseteq V(G)$ with $|Z| \leq 6k+6$, and a precoloring of Z.

Output: As described in Theorem 1.17.

Running time: $O(f(k)n^3)$ for some function $f : \mathbb{N} \to \mathbb{N}$.

Description:

Step 1. If G has a vertex of degree at most $16k-1$ in $V(G) \setminus Z$, then we delete it. We continue this procedure until there are no vertices of degree at most $16k-1$ apart from those in Z. This can be done in linear time. Let G' be the resulting graph. Proceed to Step 2.

Step 2. Test whether G' has a separation (A, B) of order at most $3k+1$ such that both $B-A-Z$ and $A-B-Z$ are nonempty. Suppose first that G' has no such separation. If $|G'| \geq g(k) := \max\{N_1(k, z) \mid 0 \leq z \leq 6k+6\}$, then it follows from Theorem 2.2 that G' contains K_k as a minor. So, we output that G has K_k as a minor. If $|G'| < g(k)$, then we can check (1), (2) and (3) of Theorem 1.17 in constant time and output the outcome.

If G' has such a separation, then go to Step 3.

Step 3. G' has a separation (A, B) of order at most $3k+1$ such that both $B-A-Z$ and $A-B-Z$ are nonempty; detect such a separation (A, B) of minimum order. This can be done in polynomial time by standard methods [40] in time $O(n^2)$.

By Fact 2.3 we may assume that $|(Z \cap B) \cup (A \cap B)| \leq 6k+4$.

Then, we test $A \cup Z$ (recursively), starting from Step 1. If $A \cup Z$ has a coloring extending the precoloring of Z such that each color class is a forest, then we precolor $A \cap B$ by this coloring. Then we test B (recursively) with $Z' = (Z \cap B) \cup (A \cap B)$ and using the corresponding precoloring of $A \cap B$ and $Z \cap B$. There is one catch here. Suppose that there are two vertices a, b in $A \cap B$ such that both a and b have the same color in $A \cup Z$. Also, suppose we can color $B \cup Z$ by extending $Z' = (Z \cap B) \cup (A \cap B)$ such that each color class is a forest. Then certainly both a and b receive the same color in B. There may be a path joining a and b in A such that each vertex on this path receives the same color. The same may happen in B. Then this is not a desired coloring. In order to avoid this situation, we do the following. If there is a path joining a and b in A such that each vertex on this path receives the same color, then we add the edge ab to $A \cap B$. Then we (recursively) test B together with all edges that have been added in the above operation, with $Z' = (Z \cap B) \cup (A \cap B)$ and using the corresponding precoloring of $A \cap B$ and $Z \cap B$. Let us observe that the resulting graph here is a minor of G. If B has a coloring extending the precoloring of $A \cap B$ and $Z \cap B$ such that each color class is a forest, this give rise to a coloring of their union, and this coloring gives rise to a coloring of G' such that each color class is a forest. Of course, we can extend this coloring of G' to G.

If either $A \cup Z$ or B contains a K_k-minor, then we obtain a K_k-minor in G'. Similarly, if outcome (3) appears for $A \cup Z$, we get the same outcome for G'. Also, if outcome (3) appears for B, we get the same outcome for G'.

This algorithm stops when either the current graph is small or when it has no separation (A, B) of order at most $3k+1$ such that both $B-A-Z$ and $A-B-Z$ are nonempty, and minimum degree is at least $16k$. By Fact 2.4, it is easy to see that this algorithm correctly gives one of outcomes in Theorem 1.17. Actually, we need the arboricity version of Fact 2.4, but Fact 2.4 still works for the arboricity (even with the additional hassle of having added edges in $A \cap B$). Also it is easy to see that if there is a vertex of degree less than $16k$, then by Fact 4.1, we can color such a vertex. The only issue we need to take care of is that there may be a path joining a and b in A such that each vertex on this path receives the same color, and there may be a path joining a and b in B such that each vertex on this path receives the same color, too. In order to avoid this situation, we add an edge ab to $A \cap B$ if there is a path joining a and b in A such that each vertex on this path receives the same color. Then we test B (recursively) with $Z' = (Z \cap B) \cup (A \cap B)$ and using the corresponding precoloring of $A \cap B$ and $Z \cap B$. This certainly avoids the above problem. Also, it is easy to see that the resulting graph here is a minor of G.

Finally, let us estimate time complexity of the algorithm. All steps used in the algorithm can be done in quadratic time. Another factor of n pops up because of applying the recursion in Step 3. This completes the analysis of the correctness and of the stated time complexity of the algorithm. $\square$

5. ALGORITHMIC ASPECT OF HADWIGER CONJECTURE

In this section we describe an algorithm for Theorem 1.15. We shall need the following results.

Highly linked graphs

A graph L is said to be *k-linked* if it has at least $2k$ vertices and for any ordered k-tuples $(s_1, \dots, s_k)$ and $(t_1, \dots, t_k)$ of $2k$ distinct vertices of L, there exist pairwise disjoint paths $P_1, \dots, P_k$ such that for $i = 1, \dots, k$, the path P_i connects s_i and t_i. Such collection of paths is called a *linkage* from $(s_1, \dots, s_k)$ to $(t_1, \dots, t_k)$.

An important tool is the following theorem due to Thomas and Wollan [66].

Theorem 5.1. *Every $2k$-connected graph G with at least $5k|V(G)|$ edges is k-linked.*

Theorem 5.1 implies that every $10k$-connected graph is k-linked. Prior to this result, Bollobás and Thomason [10] proved that every $22k$-connected graph is k-linked, and Kawarabayashi, Kostochka and Yu [45] proved that every $12k$-connected graph is k-linked.

The following result is a variation of an old theorem of Mader [56]. Its non-algorithmic counterpart appeared in [9, 51]. For completeness, we include its proof which is needed to establish the algorithmic part.

Theorem 5.2. *Let G be a graph and k an integer such that*

(a) $|V(G)| \geq \frac{5}{2}k$ and

(b) $|E(G)| \geq \frac{25}{4}k|V(G)| - \frac{25}{2}k^2$.

Then $|V(G)| \geq 10k+2$ and G contains a $2k$-connected subgraph H with at least $5k|V(H)|$ edges. If G has n vertices, then H can be found in time $O(n^3)$.

Proof. Clearly, if G is a graph on n vertices with at least $\frac{25}{4}kn - \frac{25}{2}k^2$ edges, then $\frac{25}{4}kn - \frac{25}{2}k^2 \leq \binom{n}{2}$. Hence, either $n \leq \frac{25}{4}k + \frac{1}{2} - \frac{1}{4}\sqrt{(25k+2)^2 - 400k^2} < \frac{5}{2}k$ or $n \geq \frac{25}{4}k + \frac{1}{2} + \frac{1}{4}\sqrt{(25k+2)^2 - 400k^2} > 10k+1$. Since $|V(G)| \geq \frac{5}{2}k$, we get the following:

Claim 1. $|V(G)| \geq 10k+2$.

Suppose now that the theorem is false. Let G be a graph with n vertices and m edges, and let k be an integer such that (a) and (b) are satisfied. Suppose, moreover, that

(c) G contains no $2k$-connected subgraph H with at least $5k|V(H)|$ edges, and

(d) n is minimal subject to (a), (b) and (c).

Claim 2. *The minimum degree of G is more than $\frac{25}{4}k$.*

Suppose that G has a vertex v with degree at most $\frac{25}{4}k$, and let G' be the graph obtained from G by deleting v. By (c), G' does not contain a $2k$-connected subgraph H with at least $5k|V(H)|$ edges. Claim 1 implies that $|V(G')| = n-1 \geq \frac{5}{2}k$. Finally, $|E(G')| \geq m - \frac{25}{4}k \geq \frac{25}{4}k|V(G')| - \frac{25}{2}k^2$. Since $|V(G')| < n$, this contradicts (d) and the claim follows.

Claim 3. *$m \geq 5kn$.*

The claim follows easily from (b) by using Claim 1.

By Claim 3 and (c), G is not $2k$-connected. Since $n > 2k$, this implies that G has a separation (A_1, A_2) such that $A_1 \setminus A_2 \neq \emptyset \neq A_2 \setminus A_1$ and $|A_1 \cap A_2| \leq 2k-1$. By Claim 2, $|A_i| \geq \frac{25}{4}k+1$. For $i \in \{1,2\}$, let G_i be a subgraph of G with vertex set A_i such that $G = G_1 \cup G_2$ and $E(G_1 \cap G_2) = \emptyset$. Suppose that $|E(G_i)| < \frac{25}{4}k|V(G_i)| - \frac{25}{2}k^2$ for $i = 1, 2$. Then

$$\frac{25}{4}kn - \frac{25}{2}k^2 \leq m = |E(G_1)| + |E(G_2)| <$$

$$\frac{25}{4}k(n + |A_1 \cap A_2|) - 25k^2 \leq \frac{25}{4}kn - \frac{25}{2}k^2,$$

a contradiction. Hence, we may assume that $|E(G_1)| \geq \frac{25}{4}k|V(G_1)| - \frac{25}{2}k^2$. Since $n > |V(G_1)| \geq \frac{25}{4}k+1$ and G_1 contains no $2k$-connected subgraph H with at least $5k|V(H)|$ edges, this contradicts (d), and the existence of H is established.

The above proof yields an $O(n^3)$ algorithm for finding H. First, we remove vertices of degree at most $\frac{25}{4}k$ as long as the minimum degree is more than $\frac{25}{4}k$. This can be done in linear time – first we form the list of all vertices whose degree is at most $\frac{25}{4}k$. Then we start removing one by one. At each step, degrees change only at the neighbors of the removed vertex, and it takes constant time to decrease their degrees by one and to move those, whose degree drops to $\lfloor \frac{25}{4}k \rfloor$, into the list of vertices to be removed.

Next, we check if G is $2k$-connected. This can be done in $O(n^2)$ time by an algorithm of Henzinger, Rao, and Gabow [40]. At the same time we find a separation (A_1, A_2) of order less than $2k$ if one exists. As shown above, one of the corresponding subgraphs G_1 or G_2 can be used to continue the process. The recursion brings another factor of n to the time complexity, so the subgraph H can be found in $O(n^3)$ time. $\square$

By Theorem 5.1, every $2k$-connected graph G with at least $5k|V(G)|$ edges is k-linked. Hence, Theorem 5.2 implies the following:

Corollary 5.3. *Let G be a graph of order n and k be an integer such that*

(a) $|V(G)| \geq \frac{5}{2}k$ and

(b) $|E(G)| \geq \frac{25}{4}k|V(G)| - \frac{25}{2}k^2$.

Then G contains a k-linked subgraph, and such a subgraph can be found in $O(n^3)$ time.

Connectivity and clique-sums

Let (A, B) be a separation in a graph G and let $S = V(A) \cap V(B)$. Let $S_1 \cup \cdots \cup S_r$ be a partition of S. Then we say that A can be *contracted* to $S_1, \ldots, S_r$ if A contains pairwise disjoint connected subgraphs $T_1, \ldots, T_r$ such that $S_i \subseteq V(T_i)$ for $i = 1, \ldots, r$.

The following lemma was originally proved in [44], but for completeness, we include its proof.

Lemma 5.4. *Let G be a graph with minimum degree d. Let (A, B) be a separation in G of minimum order, let $S = V(A) \cap V(B)$ and $s = |S|$. If $s \le \lfloor \frac{2d}{27} \rfloor$, then for every partition $S_1 \cup \cdots \cup S_r$ of S, A (and B) can be contracted to $S_1, \ldots, S_r$. The corresponding disjoint connected subgraphs $T_1, \ldots, T_r$ in A (resp. B) can be found in $O(n^3)$ time, where $n = |V(G)|$.*

Proof. We may assume that $A_1 = A - S$ is connected. Note that all vertices in A_1 have degree at least $d - s \ge \frac{25d}{27}$. Therefore, $|V(A_1)| \ge \frac{25d}{27}$ and $|E(A_1)| \ge \frac{25d}{54}|V(A_1)|$. Let $k = \lfloor \frac{2d}{27} \rfloor$. By Theorem 5.2, A_1 contains a $2k$-connected subgraph H_1 with $|E(H_1)| \ge 5k|V(H_1)|$. By Corollary 5.3, H_1 is k-linked. By the minimality of S and Menger's theorem, there are s disjoint paths $P_1, \ldots, P_s$ joining S and $V(H_1)$. Let $v_j \in S$ and $u_j \in V(H_1)$ be the endvertices of P_j for $j = 1, \ldots, s$. Since H_1 is $2k$-connected, there is a matching $e_1, \ldots, e_s$ in H_1, where $e_j = u_j u'_j$ for some s vertices $u'_1, \ldots, u'_s$ in H_1.

At this point, we assume that the vertices $v_1, \ldots, v_s$ of S are enumerated such that $v_1, \ldots, v_{|S_1|} \in S_1$, $v_{|S_1|+1}, \ldots, v_{|S_1|+|S_2|} \in S_2$, and so on. Since H_1 is k-linked and $s \le k$, there are s disjoint paths $P'_1, \ldots, P'_{s-1}$ in H_1 such that P'_j joins u_j and u'_{j+1} for $j = 1, \ldots, s-1$. The subgraph of A consisting of $P_1, \ldots, P_{|S_1|}, e_1, \ldots, e_{|S_1|}$ and the connecting paths $P'_1, \ldots, P'_{|S_1|-1}$ is a tree whose contraction gives rise to identification of all vertices of S_1 into a single vertex. Similarly, there are such trees for $S_2, \ldots, S_r$, and they are all disjoint. Hence we can contract A to $S_1, \ldots, S_r$.

As far as the algorithm is concerned, we can find H_1 in $O(n^3)$ time by Theorem 5.2. To find paths $P_1, \ldots, P_s$, we can apply an augmenting path algorithm ($s < 2k$ times), so we are done in $O(n^2)$ time. Finally, disjoint connecting paths $P'_1, \ldots, P'_{s-1}$ can be found by the graph minors algorithm of Robertson and Seymour [59] in $O(n^3)$ time. This completes the proof of the lemma. ☐

Remark: In the algorithm of Lemma 7.2 we apply the Robertson and Seymour's algorithm from [59] to find disjoint connecting paths $P'_1, \ldots, P'_{s-1}$. However, when we apply Lemma 7.2 in the algorithm for Theorem 1.15, we may assume that the k-linked subgraph H_1 is of bounded size. Namely, if $|V(H_1)| \ge N(k)$, where $N(k)$ is the constant from Theorem 1.6, then we know that H_1 contains K_k as a minor. Since this is one of the possible outcomes of the algorithm, we may assume that $|V(H_1)| < N(k)$, and hence $P'_1, \ldots, P'_{s-1}$ can be found in constant time.

Description of the algorithm

A *tree decomposition* of a graph G is a pair (T, Y), where T is a tree and Y is a family $\{Y_t \mid t \in V(T)\}$ of vertex sets $Y_t \subseteq V(G)$, such that the following two properties hold:

(W1) $\bigcup_{t \in V(T)} Y_t = V(G)$, and every edge of G has both ends in some Y_t.

(W2) If $t, t', t'' \in V(T)$ and t' lies on the path in T between t and t'', then $Y_t \cap Y_{t''} \subseteq Y_{t'}$.

The *width* of a tree decomposition (T, Y) is $\max_{t \in V(T)}(|Y_t| - 1)$.

Algorithm for Theorem 1.15

Input: A graph G.

Output: As described in Theorem 1.15.

Running time: $O(f(k)n^3)$ for some function $f : \mathbb{N} \to \mathbb{N}$.

Description:

Step 1. If G has a vertex of degree at most $27k - 1$, then we delete it. We continue this procedure until there are no vertices of degree at most $27k-1$. This can be done in linear time. Let G' be the resulting graph. Proceed to Step 2.

Step 2. Test if the tree-width of G' is small or not, say smaller than some value $g(k)$. For simplicity in later steps, we assume that $g(k) \ge N(k)$, where $N(k)$ is as in Theorem 1.6. This can be done in linear time by the algorithm of Bodlaender [7]. If the tree-width is at least $g(k)$, then go to Step 3. Otherwise, we use the linear-time algorithm of Arnborg and Proskurowski [3] to color G'. If G' can be colored by at most $27k$ colors, then we can extend the coloring to the whole graph G and stop. If G' cannot be colored with $27k$ colors, then we check if G' contains a K_k-minor. Again, this can be done by using the algorithm of Arnborg and Proskurowski [3] (or the algorithm of Robertson and Seymour [59]). If G' contains K_k as a minor, then we output that G contains K_k as a minor. If G' does not contain K_k as a minor, then we proceed as argued below. The whole process up to this point can be done in linear time.

Let (T, Y) be the corresponding tree-decomposition found above. The dynamic programming approach of Arnborg and Proskurowski assumes that T is a rooted tree whose edges are directed away from the root. For $tt' \in E(T)$ (where t is closer to the root than t'), define $S(t, t') = Y_t \cap Y_{t'}$ and $G'(t, t')$ be the induced subgraph of G' on vertices $\cup Y_s$, where the union runs over all nodes of T that are in the component of $T - tt'$ that does not contain the root. The algorithm of Arnborg and Proskurowski starts at all leaves of T and computes, for every $tt' \in E(T)$, the set $C(t, t')$ of all $27k$-colorings of $S(t, t')$ which can be extended to the whole $G'(t, t')$. If T has a vertex t of very large degree, then two neighbors t' and t'' have $S(t, t') = S(t, t'')$ and $C(t, t') = C(t, t'')$. Then $G'(t, t')$ can be deleted, and we still have a graph of bounded tree-width without K_k minor and without $27k$-colorings.

If all vertices of T have bounded degree, then T has a long path and there are distinct edges $t_1 t'_1$ and $t_2 t'_2$ on this path (where the second one is further from the root) such that $|S(t, t')| = |S(t, t'')|$ and $C(t, t') = C(t, t'')$. In the same way as argued in [9], we may assume that there are $|S(t, t')|$ disjoint paths joining $S(t, t')$ and $S(t, t'')$. By contracting these paths and replacing $G'(t, t')$ with $G'(t, t'')$, we get a minor of G' which is still of bounded tree-width, without K_k minor, and without $27k$-colorings. Repeating this, we eventually end up with the desired minor of G' of bounded size. (The bound is actually a doubly exponential value expressed in terms of k. More details on this part will be given in the full paper.)

Step 3. Test whether G' is $2k$-connected or not. Suppose first that G' is $2k$-connected. By the assumption in Step 2, we have $|G'| \geq g(k) \geq N(k)$. It follows from Theorem 1.6 that G' contains K_k as a minor. So, we output that G has K_k as a minor.

If G' is not $2k$-connected, then go to Step 4.

Step 4. G' is not $2k$-connected; detect a minimal separation (A, B). This can be done in polynomial time by standard methods. The best known algorithm is that of Henzinger, Rao, and Gabow [40] which needs $O(n^2)$ time for this task.

Let $S = V(A) \cap V(B)$. Then $|S| < 2k$. Let A'_1 be a component of $G' - S$ and $A'_2 = G' - A'_1 - S$. Let S_1 be a maximal independent set in the subgraph $G'(S)$ of G' induced on S. Let S_i be a maximal independent set in $G'(S) - \bigcup_{l=1}^{i-1} S_l$, for $i = 2, 3, \ldots$. Maximal independent sets can be found greedily or by means of any other method in constant time (since $|S| < 2k$). Next, we identify every nonempty set S_i into one vertex s_i $(i = 1, \ldots, r)$. Then the resulting graph on $S' = \{s_1, \ldots, s_r\}$ is a clique. Let A_1, A_2 be the corresponding graphs obtained from A'_1, A'_2 by adding the clique S' and the corresponding edges between A'_i and S'.

Finally, we test A_1, A_2 (recursively), starting from Step 1. If both graphs A_1, A_2 have $27k$-colorings, they give rise to a $27k$-coloring of their union since S' is a clique. Since vertex sets S_i $(i = 1, \ldots, r)$, that were identified into single vertices s_i in S', are independent in G', this coloring gives rise to a coloring of G'. Of course, we can extend this coloring of G' to G.

If one of the graphs, say A_1, contains K_k-minor, then we obtain a K_k-minor in G' (after contracting A_2 onto S') by using Lemma 7.2 with $d = 27k$. Similarly, if outcome (3) appears for A_1, we get the same outcome for G' by using a contraction of A_2 onto S'.

This algorithm stops when either the tree-width of the current graph is small or the current graph is $2k$-connected with minimum degree at least $27k$.

Now we shall estimate time complexity of the algorithm. All steps except the application of Lemma 7.2 can be done in time proportional to n^2. Another factor of n pops up because of applying the recursion in Step 4. Finally, Lemma 7.2 is applied only when we backtrack from the recursion. If we apply it on the graph A_1 of order n_1, we spend $O(n_1^3)$ time, but we never use it again on the same vertices. Therefore applications of Lemma 7.2 use only $O(n^3)$ time all together. This completes the proof of the correctness and of the stated time complexity of the algorithm. $\square$

6. COLORING IN ODD-MINOR-CLOSED CLASSES OF GRAPHS

We are ready to describe the algorithm of Theorem 1.8.

Algorithm for Theorem 1.8

Input: A graph G, a set $Z \subseteq V(G)$ with $|Z| \leq 192k$, and a precoloring of Z.

Output: As described in Theorem 1.8.

Running time: $O(f(k)n^4)$ for some function $f : \mathbb{N} \to \mathbb{N}$.

Description:

Step 1. If G has a vertex of degree at most $496k - 1$ in $G - Z$, then we delete it. We continue this procedure until there are no vertices of degree at most $496k - 1$ apart from Z. This can be done in linear time. Let G' be the resulting graph. Proceed to Step 2.

Step 2. Test whether G' has a $K_{32k,\, (16k-1)\binom{32k}{16k}+1}$-minor or not. If it has, then go to Step 3. Otherwise, go to Step 6. This can be done by the result of Robertson and Seymour [59] in $O(n^3)$ time.

Step 3. Find an even K_{16k}-minor by using the argument in the proof of Theorem 2.7. This can be done in linear time after detecting $K_{32k,\, (16k-1)\binom{32k}{16k}+1}$-minor in Step 2.

Step 4. Detect a separation X of order $|X| < 8k$ as described in Theorem 2.7. The proof in Geelen et al. [28] reduces our problem to the task of finding the maximum matching, which can be solved in $O(n^3)$ time, see [26, 55, 19, 27].

Step 5. We have one big component W in $G - X$ such that W contains a bipartite subgraph F and each odd cycle is contained in a component (after deleting a cut vertex to F) of $G - X$ that does not contain F. If there is either a block or a component, say B, in $G - X$ such that $|B \cap Z| + |X|$ is at least $192k$, then we first apply this algorithm to $B \cup X$ with $Z \cap B$ being the precolored set. Then we color F with two colors except possibly for the vertices in $Z \cup B$. Note that there is at most one such a component (after deleting a cut vertex to F) B by Fact 2.3. If there is no such a component (after deleting a vertex to F), then we select an arbitrary component B (after deleting a cut vertex to F), and then apply this algorithm to $B \cup X$ with $Z \cap B$ being the precolored set.

In each block and component B' of $G - X - B$ such that $|B' \cap Z| + |X| \leq 192k$, we apply this algorithm recursively with $X \cup \{v\} \cup (B' \cap Z)$ being precolored, where v is a cut vertex of $G - X - B$ if B' is a component (after deleting a cut vertex to F).

Step 6. Test whether G' has a separation (A, B) of order at most $96k$ such that both $B - A - Z$ and $A - B - Z$ are nonempty. Suppose first that G' has no such separation. If $|G'| \geq g(k) := \max\{N_1(k, x, z) \mid 0 \leq z \leq 192k\}$, where $x \geq (16k-1)\binom{32k}{16k} + 1$, then it follows from Theorem 2.2 that G' contains $K_{32k,\, (16k-1)\binom{32k}{16k}+1}$ as a minor. Hence this cannot happen by Step 2. If $|G'| < g(k)$, then we can check (1), (2) and (3) of Theorem 1.8 in constant time and stop.

If G' has such a separation, then go to Step 7.

Step 7. G' has a separation (A, B) of order at most $96k$ such that both $B - A - Z$ and $A - B - Z$ are nonempty; detect such a separation (A, B) of minimum order. This can be done in $O(n^2)$ time (Henzinger, Rao, and Gabow [40]).

By Fact 2.5, either $|(Z \cap A) \cup (A \cap B)| \leq 192k$ or $|(Z \cap B) \cup (A \cap B)| \leq 192k$. Suppose $|(Z \cap B) \cup (A \cap B)| \leq 192k$.

Then, we test $A \cup Z$ (recursively), starting from Step 1. If $A \cup Z$ has a coloring extending the precoloring of Z, then we precolor $A \cap B$ by this coloring. Then we test B (recursively) with $Z' = (Z \cap B) \cup (A \cap B)$ and using the corresponding precoloring of $A \cap B$ and $Z \cap B$. If B has a coloring extending the precoloring of $A \cap B$ and $Z \cap B$, this give rise to a coloring of their union, and this coloring gives rise to a coloring of G'. Of course, we can extend this coloring of G' to G.

If either $A \cup Z$ or B contains an odd-K_k-minor, then we obtain an odd-K_k-minor in G'. Similarly, if outcome (3) appears for $A \cup Z$, we get the same outcome for G'. Also, if (3) appears for B, we get the same outcome for G'.

This algorithm stops when either the current graph is small or when it has no separation (A, B) of order at most $96k$ such that both $B - A - Z$ and $A - B - Z$ are nonempty, and minimum degree is at least $496k$. In fact, in the second case, it has still bounded number of vertices since there is no $K_{32k,\,(16k-1)\binom{32k}{16k}+1}$-minor by Step 2 and Theorem 2.2. By Fact 2.6, it is easy to see that this algorithm correctly gives one of outcomes in Theorem 1.8. Also, in Steps 3, 4 and 5, by Theorem 2.7, since we know that G has no odd-K_k-minors, it must contain a separation X as described in Theorem 2.7. If there is either a block or a component, say B, in $G - X$ such that $|B \cap Z| + |X|$ is at least $192k$, then we can first apply this algorithm to $B \cup X$ with $Z \cap B$ precolored. Note that there is at most one such a block or a component B by Fact 2.5. Then we can color the bipartite subgraph F easily, and hence we can use our algorithm to each block and component B' in $G - X - B$ recursively. More precisely, in each block and component B' of $G - X - B$ such that $|B' \cap Z| + |X| \leq 192k$, we apply this algorithm recursively with $X \cup \{v\} \cup (B' \cap Z)$ being precolored, where v is a cut vertex of $G - X - B$ if B' is a component (after deleting a cut vertex to F).

Finally, let us estimate time complexity of the algorithm. All steps used in the algorithm can be done in quadratic time, except for detecting the minor of $K_{32k,\,(16k-1)\binom{32k}{16k}+1}$ in Step 2. This takes $O(n^3)$ by [59]. Also it takes $O(n^3)$ to detect X in Step 4, as we remarked just after the proof of Theorem 2.7. Another factor of n pops up because of applying the recursion in Step 5. So, in Step 5, we run $O(n^4)$ times. In Step 7, we need to detect the separation (A, B). This can be done by the algorithm of Henzinger, Rao, and Gabow [40] in $O(n^2)$ time. This has to be multiplied by a factor of n because of applying recursion in Step 7. Hence it takes $O(n^3)$ time in total. This completes the analysis of the correctness and of the stated time complexity of the algorithm.

Let us observe that we can detect the odd K_k-minor if the outcome (2) holds. To see this, if G' is small, then as in Step 6, we can use find one in constant time. If G' is $96k$-connected and has minimum degree at least $500k$ (see Theorem 7.1 below), then as in Step 2, we first detect a $K_{32k,\,(16k-1)\binom{32k}{16k}+1}$-minor by Robertson and Seymour [59]. Then the argument in the proof of Theorem 2.7 gives rise to detect the desired odd-minor, as we remarked just after the proof of Theorem 2.7. As we noted before, the proof of Theorem 2.8 in [28] implies polynomial time algorithm to find the desired conclusion of Theorem 2.8. Actually, it detects either a desired odd minor or a vertex set X of bounded size in G such that $G - X$ has a bipartite subgraph F and each odd cycle is contained in either components of $G - X$ that do not intersect F or blocks with a cut vertex to F. The time complexity is $O(n^3)$. Hence we can detect the desired odd-minor if the outcome (2) holds. $\square$

7. ALGORITHMIC ASPECT OF THE ODD HADWIGER CONJECTURE

In this section, we shall give sketch of the proof for Theorem 1.19.

Let us observe that Theorem 2.7 implies the following.

Theorem 7.1. *For any k, there exists a constant $N(k)$ such that every $500k$-connected graph with at least $N(k)$ vertices has either an odd K_k-minor or a vertex set X of order at most $8k$ such that $G - X$ is bipartite. In fact, the connectivity $500k$ can be replaced by $96k$ provided that the minimum degree is at least $500k$.*

To see this, by Theorem 2.1, there exists a constant $N(k)$ such that every $96k$-connected graph with minimum degree at least $500k$ and at least $N(k)$ vertices has the complete bipartite $K_{32k,\,(16k-1)\binom{32k}{16k}+1}$-minor. Thus, Theorem 2.7 implies Theorem 7.1 since G is $96k$-connected and has minimum degree at least $500k$, so there are no blocks and components as described in Theorem 2.8.

Let (A, B) be a separation in a graph G and let $S = V(A) \cap V(B)$. Let $S_1 \cup \cdots \cup S_r$ be a partition of S. We color all the vertices of S by 1. Then we say that A can be *contracted* to $S_1, \ldots, S_r$ if A contains pairwise disjoint trees $T_1, \ldots, T_r$ such that $S_i \subseteq V(T_i)$ for $i = 1, \ldots, r$, and in addition, T_i can be colored by 1 and 2 so that T_i is bichromatic and each vertex of S_i receives color 1 for all i. If there is a desired contraction in G, then we say that G has a *clique reduction*. We also say that G has a *trivial reduction* if there is a separation (A, B) of order at most $2496k - 2$ such that either $B - A$ or $A - B$ is bipartite. Our main result in this section is the following.

Lemma 7.2. *Let G be a graph with minimum degree $2496k$. Let (A, B) be a separation in G of minimum order, let $S = V(A) \cap V(B)$ and $s = |S|$. If $s \leq 96k$, then either there is a trivial reduction or for every partition $S_1 \cup \cdots \cup S_r$ of S, A (and B) can be contracted to $S_1, \ldots, S_r$. Hence there are clique reductions in both A and B.*

Our proof of Lemma 7.2 uses the method in [48]. It is concerning with the parity disjoint paths problems. The proof is lengthly, so we omit it, and refer the reader to the journal version of the first author's paper.

Let us point out that the corresponding disjoint connected trees $T_1, \ldots, T_r$ in A (resp. B) can be found in $O(n^3)$ time if there is no trivial reduction.

Lemma 7.2 has applications. Consider the following problem: *Does every $2497k$-chromatic graph have an odd-K_k-minor?*

Lemma 7.2 implies that there are only finitely many minimal counterexamples to the above problem. To see this, it is easy to prove that the minimum degree of a minimal counterexample G is at least $2497k - 1$. If there is a separation (A, B) of order at most $2497k - 3$ such that either $B - A$ or $A - B$, say $B - A$, is bipartite, then we are done since we can color A by $2497k - 1$ colors, and there are at most $2497k - 3$ colors used in $A \cap B$. Therefore, we can color $B - A$ by using two additional colors. Hence we may assume that there are no trivial reductions.

Let $S = V(A) \cap V(B)$. Then $|S| < 96k$. Let A'_1 be a component of $G' - S$ and $A'_2 = G' - A'_1 - S$. Let S_1 be a maximal independent set in the subgraph $G'(S)$ of G' induced on S. Let S_i be a maximal independent set in $G'(S) - \bigcup_{l=1}^{i-1} S_l$, for $i = 2, 3, \ldots$. Next, we identify every nonempty set S_i into one vertex s_i ($i = 1, \ldots, r$). Then the resulting graph on $S' = \{s_1, \ldots, s_r\}$ is a clique. Let A_1, A_2 be the corresponding graphs obtained from A'_1, A'_2 by adding the clique S' and the corresponding edges between A'_i and S'. Let us observe that both A_1 and A_2 are odd-minors of G since by Lemma 7.2, we can contract S into S' so that S' is a clique, and all the edges in this clique are monochromatic. If both graphs A_1, A_2 have $(2497k-1)$-colorings, they give rise to a $(2497k-1)$-coloring of their union since S' is a clique. Since vertex sets S_i ($i = 1, \ldots, r$), that were identified into single vertices s_i in S', are independent in G, this coloring gives rise to a coloring of G. Hence G is no longer counterexample, a contradiction.

This argument implies that minimal counterexample to the odd Hadwiger's conjecture for fixed k is $k/2497$-connected. This is the first result showing that every minimal counterexample has linear connectivity. Previously, it was only proved (Guenin [35]) that minimal counterexamples are 4-connected.

Having had Lemma 7.2, the algorithm for Theorem 1.19 is similar to that for Theorem 1.7. We omit it, and refer the reader to the full version of the first author's paper.

8. CONCLUSION

In this paper, we give polynomial time algorithms for list-colorings and arboreal colorings of graphs without K_k-minors, where the number of colors used is not much larger than needed. One challenging problem is, can we decide the list-coloring version of Hadwiger's conjecture? Theorem 1.15 decides the correctness of the weaker Hadwiger's conjecture in $O(n^3)$ time for every fixed k.

Our additive error of order $O(k)$ also gives a factor $O(k)$-approximation. Another challenging problem would be trying to improve the $O(k)$-approximation to $o(k)$-approximation in Theorem 1.1. This is not settled yet, but in our feeling, this may be hard. Approximating the list-chromatic number of graphs without K_k-minors within $o(k)$ may be NP-hard.

In [28], Geelen et al. proved that any graph with no odd K_k-minor is $O(k\sqrt{\log k})$-colorable. More precisely, they proved that there exists a constant c such that any graph with no odd-K_k-minor is $ck\sqrt{\log k}$-colorable. Their proof gives a polynomial time algorithm to color such a graph for fixed k with $ck\sqrt{\log k}$-colors, but for the completeness, we shall sketch the algorithm of the result. We remove vertices of degree at most $ck\sqrt{\log k}$ until there are no such vertices left. The resulting graph G' has minimum degree at least $ck\sqrt{\log k}$. By a simple local optimization we can find a spanning bipartite graph H such that the minimum degree of H is at least $ck\sqrt{\log k}/2$. This can be easily done in linear time. Let us assume that $c/32$ is greater than the constant in [67, 68, 54, 53]. Then H has a K_{16k}-minor by [67, 68, 54, 53]. Actually it is an even minor since H is bipartite. Hence, we can detect this even K_{16k}-minor by the result of Robertson and Seymour [59] in $O(n^3)$ time. As noted above, the proof of Theorem 2.8 in [28] implies polynomial time algorithm to find the desired conclusion of Theorem 2.8. Actually, it detects either a desired odd minor or a vertex set X of bounded size such that $G - X$ has a bipartite subgraph F and each odd cycle is contained in either components of $G - X$ that do not intersect F or blocks with a cut vertex to F. The time complexity is $O(n^3)$. Since we assume that G has no odd K_k-minor, the only case we need to handle is the second case. Let us observe that we have already got a kind of decomposition.

In order to use induction for the structure of this decomposition, Geelen et al. [28] used precoloring extension as we used in the proof of Theorem 1.2. See Theorem 13 of [28]. This certainly gives rise to a polynomial time algorithm to color the graph G with at most $ck\sqrt{\log k}$ colors. In fact, the complexity is $O(n^4)$.

It remains open whether or not there is a polynomial time algorithm for an $o(k)$-approximation of the chromatic number of graphs with no odd-K_k-minors.

In this paper, we also give a polynomial time algorithm for deciding whether linear lower bound on the chromatic number in terms of a parameter k is enough to force a K_k minor. Recently, Kawarabayashi and Mohar [46] proved that Theorem 1.6 can be improved as follows: For any k, there exists a constant $N(k)$ such that every $2k$-connected graph with minimum degree at least $9k$ and with at least $N(k)$ vertices has a K_k-minor. Also, if the tree-width is large (in a sense that we can apply Robertson and Seymour's result from [60] to G, see a detailed description in [9, 46]), then the minimum degree condition can be improved to $\frac{15a}{2}$. Also, Kawarabayashi proved in [42] that the order of the separation (A, B) in Lemma 7.2 can be reduced to $\frac{1}{6}k$. Together with these results, our algorithm implies that the chromatic number in Theorem 1.15 can be improved from $27k$ to $12k$.

Robertson and Seymour (private communication) have the following unpublished result, which would give rise to a polynomial-time algorithm for k-coloring K_k-minor free graphs if the Hadwiger Conjecture is true.

Theorem 8.1 (Robertson and Seymour). *For every fixed k, there is a polynomial-time algorithm for deciding either that*

(1) a given graph G is k-colorable, or

(2) G contains K_{k+1}-minor, or

(3) G contains a minor H without K_{k+1}-minors, of order at most $N(k)$, and with no k-coloring.

Neil Robertson (private communication) pointed out that in order to prove the above theorem, Robertson and Seymour used the following lemma, which is of independent interest, and perhaps would be the strongest result in this direction.

Lemma 8.2. *Let $k \geq 4$ be an integer. For any graph G with no K_{k+1}-minor, one of the followings holds:*

(1) There exists an integer $f(k)$ such that G has tree-width at most $f(k)$.

(2) G contains a vertex of degree at most k.

(3) G contains a vertex v of degree $k+1$ whose neighbors include three mutually nonadjacent vertices.

(4) G has a separation (A, B) of order at most k with $V(A) \neq V(G)$ such that A can be contracted to a clique on $A \cap B$ such that each vertex of $A \cap B$ is contained in the different node of this clique minor.

(5) G has a vertex set X, $|X| \leq k-4$, such that $G - X$ is planar.

Note that (2), (3) and (4) cannot happen in minimal counterexamples to Hadwiger's conjecture, and (5) is no longer counterexample, assuming the Four Color Theorem [1, 2, 58]. The proof is complicated and uses the graph minor structure theory (cf., e.g., [61, 60]) heavily (but does not use the well-quasi-ordering result). Paul Seymour also pointed out that outcome (3) can be eliminated on the expense of a considerably longer proof.

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