Quadrangulations and 5-critical Graphs on the Projective Plane

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Summary. Let Q be a nonbipartite quadrangulation of the projective plane. Youngs [You96] proved that Q cannot be 3-colored. We prove that for every 4-coloring of Q and for any two colors a and b, the number of faces F of Q, on which all four colors appear and colors a and b are not adjacent on F, is odd. This strengthens previous results that have appeared in [You96, HRS02, Moh02, CT04]. If we form a triangulation of the projective plane by inserting a vertex of degree 4 in every face of Q, we obtain an Eulerian triangulation T of the projective plane whose chromatic number is 5. The above result shows that T is never 5-critical. We show that sometimes one can remove two, three, or four, vertices from T and obtain a 5-critical graph. This gives rise to an explicit construction of 5-critical graphs on the projective plane and yields the first explicit family of 5-critical graphs with arbitrarily large edge-width on a fixed surface.

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1 Introduction

Youngs [You96] proved that a quadrangulation of the projective plane which is not bipartite is never 3-colorable, and its chromatic number is 4. Youngs' proof also implies that in any 4-coloring of a nonbipartite quadrangulation Q of the projective plane, there is a 4-face with all four vertices of distinct colors. This fact appears in a slightly extended version (where 4-colorings are replaced by k-colorings, $k \geq 3$) in [HRS02]. A strengthening of that result, proved in [Moh02], states that under every k-coloring of Q, there are at least three faces on which all four vertices have distinct colors. This in particular implies that $k \geq 4$. Collins and Tysdal [CT04] found that every 4-coloring of Q has a

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face F on which all four colors appear and in which colors 1 and 2 are used on opposite vertices of F. In this paper we give further extensions of these results in Theorems 2.1 and 3.1.

Let G be a graph and k a positive integer. We say that G is *k-critical* if its chromatic number is k, but every proper subgraph of G has chromatic number at most $k-1$.

For a fixed surface S and every $k \geq 8$, there are only finitely many k-critical graphs that can be embedded in S. This follows easily from Euler's formula and the fact that a k-critical graph cannot have vertices of degree smaller than $k-1$. Slightly more involved arguments imply that the same holds for 7-critical graphs on a fixed surface. See [Edw92, Moh93] or [MT01, Section 8.4]. Thomassen [Tho97] extended this to 6-critical graphs (and the proofs became much more complicated at this point). A corollary of these results is that for every surface S, there is a polynomial time algorithm to decide if a given graph G embedded in S has chromatic number at least 5, and if so, the algorithm also outputs $\chi(G)$.

On the other hand, 3-coloring planar graphs is NP-hard [GJ79], so it is NP-hard to 3-color graphs on any fixed surface. It is an open problem if 4-coloring on fixed surfaces is polynomially decidable. It is known to be so for planar graphs [RSST96], but unknown for other surfaces.

It is known that every nonplanar surface S contains infinitely many 5-critical graphs. There is a rather "straightforward" argument showing this. The proof goes as follows (see also [MT01, Corollary 8.4.13]). Let T be a triangulation of S such that all vertices of T have even degree except two of them whose degree is odd and are adjacent. Such triangulations are easy to construct on every nonplanar surface. They are called *Fisk triangulations* after a result of Fisk, who proved in [Fisk78] that T cannot be 4-colorable, since every 4-coloring of a triangulation with precisely two vertices of odd degree uses the same color on both vertices of odd degree. For every nonplanar surface S and every integer k, there exists a Fisk triangulation T_k on S whose *edge-width* (the length of a shortest noncontractible cycle) is at least k. Now, let R_k be a 5-critical subgraph of T_k. Since $\chi(R_k) \geq 5$, R_k is nonplanar and hence it contains a cycle that is noncontractible in the induced embedding. That cycle is also noncontractible in T_k and hence it has at least k vertices. In particular, $|R_k| \geq k$, and hence there are infinitely many nonisomorphic graphs in the sequence $R_1, R_2, R_3, \ldots$. The reader may have observed that the above simple argument is not entirely elementary since it uses the 4-color-theorem.

Fisk triangulations are never 5-critical. To see this, let T be a Fisk triangulation, and let x, y be its vertices of odd degree. Let M be the subgraph of T obtained by deleting the two edges xz, yz in a facial triangle containing the edge xy. The new face of M is bounded by a cycle C of length 5, and all vertices of M have even degree. Suppose now that M has a 4-coloring. Then some vertex $u \in V(C)$ has a color which does not appear on other vertices of C. By adding two edges connecting u with the two vertices which are "opposite" on C, we obtain a Fisk triangulation with a 4-coloring, a contradiction.

Let c be a coloring of a graph G. If G is embedded in some surface and F is a face, we say that F is *multicolored* if all vertices of F have distinct colors under the coloring c. The results about nonbipartite quadrangulations of the projective plane mentioned above show that by triangulating every face of such a quadrangulation by inserting a new vertex of degree 4, we obtain a triangulation whose chromatic number is 5, but such a graph is never 5-critical.

In Section 4, we describe some 5-critical subgraphs of triangulated quadrangulations, and we use them to show the following:

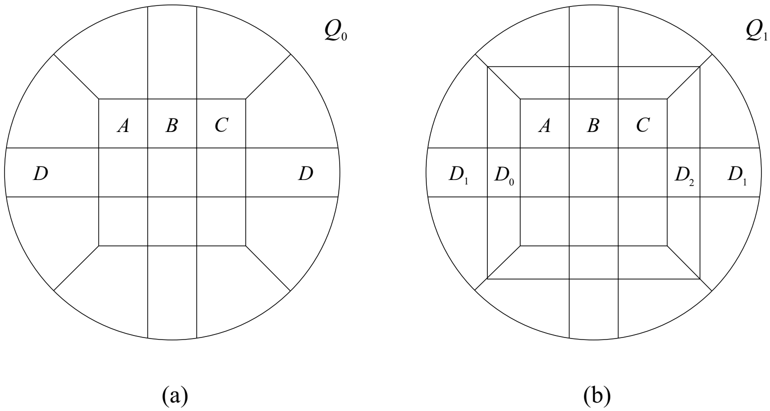

Fig. 1. Projective quadrangulations Q_0 and Q_1

Theorem 1.1. *Let Q_1 be the quadrangulation of the projective plane shown in Figure 1(b), and let A, B, C be the faces of Q_1 as shown in the same figure. Let Q be a quadrangulation of the projective plane that can be obtained by successively replacing a face distinct from A, B, C by one of the graphs K or L shown in Figure 2. Denote by $\mathcal{T}(Q; A, B, C)$ the graph obtained from Q by adding a new vertex in each face F distinct from A, B, C and joining that vertex to all four vertices on the facial walk of F. Then $\mathcal{T}(Q; A, B, C)$ is* 5*-critical.*

The proof of Theorem 1.1 is given in Section 4.

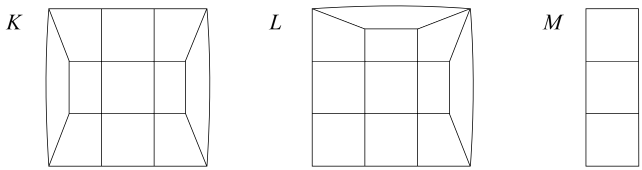

Fig. 2. Subdividers K, L, and M

Of course, replacements of faces of Q_1 by the graphs K and L from Figure 2 has to be done in such a way that a quadrangulation is obtained. In order to achieve this, replacement of a face F by K requires to replace also the faces above and below F (or left and right of F if K is rotated by angle $\pi/2$ before the replacement); similarly for L. After making the first round of replacements, we can repeat the procedure with the faces of the resulting quadrangulation, etc. In this way we can obtain quadrangulations whose edge-width is as large as we want. Henceforth, the family of graphs described in Theorem 1.1 is the first explicitly described family of 5-critical graphs with arbitrarily large edge-width on a fixed surface.

Other constructions of 5-critical graphs on the projective plane are given in Section 4.

Graphs in this paper are finite; multiple edges are allowed, while loops are excluded. If G is a graph, then $|G|$ denotes its order. A *quadrangulation* is a connected graph together with a 2-cell embedding in a closed surface such that every facial walk is of length 4. Note that a pair of edges joining the same pair of vertices can form a contractible curve on the surface of the quadrangulation, but it does not bound a face since all faces are of lenth 4.

2 Quadrangulations of the Sphere

Let Q be a quadrangulation of the sphere. It is easy to see that Q is bipartite and hence 2-colorable. Therefore, it may seem surprising if we would be able to say something about global properties of 4-colorings of Q. In this section, we uncover some surprising properties.

Let c be a k-coloring of an embedded graph. We say that a face F is *multicolored* if all its vertices have distinct colors. Suppose that the coloring uses colors $0, 1, \ldots, k-1$. If $F = xyzw$ is a multicolored 4-face and $c(x) = 0$, then the color $c(z)$ of z is said to be the *type* of F. If the surface is orientable, we may assume that all faces are equipped with the positive orientation. Assuming this and assuming that $k = 4$, we can refine the notion of the type of multicolored 4-faces. We say that the *type* of F is $1^-, 1^+, 2^-, 2^+, 3^-$, or 3^+ if the clockwise cyclic order of colors on F is $0213, 0312, 0123, 0321, 0132$, or 0231, respectively. (So, the "minus" in the type says that $c(y) < c(w)$, and the "plus" says the converse.)

Theorem 2.1. *Let Q be a quadrangulation of the sphere and c its 4-coloring. Then for every type $t \in \{1, 2, 3\}$, the number of multicolored faces of type t^- has the same parity as the number of multicolored faces of type t^+. In particular, the number of multicolored faces of type t is even.*

Proof. The proof is by induction on the number N which is the sum of the number of vertices $n = |Q|$ and the number of multicolored faces of Q. If Q has no multicolored faces, there is nothing to prove. So, we assume that there is at least one multicolored face (and hence we have $n \geq 4$).

If Q has a face $F = xyzw$ in which $c(x) = c(z)$, then we identify x and z (and delete F and replace resulting parallel edges by single edges). This operation gives rise to a smaller 4-colored quadrangulation Q'. We say that Q' has been obtained by *squeezing* F. Clearly, all other faces and their coloring remain the same as in Q, so we just apply the induction hypothesis to Q' and thus complete the proof.

In the next paragraph we will apply an operation for which we need that there are no parallel edges in Q. If there were, we could apply the induction hypothesis, first to the interior of the disk bounded by a couple of parallel edges between vertices x, y, and then to the exterior of the same disk.

We may now assume that all faces of Q are multicolored. Let xy be an edge of Q, and let $F = xyzw$ and $F' = xw'z'y$ be the facial walks containing the edge xy. Supose that $c(z) = c(z')$ (and hence $c(w) = c(w')$), i.e., F and F' are of the same type. In this case we remove faces F and F', identify w with w' and y with y' and keep only single edges between x and w and y and z, respectively. This gives rise to a smaller 4-colored quadrangulation Q' in which the faces have the same types as before. By applying the induction hypothesis and observing that F and F' were of the same type, but with different signs, we easily complete the proof.

From now on we also assume that no adjacent faces are of the same type. Then it is easy to see that every face of type t^+ or t^- is adjacent to four faces of precisely the same four types as shown in Figure 3. This implies that the dual graph of Q is a covering graph of the dual graph of the cube (which is the octahedron graph) shown in the figure. In particular, the number of faces of any of the types t^+ or t^- is equal to the degree of the corresponding covering projection. This completes the proof. □

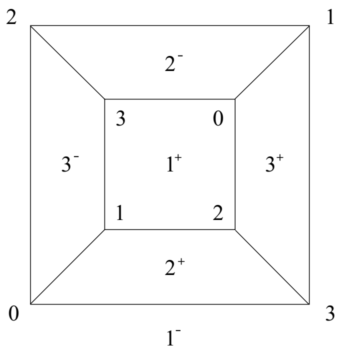

Fig. 3. Types of faces in a 4-colored cube

If Q is a connected plane graph whose interior faces are all 4-sided and the exterior face is of length k, then we say that Q is a *k-near-quadrangulation*. Note that k must be even since Q is bipartite.

Corollary 2.2. *Let Q be a planar 6-near-quadrangulation and let c be a 4-coloring of Q. For $t \in \{1,2,3\}$, let m_t be the number of multicolored 4-faces of Q whose type is t. Then $m_1 \equiv m_2 \equiv m_3 \equiv 1 \pmod 2$ if and only if under the coloring c, every pair of opposite vertices on the exterior face are colored the same.*

Proof. If a pair of opposite vertices x, y have distinct colors, we add the edge xy and obtain a 4-colored quadrangulation. With respect to the coloring of this quadrangulation, at most two of the values m_t can change. Therefore, the one that has not been changed is even by Theorem 2.1. On the other hand, if opposite vertices have the same colors, we quadrangulate the outer face by adding a vertex of degree 3 and color this vertex by the color that does not appear on the exterior face. Then the new faces are all multicolored and of all three types. Again, Theorem 2.1 shows that the values m_1, m_2, and m_3 (in Q) are all odd. This completes the proof. □

We will need another result which will enable us to construct colorings with few multicolored faces.

Lemma 2.3. *Let Q be an 8-near-quadrangulation that is isomorphic to one of K or L shown in Figure 2, and let x, y, z, w be the four corner vertices. Suppose that c_0 is a 4-coloring of x, y, z, w such that $c_0(x) \neq c_0(y) \neq c_0(z) \neq c_0(w) \neq c_0(x)$. For every interior 4-face F of Q, there exists a 4-coloring c of Q which extends c_0 and has the following properties:*

(a) *The vertices on the segment of the outer face from x to y are 2-colored with colors $c_0(x)$ and $c_0(y)$, the vertices from y to z with colors $c_0(y)$ and $c_0(z)$, and similarly for the segments from z to w, and from w to x. There is one exception to this rule if F is the "middle" face having an edge on one of these segments, as shown in Figure 4(d_1) and (d_2) for the segment from x to y. In that case, one of the vertices is not colored as stated, and we may choose either of the two vertices to be this exception. See Figure 4(d_1) and (d_2), where the exceptions are emphasized by little circles.*
(b) *No interior face different from F is multicolored.*
(c) *The colors on F in the clockwise cyclic order are either $c_0(x), c_0(y), c_0(z), c_0(w)$, or the reverse of this.*

Proof. Extensions (up to symmetries) are shown in Figure 4, where a $= c_0(x)$, b $= c_0(y)$, c $= c_0(z)$, and d $= c_0(w)$; the face F is shaded. □

Lemma 2.3 has a generalization which we include for the sake of completeness.

Proposition 2.4. *Let Q be a quadrangulation of the sphere without multiple edges. If F_1, F_2 are distinct faces of Q, then there is a 4-coloring of Q such*

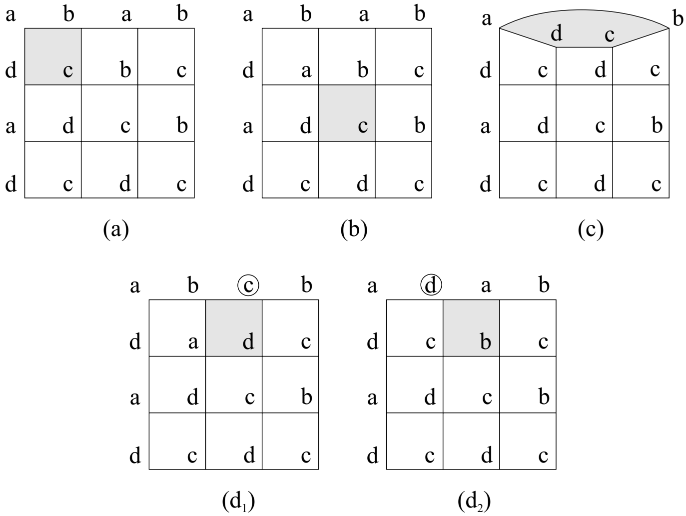

Fig. 4. Extensions of c_0 in K or L

that F_1 and F_2 are multicolored but no other face is multicolored. If the sequence of colors in the clockwise direction on F_1 is $c_1c_2c_3c_4$, then the cyclic sequence of colors on F_2 is its reverse $c_1c_4c_3c_2$. In particular, F_1 and F_2 are of the same type $t \in \{1,2,3\}$, but of different signs, i.e., one is of type t^+, the other one of type t^-.

Proof. The proof is by induction on $n = |Q|$. If Q is just a 4-cycle, the statement clearly holds. If Q contains a 4-cycle C that is not facial, let Q_1 be its interior. We may assume that F_1 is a face of Q_1. Let Q_2 be the exterior of C. Now, C is facial in Q_1 and in Q_2. If F_2 is a face in Q_2, then we apply induction to Q_1 (with faces F_1 and C to be multicolored) and then to Q_2 (with multicolored C and F_2). By permuting the colors in Q_2, the two colorings coincide on C, and their union gives a required coloring of Q.

Suppose now that F_2 is in Q_1. Then we first apply induction on Q_1 with F_1, F_2 to be multicolored. The face C is not multicolored. We may assume that its coloring is 1213 or 1212. Let $x \in V(C)$ be the vertex of color 3 (if it exists). Now we color Q_2 with colors 1 and 2 (except for the vertex x) so that its coloring on C coincides with the coloring from Q_1. The colorings of Q_1 and Q_2 can now be combined to obtain a required coloring of Q.

Suppose now that every 4-cycle in Q is facial. There is a face $F = xyzw$ distinct from F_1 and F_2. Now, let Q' be obtained by identifying x and z and squeezing F. Since every 4-cycle is facial, Q' is a quadrangulation. However, we have to be certain that Q' does not contain parallel edges (since then the claim may not hold any more). By excluding parallel edges, we also make sure that each of the faces F_1 and F_2 has four distinct vertices in Q'. If Q' has parallel edges, then x and z have a common neighbor $u \notin \{y, w\}$. Since Q is planar and bipartite, y and w cannot have a common neighbor distinct from x and z. Therefore, we can squeeze F by identifying y and w instead.

By applying the induction hypothesis to Q', we get a coloring of Q' which can also be used as a required coloring of Q. □

3 Quadrangulations of the Projective Plane

Let G be a graph embedded in a surface S. If C is a cycle in G that is contractible in the surface, then C bounds a disk in S. That disk (together with its boundary C) is called the *interior* of C.

In this section we prove an extension of known results about 4-colorings of nonbipartite quadrangulations of the projective plane obtained in [You96, HRS02, Moh02, CT04]. Claim (a) of Theorem 3.1 is reproduced from [Moh02]. Part (b) extends a result that was obtained previously (with an essentially different proof) by Collins and Tysdal [CT04].

In the proof of Theorem 3.1, we will use the following easy fact. In a nonbipartite quadrangulation of the projective plane, all contractible cycles have even length and all noncontractible cycles have odd length. In particular, if we allow multiple edges, they can never form a noncontractible 2-cycle.

Theorem 3.1. *Let Q be a nonbipartite quadrangulation of the projective plane, and let k be an integer.*

(a) *If Q is k-colored, then there are at least three multicolored faces. In particular, $k \geq 4$.*
(b) *If $k = 4$, then for every type $t \in \{1, 2, 3\}$, the number of multicolored faces of type t is odd.*

Proof. Suppose that Q is not bipartite, that it is k-colored, and that it is a counterexample to either (a) or (b) with the minimum number of vertices. Let $\mathcal{F}_1$ be the set of multicolored faces. Denote by $\mathcal{F}$ the set of all faces which are not in $\mathcal{F}_1$.

We allow multiple edges. However, we shall prove that they do not appear in Q. If Q would have a pair of edges joining vertices x and y, the corresponding 2-cycle C would be contractible since all noncontractible cycles are odd. Hence, C would bound a disk D. By deleting the interior of D and identifying the two parallel edges, we would get a smaller counterexample (where we apply Theorem 2.1 to D when proving (b)), which would contradict our choice of Q.

Suppose now that Q has a facial walk $xyzw \in \mathcal{F}$ such that x and z have the same color. Clearly, $x \neq z$ since Q has no parallel edges. Then we can squeeze the face by identifying x and z. The resulting graph is a nonbipartite k-colored quadrangulation of the projective plane with the same multicolored faces, which yields a contradiction to the minimality of Q.

The conclusion of the above is that $\mathcal{F} = \emptyset$ (and that Q does not have multiple edges). Since $\mathcal{F}_1 \neq \emptyset$, the quadrangulation Q has $n \geq 4$ vertices. For quadrangulations on the projective plane, Euler's formula implies that the number of faces is $n-1 \geq 3$. This proves (a). As it only remains to prove (b), we assume henceforth that $k = 4$.

Now we proceed in the same way as in the proof of Theorem 2.1. First we exclude the case when two faces of the same type share an edge. Having done that, we conclude that the dual graph of Q covers the dual graph R of the K_4-quadrangulation. Observe that R has three vertices and that any pair of them is joined by two edges forming a noncontractible 2-cycle in the projective plane. It follows that the number of faces of Q is $3d$, where d is the degree of the covering projection. If $d = 1$, then $Q = K_4$, which has a unique 4-coloring, with one multicolored face of each type. This concludes the proof for $d = 1$.

If $d > 1$, then Q has a vertex u of degree 3 which is adjacent to a vertex v whose degree is more than 3. (This follows from Euler's formula by using standard counting arguments.) In this case, the coloring around u and v is as shown in Figure 5, and we can make a reduction shown in that figure, and finally apply the induction hypothesis. There are some minor technical details about this reduction that are worth mentioning. First, the reduction shown in Figure 5 gives rise to a loopless 4-colored graph since the added edges join vertices of distinct colors. However, there is a possible trouble if $x = u$. In that case, the vertex x is not present after the deletion of u. If this happens, we apply a similar reduction at the vertex y (see Figure 5). Let us observe that in this case $y \neq u$, since if it were, u would be contained in four quadrangular faces, contradicting the fact that its degree is 3. Lastly, by performing this reduction, four multicolored faces of types 1, 2, 2, 3 are replaced with two multicolored faces, whose types are 1 and 3, and one face which is not multicolored. Hence, the parities of the numbers of multicolored faces of specific types remain unchanged.

This completes the proof. □

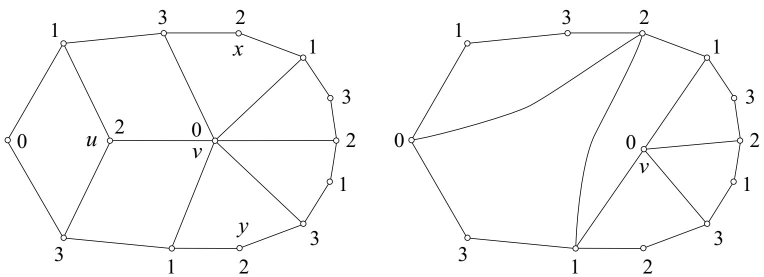

Fig. 5. Another reduction

Let Q be a quadrangulation and $\mathcal{F}$ a collection of some of the faces of Q. The pair $(Q, \mathcal{F})$ is called a *bordered quadrangulation*, and we think of it as being the 2-cell complex obtained from Q by deleting the 2-cells corresponding to the faces in $\mathcal{F}$. A 4-coloring of the bordered quadrangulation is *soft* if all faces of Q distinct from those in $\mathcal{F}$ have at most three colors. Observe that this notion is natural when passing from (bordered) quadrangulations to (bordered) triangulations in which each face is triangulated. A 4-coloring of Q can be extended to a 4-coloring of the corresponding triangulations if and only if it is soft. See Section 4.

A corollary of Theorem 3.1(b) is the following

Lemma 3.2. *Let A, B, C be distinct faces of a nonbipartite quadrangulation Q of the projective plane. If $B = xyzw$ and $xy \subseteq A \cap B$, $zw \subseteq B \cap C$, then $(Q, \{A, B, C\})$ has no soft 4-coloring.*

Proof. If c is a soft 4-coloring of $(Q, \{A, B, C\})$, then A, B, C are all multicolored by Theorem 3.1(b), and they are of diffierent types. We may assume that $c(x) = 0$, $c(y) = 1$, $c(z) = 2$, $c(w) = 3$, so B is of type 2. Then A must be of type 3. Now, color 1 is opposite to z or opposite to w in C. It follows that C cannot be of type 1, a contradiction. □

Now we shall add some specific examples.

Lemma 3.3. *Let A, B, C, D be the faces of the projective planar quadrangulation Q_0 as shown in Figure 1(a). Then the bordered quadrangulation $(Q_0, \{A, B, C, D\})$ has no soft 4-colorings. On the other hand, if $F \notin \{A, B, C, D\}$ is another face of Q_0, then $(Q_0, \{A, B, C, F\})$ admits a soft 4-coloring.*

Proof. Soft colorings of $(Q_0, \{A, B, C, F\})$ are shown in Figure 6 for different choices of F (up to symmetries). The types of the three multicolored faces are shown inside small circles.

To prove the first part of the lemma, let $A = xyzw$, where $x \in V(D)$, $B = wzts$ and $C = stuv$, where $v \in V(D)$ is opposite to x in D. Suppose now that $(Q_0, \{A, B, C, D\})$ has a soft 4-coloring c. By Theorem 3.1(b), precisely three of the faces A, B, C, D are multicolored. By Lemma 3.2, D is necessarily one of them, and we will assume that it is of type 1 and that $c(x) = 0$ and $c(v) = 1$.

Suppose first that A, B, D are multicolored. Since A is not of type 1 and colors 2, 3 have not yet been introduced, we may assume that A is of type 2, so that $c(z) = 2$. Then B is of type 3, and hence $c(t) = 0$ and $c(w) = 3$. Since B is multicolored, it follows that $c(s) = 1$, a contradiction since its neighbor v is colored 1.

The case when B, C, D are multicolored is symmetric to the above, so it remains to consider the case when A, C, D are multicolored. Again, we may assume that A is of type 2, so $c(z) = 2$. Since C is of type 3, we conclude that $\{c(s), c(u)\} = \{0, 3\}$ and $c(t) = 2$. But this is a contradiction since the neighbor z of t is also colored 2. □

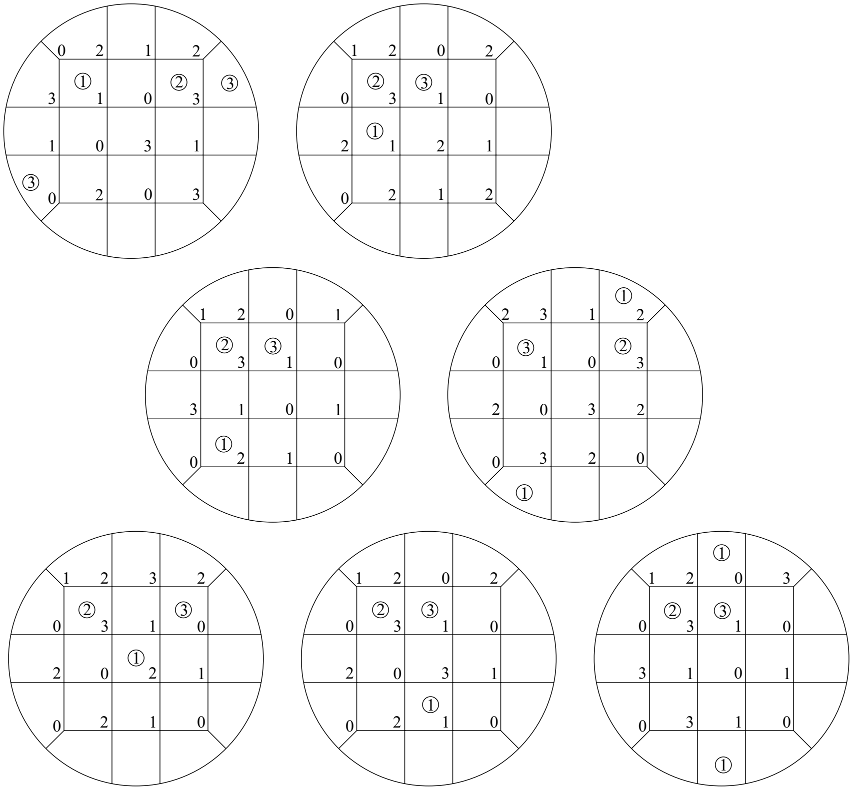

Fig. 6. Soft colorings of $(Q_0, \{A, B, C, F\})$

A bordered quadrangulation $(Q, \mathcal{F})$ is *soft-4-critical* if it does not have a soft 4-coloring but for every face F of Q, where $F \notin \mathcal{F}$, $(Q, \mathcal{F} \cup \{F\})$ has a soft 4-coloring.

Some soft-4-critical bordered quadrangulations of the form $(Q_0, \mathcal{F})$ are presented in Figure 7, where the faces in $\mathcal{F}$ are represented by circles. Criticality of the first one is a direct consequence of Lemma 3.3. For the other two, the proof is similar, and we leave details to the reader.

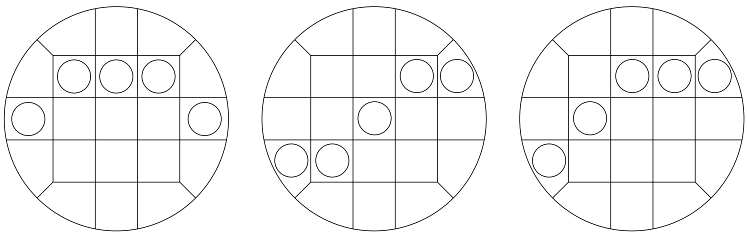

Fig. 7. Some soft-4-critical bordered quadrangulations

An unfortunate property of the above examples is that they cannot be used directly to construct soft-4-critical bordered quadrangulations of large edge-width. This is repaired in the examples given in the sequel.

Let Q_0 be the quadrangulation shown in Figure 1(a). For $k \geq 1$, let Q_k be the projective planar quadrangulation obtained from Q_0 as follows. First, subdivide the six edges e_i ($1 \leq i \leq 6$) passing "through the crosscap" (the outside edges), each with $2k$ new vertices $v_{i,j,l}$, where $1 \leq j \leq k$ and $l = 1, 2$. These vertices subdivide e_i in the respective order $v_{i,1,1}, v_{i,2,1}, \ldots, v_{i,k,1}, v_{i,k,2}, \ldots, v_{i,1,2}$. Finally, we add k cycles of lenth 12, each surrounding the "outer" 12-cycle of the 3×3 grid in Q_0. The jth cycle passes through vertices $v_{1,j,1}, \ldots, v_{6,j,1}, v_{1,j,2}, \ldots, v_{6,j,2}$. In Figure 1, quadrangulations Q_0 and Q_1 are shown.

Theorem 3.4. *Let $k \geq 1$ be an integer, and let A, B, C be the faces of the projective quadrangulation Q_k as shown in Figure 1. Then the bordered quadrangulation $(Q_k, \{A, B, C\})$ is soft-4-critical.*

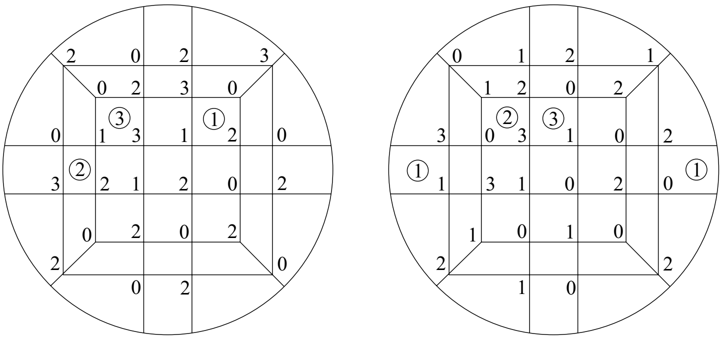

Fig. 8. Soft colorings for D_0 and D_1

Proof. Let us first observe that Q_1 is obtained from Q_0 by replacing each of the six "exterior" faces $F_1, \ldots, F_6$ by three quadrangles, i.e., they are replaced by subdividers M shown in Figure 2. Similarly, Q_k ($k \geq 2$) is obtained from Q_{k-1} by replacing six faces by subdividers M. In the sequel, we give details for the proof that Q_1 is soft-4-critical, and leave the general case to the reader.

Let $F_1, \ldots, F_6$ be the "exterior" faces of Q_0, and let $e_i = a_i b_i$ be the edge shared by F_{i-1} and F_i ($1 \leq i \leq 6$; all indices are considered modulo 6). In other words, the "outer" 12-cycle of the central 3×3-grid in Q_0 is $a_1 a_2 \ldots a_6 b_1 b_2 \ldots b_6$. In Q_1, e_i is subdivided by two vertices a_i', b_i', where we assume that b_i' is adjacent to a_i, and a_i' is adjacent to b_i. Here we slightly adapt the convention about the notation modulo 6 and assume that $(a_1, b_1', a_1', b_1) = (b_7, a_7,' b_7', a_7)$.

Suppose that c is a 4-coloring of Q_0 with the following properties:

(a) At most one of the faces F_i, $1 \le i \le 6$, is multicolored.
(b) If $F = F_i = a_i a_{i+1} b_{i+1} b_i$ is multicolored, then either the color $c(a_i)$ appears twice in F_{i-1} and $c(b_{i+1})$ appears twice in F_{i+1}, or $c(b_i)$ appears twice in F_{i-1} and $c(a_{i+1})$ appears twice in F_{i+1}.

Now we extend c to Q_1 as follows. If neither F_i nor F_{i-1} is multicolored, we set $c(a_i') := c(a_i)$ and $c(b_i') := c(b_i)$. If F_i is multicolored, then we assume that the color $c(a_i)$ appears twice in F_{i-1} and $c(b_{i+1})$ appears twice in F_{i+1}. (The other possibility provided by (b) can be handled similarly.) Then we set $c(a_i') := c(a_i)$ and $c(b_{i+1}') := c(b_{i+1})$. This choice guarantees that the new faces in F_{i-1} and F_{i+1} will not be multicolored. For the colors of b_i' and a_{i+1}' we use one of the following three possibilities: twice the color $c(b_i)$, colors $c(a_{i+1})$ and $c(b_i)$ (respectively), or twice the color $c(a_{i+1})$. Each of these three possibilities gives rise to a different multicolored subface of F_i.

The described extension of the colorings from Figure 6 (which satisfy (a) and (b)) prove that $(Q_1, \{A, B, C, F\})$ has a soft 4-coloring for all faces $F \notin \{A, B, C, D_0, D_1, D_2\}$ (where the faces D_0, D_1, D_2 are those shown in Figure 1(b). Finally, for $F \in \{D_0, D_1\}$, the corresponding soft 4-colorings are exhibited in Figure 8, and the case of $F = D_2$ is symmetric to the case when $F = D_0$. The proof is complete. □

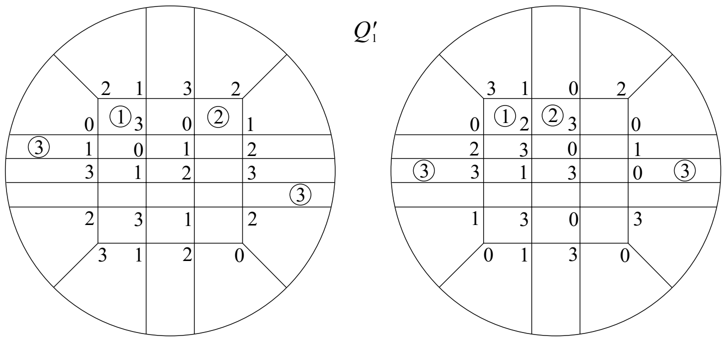

Fig. 9. Two 4-colorings of the projective quadrangulation Q_1'

Another family of quadrangulations can be derived from Q_0. If we replace each of the four faces in the "middle" by the subdivider M (see Figure 2), we obtain Q_1', which is shown in Figure 9. If we repeat the same with the new four faces forming the middle Möbius strip, we obtain Q_2'. By doing this k times all together, we get Q_k' $(k \ge 1)$.

Theorem 3.5. *Let $k \ge 1$ be an integer, and let A, B, C be the faces of the projective quadrangulation Q_k' corresponding to the faces of Q_0 shown in Figure 1. Then the bordered quadrangulation $(Q_k', \{A, B, C\})$ is soft-4-critical.*

Proof. The proof is similar to the proof of Theorem 3.4. The necessary soft 4-colorings of $(Q_1', \{A, B, C, D_0\})$ and $(Q_1', \{A, B, C, D_1\})$ are shown in Figure 9 (where D_0 and D_1 are the multicolored faces shown in the figure). The details are omitted. □

4 Eulerian Triangulations

A graph is *Eulerian* if all its vertices have even degree. It is well known that Eulerian triangulations of the plane are 3-colorable. However, Eulerian triangulations on other surfaces may have arbitrarily large chromatic number. It is easy to find examples on the projective plane whose chromatic number is equal to 3, 4, or 5, respectively, and it is easy to see that the chromatic number of an Eulerian triangulation of the projective plane cannot be more than 5. In [Moh02], a simple characterization and a polynomial time algorithm are given to decide if an Eulerian triangulation of the projective plane is 3-, 4-, or 5-colorable.

The class of graphs embedded in some surface S such that all facial walks have even length (called *locally bipartite embeddings*) is closely related to Eulerian triangulations of S. Namely, if we insert a new vertex in each of the faces of a locally bipartite embedded graph G, and join it to all vertices on the corresponding facial walk, we obtain an Eulerian triangulation $\mathcal{T}(G)$ which contains G as a subgraph. We say that $\mathcal{T}(G)$ is a *face subdivision* of G and that the set of added vertices $U = V(\mathcal{T}(G)) \setminus V(G)$ is a *color factor* of $\mathcal{T}(G)$. Since U is an independent set, $\chi(G) \leq \chi(\mathcal{T}(G)) \leq \chi(G) + 1$, where $\chi(\cdot)$ denotes the chromatic number of the corresponding graph.

Theorem 3.1 applied to a nonbipartite projective planar quadrangulation Q implies that the chromatic number of its face subdivision $\mathcal{T}(Q)$ is equal to 5. Theorem 3.1 also implies that $\mathcal{T}(Q)$ is not 5-critical since the removal of any two vertices of degree 4 in $\mathcal{T}(Q)$ leaves a graph which is not 4-colorable.

Eulerian triangulations of the projective plane with chromatic number 5 may have arbitrarily large face-width and they show that nonorientable surfaces behave differently than the orientable ones. Namely, Hutchinson, Richter, and Seymour [HRS02] proved that Eulerian triangulations of orientable surfaces of sufficiently large face-width are 4-colorable.

The concept of face subdivisions extends to bordered surfaces. Given a bordered quadrangulation $(Q, \mathcal{F})$, we define $\mathcal{T}(Q, \mathcal{F})$ in the same way as above, except that we do not subdivide the faces in $\mathcal{F}$. If c is a 4-coloring of $\mathcal{T}(Q, \mathcal{F})$, then its restriction to Q is a soft 4-coloring of $(Q, \mathcal{F})$. Conversely, every soft 4-coloring of $(Q, \mathcal{F})$ can be extended to a 4-coloring of $\mathcal{T}(Q, \mathcal{F})$.

Our next result shows that soft-4-criticality of bordered quadrangulations is essentially equivalent to 5-criticality of their face subdivisions.

Theorem 4.1. *Let $(Q,\mathcal{F})$ be a soft-4-critical bordered quadrangulation of the projective plane. If every (contractible) 4-cycle of Q bounds a face of Q, then the graph of the face subdivision $\mathcal{T}(Q,\mathcal{F})$ is 5-critical.*

Proof. Since $(Q,\mathcal{F})$ has no soft-4-colorings, $T = \mathcal{T}(Q,\mathcal{F})$ cannot be 4-colored. Thus, it suffices to see that the removal of any edge uv of T yields a 4-colorable graph.

Suppose first that u is a vertex which is not in Q, i.e., u subdivides some face $F \notin \mathcal{F}$. Soft-4-criticality of $(Q,\mathcal{F})$ implies that there is a soft 4-coloring of $(Q,\mathcal{F}\cup\{F\})$. This 4-coloring can be extended to a 4-coloring of $T-u$ since it is soft, and it can further be extended to u in $T-uv$ since u has degree 3 in $T-uv$. This proves that $T-uv$ is 4-colorable.

Suppose now that $uv \in E(Q)$. Since every contractible 4-cycle of Q bounds a face in Q, $Q-uv$ is 3-colorable, as proved by Gimbel and Thomassen [GT97] (cf. also [MS02]). Obviously, a 3-coloring of $Q-uv$ can be extended to a 4-coloring of $T-uv$. This completes the proof. □

Theorem 4.1 (together with results of Section 3) gives rise to 5-critical graphs on the projective plane. By Theorems 3.4 and 3.5, graphs $\mathcal{T}(Q_k, \{A,B,C\})$ and $\mathcal{T}(Q'_k,\{A,B,C\})$ are 5-critical for every $k \geq 1$. Theorem 4.1 and Lemma 2.3 imply that adding subdividers K and L in faces distinct from A,B,C in such a way that another quadrangulation is produced, yields new soft-4-critical bordered quadrangulations. Consequently, new 5-critical graphs are obtained as their face subdivisions. This in particular proves Theorem 1.1.

Let us observe that the edge-width of $\mathcal{T}(Q'_k,\{A,B,C\})$ can also be made arbitrarily large by using the subdividers K and L (indeed only K suffices). Both of these constructions yield the first explicit families of 5-critical graphs of arbitrarily large edge-width on a fixed surface.

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