On polyhedral embeddings of cubic graphs

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January 29, 2004

Abstract

Polyhedral embeddings of cubic graphs by means of certain operations are studied. It is proved that some known families of snarks have no (orientable) polyhedral embeddings. This result supports a conjecture of Grünbaum that no snark admits an orientable polyhedral embedding. This conjecture is verified for all snarks having up to 30 vertices using computer. On the other hand, for every non-orientable surface S, there exists a non 3-edge-colorable graph, which polyhedrally embeds in S.

Keywords: polyhedral embedding, cubic graph, snark, flower snark, Goldberg snark.

1 Introduction

In this paper we study embeddings of cubic graphs in closed surfaces. We refer to [5] for basic terminology and properties of embeddings. Following the approach of [5], all embeddings are assumed to be 2-cell embeddings. Two embeddings of a graph are considered to be (combinatorially) equal, if they have the same set of facial walks. If S is a surface with Euler characteristic $\chi(S)$, then $\epsilon(S) := 2 - \chi(S)$ is a non-negative integer, which is called the *Euler genus* of S.

If an embedding of a graph G in a non-orientable surface is given by a rotation system and a signature $\lambda : E(G) \to \{+1, -1\}$ and H is an acyclic subgraph of G, then we can assume that the edges of H have positive signature, $\lambda(e) = 1$ for all $e \in E(H)$. We shall assume this in several instances without explicitly mentioning it. Instead of describing an embedding by specifying rotation system an signature, it suffices to list all facial walks.

An embedding of a graph G is called *polyhedral*, if all facial walks are cycles and any two of them are either disjoint or their intersection is a vertex or an edge. If G is a cubic graph, then any two facial walks are either disjoint or they intersect in an edge. It is also easy to see (cf. [5]) that every facial cycle of a polyhedral embedding is induced and non-separating. There is another way of looking at polyhedral embeddings (cf., e.g. [5], [6]).

Proposition 1.1 *An embedding of a graph G is polyhedral if and only if G is 3-connected and the embedding has face-width at least 3.*

If 1967 Grünbaum proposed a far-reaching generalization of the four color theorem (which had not yet been proved at that time). It is well known that the four color theorem is equivalent to the statement that every 3-connected planar cubic graph is 3-edge-colorable. This is no longer true for 3-connected cubic graphs on the torus since the Petersen graph P embeds in this surface. However no embedding of P in the torus is polyhedral. The lack of orientable polyhedral embeddings of other non 3-edge-colorable cubic graphs known at that time led Grünbaum to the following

Conjecture 1.2 (Grünbaum [2]) *If a cubic graph admits a polyhedral embedding in an orientable surface, then it is 3-edge-colorable.*

This Conjecture has been checked for all cubic graphs having up to 30 vertices (see Section 6). Cubic graphs that do not have 3-edge-colorings are said to be of *class 2*.

By Proposition 1.1 it suffices to check this Conjecture for 3-connected cubic graphs. It is not difficult to see that we may also restrict our attention to cyclically 4-edge-connected cubic graphs (cf. Theorem 3.1). Cyclically 4-edge-connected cubic graphs of class 2 and with girth at least 5 are commonly known as *snarks*.

In Section 2 it is proved that short cycles are necessarily facial in polyhedral embeddings of cubic graphs. In Section 3 we study reductions of graphs with nontrivial k-edge-cuts for $k \leq 5$. In Section 4 it is proved that Isaacs graphs [4], except for the smallest one J_3 (which is just a one-vertex-truncation of the Petersen graph and has a polyhedral embedding in the projective plane), have polyhedral embeddings neither in orientable nor in non-orientable surfaces. In Section 5 Goldberg graphs are considered. They have no polyhedral embeddings in orientable surfaces, but all of them have polyhedral embeddings in non-orientable surfaces. It is also proved that for every non-orientable surface S there exists a cubic graph of class 2, which polyhedrally embeds in S.

2 Short cycles

If a cubic graph with a short cycle C has a polyhedral embedding, then C is very likely to be a facial cycle. This is established by the following results.

Lemma 2.1 *Let G be a cubic graph and C a 3-cycle of G. Then C is a facial cycle in every polyhedral embedding of G.*

Proof. Let $C = v_0v_1v_2v_0$ be a 3-cycle of G. Denote the neighbour of v_i not in C with v_i', $i = 0, 1, 2$. A facial cycle in a polyhedral embedding of G cannot contain any of the paths $v_i'v_iv_{i+1}v_{i+2}v_{i+2}'$, $i = 0, 1, 2$, indices modulo 3, since it must be induced. This implies that we have three facial cycles at C, which contain $v_i'v_iv_{i+1}v_{i+1}'$, $i = 0, 1, 2$, indices modulo 3. Then C is a facial cycle. □

Lemma 2.2 *Let G be a cubic graph other than K_4 and let C be a 4-cycle of G. Then C is a facial cycle in every polyhedral embedding of G.*

Proof. If G has a polyhedral embedding and G is not K_4, then every 4-cycle of G is induced, since G is 3-connected by Proposition 1.1.

Let $C = v_0v_1v_2v_3v_0$ be a 4-cycle of G and let v_i' be the neighbour of v_i not in C, $i = 0, 1, 2, 3$. Suppose that all facial cycles, which intersect C, intersect C in one edge only. Then it is easy to see that C is a facial cycle. Otherwise there is at least one facial cycle $C_1 \neq C$ that intersects C in more than one edge. Facial cycles in polyhedral embeddings are induced. Hence we may assume that C_1 contains the path $v_0'v_0v_1v_2v_2'$. The other facial cycle C_2, which contains the edge $v_0'v_0$, must contain the path $v_0'v_0v_3v_3'$ in order not to intersect C_1 at v_2. The third facial cycle through v_0 then contains edges v_0v_1, v_0v_3 and v_3v_2, which is a contradiction. □

Let a graph G be embedded in a surface S, let F be a facial cycle and let C be a cycle of G. We say that F is *k-forwarding* at C, if F and C intersect precisely in k consecutive edges on C.

Lemma 2.3 *Let G be a cubic graph and C an induced 5-cycle of G. If G has a polyhedral embedding in a surface S, then the following holds.*

(a) *If S is orientable, then C is a facial cycle.*

(b) *If S is non-orientable, then either C is a facial cycle or all facial cycles that intersect C are 2-forwarding at C.*

Proof. Let $C = v_0v_1v_2v_3v_4v_0$ be a 5-cycle of G. Suppose that no facial cycle (other than possibly C) intersects C in more than one consecutive edge on C. Then it is easy to see that C is a facial cycle.

Now let F be a facial cycle that intersects C in at least two consecutive edges on C. Facial cycles in polyhedral embeddings are induced. Therefore F is either 3-forwarding or 2-forwarding at C.

If F is 3-forwarding, we can assume that the path $v_0'v_0v_1v_2v_3v_3'$ is in F. Then the facial cycle, which contains the path $v_0v_4v_3$, intersects twice with F. This contradiction implies that no facial cycle is 3-forwarding at C.

We may assume that F contains the path $v_0'v_0v_1v_2v_2'$. The facial cycle, which contains the path $v_1'v_1v_2$, must contain the path $v_1'v_1v_2v_3$ so it is 2-forwarding. If we continue along the cycle C, we see that all facial cycles at C are 2-forwarding at C.

To complete the proof, we will show that S is not orientable, if all facial cycles at C are 2-forwarding. Suppose that S is orientable and let C_i be the facial cycle, which contains the path $v_iv_{i+1}v_{i+2}$, $i = 0, 1, 2, 3, 4$, indices modulo 5. We can assume that in the orientation of C_0, induced by the orientation of S, vertices $v_0v_1v_2$ are in clockwise order. Then the vertices $v_3v_2v_1$ are in this clockwise order on C_1. If we continue along C, we see that in C_4 vertices $v_4v_0v_1$ are in clockwise order. But then C_0 and C_4 induce the same orientation of the edge v_0v_1, which is a contradiction with the assumption that S is orientable. □

Corollary 2.4 *If a cubic graph G contains two induced 5-cycles, whose intersection is nonempty and is not just a common edge, then G has no orientable polyhedral embeddings.*

Proof. Suppose we have an orientable polyhedral embedding of G. By Lemma 2.3 both 5-cycles are facial. This is a contradiction with the fact that their intersection contains more than just one edge. □

In the Petersen graph P every edge is contained in four induced 5-cycles. Lemma 2.3 therefore implies that P has no orientable polyhedral embeddings. However, P has a polyhedral embedding in the projective plane (see Figure 1).

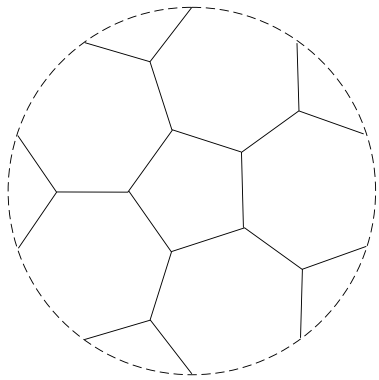

Figure 1: The Petersen graph embedded in the projective plane.

Lemma 2.3 and its Corollary 2.4 can be applied on many other snarks, for example the Szekeres snark that is shown in Figure 2.

Theorem 2.5 *The Szekeres snark has no polyhedral embeddings.*

Proof. Each of the five "parts" of the Szekeres snark (see Figure 2) contains a path $v_1v_2\dots v_9$ on 9 vertices and a vertex v_0 that is adjacent with v_2, v_5, v_8 and further there are edges v_1v_6 and v_4v_9. There are four induced 5-cycles $C_1 = v_0v_2v_1v_6v_5v_0$, $C_2 = v_0v_2v_3v_4v_5v_0$, $C_3 = v_0v_8v_9v_4v_5v_0$ and $C_4 = v_0v_8v_7v_6v_5v_0$. Cycles C_1 and C_2 intersect at two edges adjacent to v_0. Therefore they are not both facial cycles. If none of C_1, C_2 is facial, then the 2-forwarding facial cycles at C_1 and C_2, which contain their intersection $C_1 \cap C_2$, are distinct and intersect in two edges. So one of them is facial and the other is not. Similarly, one of the cycles C_3, C_4 is facial and the other one is not.

Suppose the cycle C_2 is facial. Then it is 1-forwarding at C_4, so C_4 is facial and C_1 and C_3 are not facial. This implies that there is a facial cycle that contains the path $v_1v_6v_5v_4v_9$ and another facial cycle that contains the path $v_1v_2v_0v_8v_9$, which is a contradiction.

Suppose now that C_2 is not facial. Then C_1 is facial and is 1-forwarding at C_4. So C_4 is a facial cycle and C_3 is not. This implies that there is a facial cycle that contains the path $v_3v_2v_0v_8v_7$ and another facial cycle that contains the path $v_3v_4v_5v_6v_7$, which is a contradiction. □

Nonexistence of orientable polyhedral embeddings of the Szekeres snark has been proved earlier by Szekeres [7].

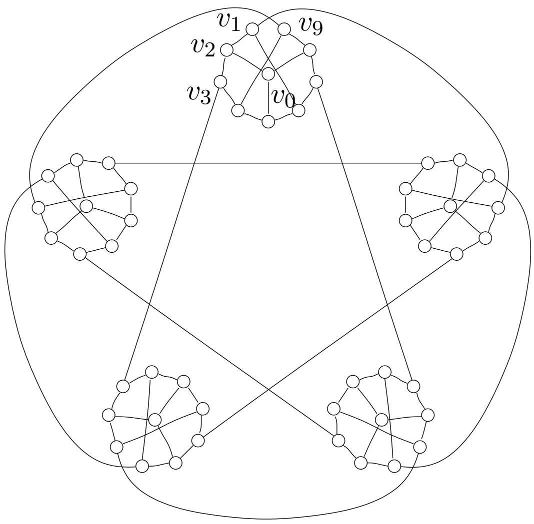

Figure 2: The Szekeres snark.

3 Small edge-cuts

Let G_1 and G_2 be cubic graphs and $v_1 \in V(G_1)$, $v_2 \in V(G_2)$. Denote the three neighbours of v_1 in G_1 by z_0, z_1, z_2 and the three neighbours of v_2 in G_2 by u_0, u_1, u_2. Let $G = G_1 * G_2$ be the cubic graph obtained from graphs G_1 and G_2 by deleting vertices v_1 and v_2 and connecting vertices u_i with z_i for $i = 0, 1, 2$. We call G the *star product* of G_1 and G_2. It is easy to see that the graph G is 3-edge-colorable if and only if both G_1 and G_2 are 3-edge-colorable.

Theorem 3.1 *The star product $G = G_1 * G_2$ has a polyhedral embedding in an (orientable) surface if and only if both G_1 and G_2 have polyhedral embeddings in some (orientable) surfaces.*

Proof. Suppose we have polyhedral embeddings of G_1 and G_2. At vertex v_1 we have three facial cycles $C_i = z_i v_1 z_{i+1} P_i z_i$ for $i = 0, 1, 2$, indices modulo 3. At vertex v_2 we have three facial cycles $D_i = u_i R_i u_{i+1} v_2 u_i$ for $i = 0, 1, 2$. Since the embeddings are polyhedral, paths P_0, P_1, P_2 and paths R_0, R_1, R_2 are pairwise disjoint. In the embedding of the star product $G = G_1 * G_2$ we keep all facial cycles from embeddings of G_1 and G_2, which do not contain vertices v_1 and v_2, and add three new facial cycles $F_i = z_i u_i R_i u_{i+1} z_{i+1} P_i z_i$, $i = 0, 1, 2$, indices modulo 3. Facial cycles in G, which are facial cycles in

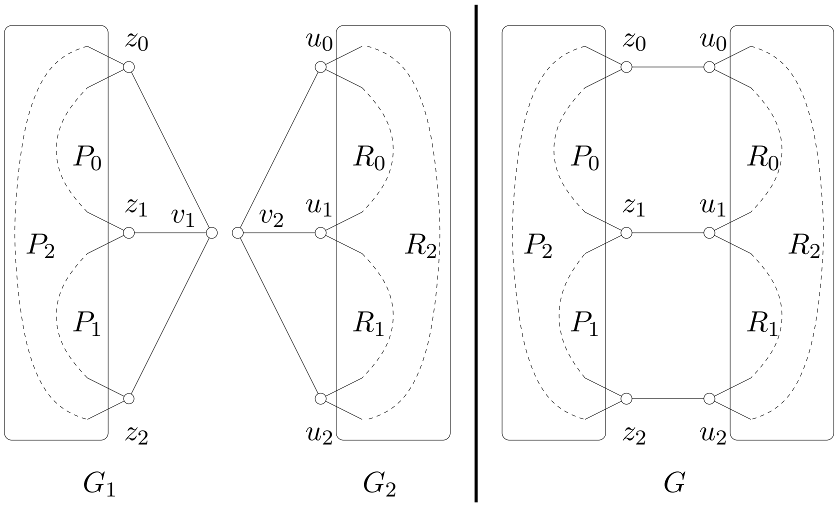

Figure 3: The star product G of graphs G_1 and G_2.

G_1 or G_2, intersect pairwise at most once. A facial cycle F, which is also a facial cycle in G_1 or G_2, intersects the facial cycle F_i, $i = 0, 1, 2$, only on the path P_i or only on the path R_i. So it intersects F_i at most once. Facial cycles F_i and F_{i+1} intersect only in the edge $u_{i+1}z_{i+1}$, $i = 0, 1, 2$, indices modulo 3, since the paths P_0, P_1, P_2 and R_0, R_1, R_2 are pairwise disjoint. So the embedding of G is polyhedral. It is easy to see that the embedding of G is orientable if and only if the embeddings of G_1 and G_2 are orientable.

Suppose now that G has a polyhedral embedding. The three edges z_iu_i, $i = 0, 1, 2$, form a 3-cut in G. Since the embedding is polyhedral, we have three facial cycles $F_i = u_iR_iu_{i+1}z_{i+1}P_iz_iu_i$, such that F_i and F_{i+1} intersect in the edge $z_{i+1}u_{i+1}$, $i = 0, 1, 2$, indices modulo 3. We may assume that there are no negative signatures on edges z_iu_i, $i = 0, 1, 2$. In the embedding of G_1 (and G_2) we keep all facial cycles, which do not intersect G_2 (respectively G_1), and add vertices v_1, v_2 with such local rotations that we obtain new facial cycles $C_i = z_iv_1z_{i+1}P_iz_i$ in G_1 and $D_i = u_iR_iu_{i+1}v_2u_i$ in G_2, $i = 0, 1, 2$, induces modulo 3. Since we have no new intersections between facial cycles (intersections on z_iu_i become intersections on z_iv_1 and u_iv_2), the embeddings of G_1 and G_2 are polyhedral. It is also clear that both embeddings are in orientable surfaces if and only if the embedding of G is orientable, since we did not change local rotation at any vertex or change the signature of any edge. □

If the embedding of $G = G_1 * G_2$ in a surface S is constructed as in the proof of Theorem 3.1 from embeddings of G_1 and G_2 in surfaces S_1 and S_2 of Euler genus $\epsilon(S_1) = k_1$ and $\epsilon(S_2) = k_2$, respectively, then the Euler genus of S is $\epsilon(S) = k_1 + k_2$. This is easily proved by using Euler's formula for G, G_1 and G_2.

Let G_1 and G_2 be cubic graphs. Choose an edge $e = xy$ in G_1 and two nonadjacent edges $f_1 = u_0u_1$ and $f_2 = u_2u_3$ in G_2. Denote the neighbours of x in G_1 by v_0, v_1, and the neighbours of y by v_2, v_3. Let G be the graph obtained from G_1 and G_2 by deleting vertices x, y in G_1 and edges f_1, f_2 in G_2 and joining pairs v_iu_i, $i = 0, 1, 2, 3$. The graph $G = G_1 \cdot G_2$ is called the *dot product* of G_1 and G_2. If both G_1 and G_2 are snarks, then their dot product is also a snark.

Theorem 3.2 *Let G_1 and G_2 be cubic graphs. If G_1 and G_2 have polyhedral embeddings in (orientable) surfaces S_1 and S_2, such that the geometric dual of G_2 is not a complete graph, then a dot product $G = G_1 \cdot G_2$ exists, which has a polyhedral embedding in an (orientable) surface S. If the Euler genus of surfaces S_1 and S_2 are $\epsilon(S_1) = k_1$ and $\epsilon(S_2) = k_2$, then the Euler genus of S is $\epsilon(S) = k_1 + k_2$.*

Proof. Suppose that we have polyhedral embeddings as described. We claim that G_2 contains facial cycles D_0, D_1, D_2, such that D_1 intersects D_0 and D_2 but D_0 and D_2 do not intersect. To see this, consider the dual graph R. Since it is not a complete graph, it has two vertices c_0 and c_2 that are at distance two in R. If c_1 is their common neighbor, then we can take D_0, D_1, D_2 to be the facial cycles corresponding to c_0, c_1 and c_2, respectively.

Let $f_1 = u_0u_1$ and $f_2 = u_2u_3$ be the intersections between D_0, D_1 and D_1, D_2, respectively, and choose an arbitrary edge $e = xy$ in G_1. Denote the neighbours of x and y in G_1 so that the facial cycles, which contain x or y, are $C_0 = v_0xv_1P_0v_0$, $C_1 = v_1xyv_2P_1v_1$, $C_2 = v_2yv_3P_2v_2$, and $C_3 = v_3yxv_0P_3v_3$. Since the embedding of G_1 is polyhedral, paths P_0, P_1, P_2, P_3 are pairwise disjoint, except that P_0 and P_2 may intersect. In G_2 we will use the following notation for facial cycles: $D_0 = u_0R_0u_1u_0$, $D_1 = u_0u_1R_1u_2u_3R_3u_0$ and $D_2 = u_2R_2u_3u_2$. The paths R_0, R_1, R_2, R_3 are pairwise disjoint. In the embedding of G we keep all local rotations at vertices of G_1 and G_2, which are not deleted (with added edges naturally replacing deleted edges), and all edge signatures. Instead of facial cycles C_i, D_i we get a facial cycle $F_i = v_iu_iR_iu_{i+1}v_{i+1}P_iv_i$, $i = 0, 1, 2, 3$, indices modulo 4. Since the paths P_i, R_i are pairwise disjoint, except for the possible intersection between P_0 and P_2, all intersections between facial cycles F_i, $i = 0, 1, 2, 3$, are the

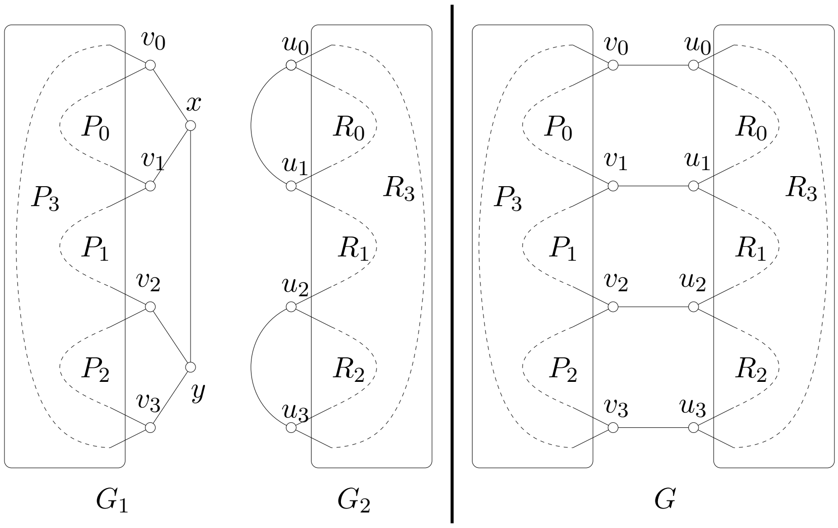

Figure 4: The dot product G of graphs G_1 and G_2.

intersections of F_i and F_{i+1} in edges $v_{i+1}u_{i+1}$, $i = 0, 1, 2, 3$, indices modulo 4, and possibly one more intersection between F_0 and F_2. It is clear that any facial cycle F that does not contain any of the vertices v_i, u_i intersects at most once with any F_i and that two such facial cycles intersect at most once. So the embedding of G is polyhedral. It is also clear that if the embeddings of G_1 and G_2 are in orientable surfaces, the embedding of G is also in an orientable surface.

The Euler genus of S is obtained from Euler's formula and equalities

$$\begin{aligned} |V(G)| &= |V(G_1)| + |V(G_2)| - 2 \\ |E(G)| &= |E(G_1)| + |E(G_2)| - 3 \\ |F(G)| &= |F(G_1)| + |F(G_2)| - 3 \end{aligned}$$

from which we conclude that $\epsilon(S) = k_1 + k_2$. □

Theorem 3.3 *Let G be a cubic graph and S a 4-cut in G. If G admits a polyhedral embedding (in an orientable surface), then there exist graphs G_1 and G_2, such that $G = G_1 \cdot G_2$ and G_1 admits a polyhedral embedding (in an orientable surface).*

Proof. Suppose that the edges u_iv_i, $i = 0, 1, 2, 3$, form a 4-cut S in G. If a facial cycle contains more than two edges of S, the embedding of G can not be polyhedral. So we have four distinct facial cycles F_0, F_1, F_2, F_3 that contain edges of S. Since S is a cut, every cycle F_i, $i = 0, 1, 2, 3$, contains two edges of S.

Since the embedding is polyhedral, each of the F_i intersects two other F_j, F_k. In the dual a subgraph induced by the vertices corresponding to F_i, $i = 0, 1, 2, 3$, is a simple graph on four vertices in which all vertices are of degree 2. It must be a 4-cycle. Therefore we can assume that faces F_i and F_{i+1} intersect in the edge $v_{i+1}u_{i+1}$, $i = 0, 1, 2, 3$, indices modulo 4. Each facial cycle F_i is then of the form $F_i = v_i u_i R_i u_{i+1} v_{i+1} P_i v_i$. Since F_0 and F_2 intersect at most once, we can assume they do not intersect at the paths P_0 and P_2. Let G_1 be the component of $G - S$, which contains paths P_i. If we set rotations of all vertices in G_2 as they are in G (and replace deleted edges naturally with added edges), we can set rotations around vertices x and y so that the facial cycles in G_1, which do not contain x or y, remain unchanged and we have four new facial cycles $C_0 = v_0 x v_1 P_0 v_0$, $C_1 = v_1 x y v_2 P_1 v_1$, $C_2 = v_2 y v_3 P_2 v_2$, and $C_3 = v_3 y x v_0 P_3 v_3$. Since we added no new intersections between facial cycles, which were already in G, and facial cycles C_i, $i = 0, 1, 2, 3$ intersect pairwise only once, the embedding of G_1 is polyhedral. If the embedding of G is in an orientable surface, it is clear that the embedding of G_1 is in an orientable surface. □

Suppose we have polyhedral embeddings of cubic graphs G_1 and G_2, at least one of which is in a non-orientable surface. Let us construct the embedding of the dot product $G = G_1 \cdot G_2$ as in the proof of Theorem 3.2. If the embedding of G is in orientable surface, then we may assume that all signatures of edges are positive. Now we can construct embeddings of G_1 and G_2 similarly as the embedding of G_1 in the proof of Theorem 3.3, which are both in orientable surfaces and have the same set of facial cycles as the embeddings of G_1 and G_2 with which we started. Since at least one of these two is an embedding in a non-orientable surface, we have a contradiction. This shows

Corollary 3.4 *If we have polyhedral embeddings of G_1 and G_2 at least one of which is non-orientable and construct a polyhedral embedding of $G = G_1 \cdot G_2$ as in the proof of Theorem 3.2, then the embedding of G is non-orientable.*

Let G_1 and G_2 be cubic graphs. Choose a vertex v in G_1, an edge v_3v_4 in G_1 and a vertex z_0 in G_2. Let the three neighbours of v be v_0, v_1, v_2 and let z_1, z_2, u_4 be the neighbours of z_0. Let the neighbours of z_1, z_2 other than

u be u_0, u_1 and u_2, u_3, respectively. If all these vertices are distinct, remove the vertex v from G_1, vertices z_0, z_1, z_2 from G_2 and the edge v_3v_4 from G_1. If we join pairs v_iu_i, $i = 0, 1, 2, 3, 4$, we get a cubic graph $G = G_1 \Diamond G_2$, which is called a *square product* of graphs G_1 and G_2 (see also Figure 5). The cut $Q = \{v_iu_i \mid i = 0, \ldots, 4\}$ in G is said to be the *product cut*. It is well known (cf., e.g. [3]) that if G_1 and G_2 are snarks, then their square product is also a snark.

Theorem 3.5 *Let G be a cubic graph with a matching Q, which is a 5-cut of G. If G admits a polyhedral embedding (in an orientable surface), then there exist graphs G_1 and G_2 such that $G = G_1 \Diamond G_2$ and Q is the corresponding product cut and such that G_2 admits a polyhedral embedding (in an orientable surface).*

Proof. Suppose that G has a polyhedral embedding. Since Q is a cut, every facial cycle contains an even number of edges in Q. It is easy to see that none of them contains four edges of Q (since the embedding is polyhedral). This implies that there are precisely 5 facial cycles $F_0, \ldots, F_4$ that intersect Q and that the edges v_iu_i of Q, $i = 0, \ldots, 4$, can be enumerated so that F_i contains edges v_iu_i and $v_{i+1}u_{i+1}$, indices modulo 5, and $v_0, \ldots, v_4$ are in the same component of $G - Q$.

The facial cycles F_i are of the form $F_i = v_iu_iR_iu_{i+1}v_{i+1}P_iv_i$, $i = 0, \ldots, 4$, indices modulo 5. Since the embedding is polyhedral, every one of the pairs of paths P_i, P_{i+1} and R_i, R_{i+1} is disjoint.

Suppose that the facial cycles F_i and F_{i+2} are disjoint for some i. Then both pairs P_i, P_{i+2} and R_i, R_{i+2} are disjoint. One of the pairs P_{i+2}, P_{i+4} and R_{i+2}, R_{i+4}, $i = 0, \ldots, 4$, is disjoint. Because of the symmetry, we can assume that the pair R_{i+2}, R_{i+4} is disjoint.

Suppose now that all pairs of cycles F_i, F_{i+2}, $i = 0, \ldots, 4$, intersect. In at least three out of five pairs, F_i and F_{i+2} intersect on the same "side" (P_i and P_{i+2} or R_i and R_{i+2}). By symmetry, we may assume that intersections are between P_i and P_{i+2}. Since facial cycles F_i and F_{i+2} intersect at most once, it follows that there exists an index j such that R_j, R_{j+2}, R_{j+4} are pairwise disjoint.

By above, we can assume that R_4, R_1, R_3 are pairwise disjoint. Now we can add to $G - Q$ new vertices v, z_0, z_1, z_2 and edges v_0v, v_1v, v_2v, v_3v_4 and u_0z_1, u_1z_1, u_2z_2 ,u_3z_2, z_1z_0, z_2z_0, u_4z_0 so that the graph G is a square product of G_1 and G_2. In the embedding of G_2 we keep all rotations and signatures of vertices and edges that were already in G and we naturally replace deleted edges with the added ones. Around vertices z_0, z_1, z_2 we

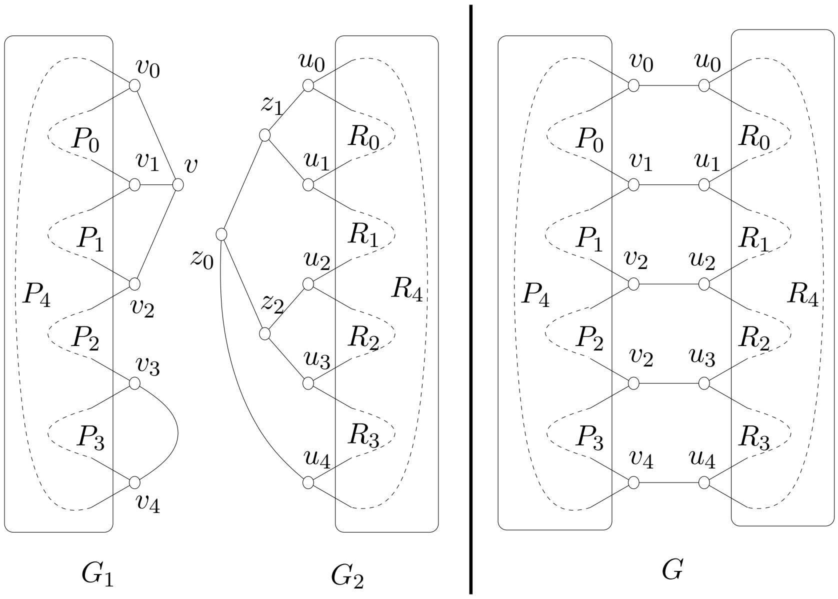

Figure 5: The square product of G_1 and G_2.

can set rotations so that facial cycles in G_2, which were not already in G, are $D_0 = u_0R_0u_1z_1$, $D_1 = z_0z_1u_1R_1u_2z_2z_0$, $D_2 = z_2u_2R_2u_3z_2$, $D_3 = z_0z_2u_3R_3u_4z_0$ and $D_4 = z_0u_4R_4u_0z_1z_0$. The only new intersections of facial cycles of G_2 are between D_4 and D_1 and between D_1 and D_3. Hence the embedding of G_2 is polyhedral and if the embedding of G is in an orientable surface, so is the embedding of G_2. □

4 Flower snarks

Let J_k be the graph with vertices a_i, b_i, c_i, d_i and edges a_ia_{i+1}, a_ib_i, b_ic_i, b_id_i, c_id_{i+1}, d_ic_{i+1} for $i = 0, \ldots, k-1$, indices modulo k. If $k \geq 5$ is odd, then the graph J_k is a snark and is called the *flower snark* of order k [4]. The graph J_5 is shown in Figure 6. Szekeres proved that flower snarks have no polyhedral embeddings in orientable surfaces [8]. The goal of this section is to prove the following

Theorem 4.1 *For $k \geq 4$ the flower graph J_k has no polyhedral embeddings.*

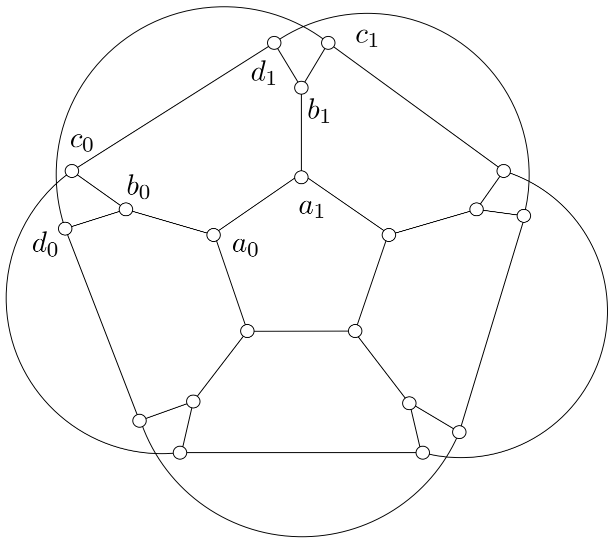

Figure 6: The Flower snark J_5.

The rest of this section is devoted to the proof of Theorem 4.1.

The subgraph Y_j of J_k induced on vertices $\{a_j, b_j, c_j, d_j\}$ is called the *jth star* or simply just a *star* in J_k.

Suppose that we have a polyhedral embedding of J_k. Let us look at how facial cycles can traverse Y_j. If we walk along a facial cycle C, come to Y_j from Y_{j-1} and then leave Y_j going back to Y_{j-1}, we say that C is a *backward face at* j. Similarly we define a *forward face at* j, which is a facial cycle that enters Y_j from Y_{j+1} and leaves it towards Y_{j+1}.

If a cubic graph G has a polyhedral embedding, then at every vertex $v \in V(G)$ with neighbours v_1, v_2, v_3, each path $P = v_i v v_j$, $j \neq i$, defines a unique facial cycle, which we will denote by $F(P)$.

Lemma 4.2 *If C is a facial cycle that contains at least two vertices of Y_j, then the intersection of C with Y_j is one of the three possible paths: $a_j b_j c_j$, $a_j b_j d_j$ or $c_j b_j d_j$.*

Proof. A cycle C can enter and exit Y_j only through vertices a_j, c_j or d_j. Suppose now that $a_j, c_j \in V(C)$. The facial cycle $C' = F(a_j b_j c_j)$ intersects C in two nonadjacent vertices a_j and c_j, so $C = C'$ and C' contains the path $a_j b_j c_j$. Similar conclusion holds if a_j and d_j are on C or if c_j and d_j are on C. Since all facial cycles are induced, the intersection $C \cap Y_j$ can consists only of one of the three paths. □

A facial cycle, which is neither forward nor backward at Y_j, is called a *cross face*. It follows from Lemma 4.2 that each facial cycle, which intersects Y_j, is either a backward, forward or a cross face.

Lemma 4.3 *At Y_j there can be at most one backward (forward) face. If there is one backward face, then there is also one forward face and four distinct cross faces. The backward face at Y_j is forward at Y_{j-1} and the forward face at Y_j is backward at Y_{j+1}.*

Proof. Suppose we have two backward (forward) faces. By Lemma 4.2 they intersect at an edge adjacent to b_j. If they intersect at $b_j a_j$, they also intersect at $a_{j-1}a_j$, which is a contradiction. Similarly we get a contradiction, if they intersect at $b_j c_j$ or $b_j d_j$. This shows that there is at most one backward (forward) face.

Suppose now that C is a backward face. The edges between Y_j and Y_{j+1} are traversed twice by C and four times by cross faces. The cross faces therefore traverse the edges between Y_j and Y_{j+1} at most four times, hence there must be a forward face at Y_j.

If C contains the path $a_j b_j c_j$, then $\{a_{j-1}, d_{j-1}\} \subseteq C \cap Y_{j-1}$. By Lemma 4.2, $C \cap Y_{j-1} = a_{j-1}b_{j-1}d_{j-1}$, so C is a forward face at Y_{j-1}. A similar conclusion holds if $C \cap Y_j$ is either $a_j b_j d_j$ or $c_j b_j d_j$. Similarly we also show that a forward face at Y_j is backward at Y_{j+1}.

Out of facial cycles $F(a_j b_j c_j)$, $F(a_j b_j d_j)$ and $F(c_i b_j d_j)$ one is a backward face, one is a forward face and one is a cross face. Since the one that is a cross face is the only cross face, which contains more than one vertex of Y_j, all cross faces are distinct. □

A backward face at j is called a *bottom face* if it contains the edge $a_{j-1}a_j$ and is called a *top face* if it does not contain $a_{j-1}a_j$. A top face at Y_j is of the form $c_{j-1}b_{j-1}d_{j-1}c_j b_j d_j c_{j-1}$. So it is clear that we cannot have backward top faces at Y_j and Y_{j+1} at the same time.

The star Y_j is of *type 0*, if all facial cycles, which intersect it, are cross faces. It is of *type 1*, if there is one forward and one backward face at Y_j.

Lemma 4.2 implies that if the graph J_k has a polyhedral embedding, then all stars are of type 0 or all stars are of type 1.

Lemma 4.4 *If J_k has a polyhedral embedding, then $k \leq 6$ and all stars are of type 1.*

Proof. By Lemma 4.2 every polyhedral embedding of J_k has at least four cross faces. For each $j = 0, \ldots, k-1$ we have at least one intersection between four selected cross faces on edges from Y_j to Y_{j+1}. Since we can have at most 6 such intersections, we have $k \leq 6$.

If all stars are of type 0, then J_k has precisely 6 facial cycles. The geometric dual of G on S has 6 vertices and $\frac{4k \cdot 3}{2} = 6k$ edges. Since the dual is a simple graph, it has at most 15 edges, so $6k \leq 15$. This implies that $k \leq 2$. □

Lemma 4.5 *The graph J_4 has no polyhedral embeddings.*

Proof. Suppose we have a polyhedral embedding of J_4. All stars are of type 1, so there are precisely 4 cross faces. We have three 4-cycles $C_1 = a_0a_1a_2a_3a_0$, $C_2 = d_0c_1d_2c_3d_0$, $C_3 = c_0d_1c_2c_3c_0$ in J_4, which are facial cycles by Lemma 2.2. These cycles are all cross faces. As in the proof of Lemma 4.4, we see that there are at least four intersections of cross faces. But since C_1, C_2, C_3 are pairwise disjoint, this is not possible. □

Lemma 4.6 *The flower snark J_5 has no polyhedral embeddings.*

Proof. Suppose we have a polyhedral embedding of J_5. Each star must be of type 1. If all backward faces are bottom faces, then the inner cross face $a_0a_1a_2a_3a_4a_0$ does not intersect any other cross faces. So we have 5 intersections between three cross faces, which is not possible.

Since we cannot have two consecutive top faces, we must have two consecutive bottom faces at stars j and $j + 1$ and a top face at star $j + 2$. We can assume $j = 1$. The facial cycle $F(a_0a_1a_2)$ contains the path $a_0a_1a_2a_3a_4$. If not, it would intersect twice with one of the bottom faces at stars 1 or 2. So it must be $a_0 \dots a_4a_0$. The facial cycle, which contains b_2a_2 and is different from the backward face at star 2, must contain the path $b_2a_2a_3b_3$. This facial cycle intersects twice with the facial cycle $d_2b_2c_2d_3b_3c_3d_2$, which is a contradiction. □

Lemma 4.7 *The graph J_6 has no polyhedral embeddings.*

Proof. All stars in J_6 are of type 1. We have three 6-cycles $C_1 = a_0a_1 \dots a_5a_0$, $C_2 = c_0d_1c_2 \dots d_5c_0$ and $C_3 = c_0d_1c_2 \dots d_5c_0$. From previous proofs it follows that at each star Y_j one of the four cross faces goes from one of C_1, C_2, C_3 to another. We say that this cross face has made a transition at Y_j. It is obvious that if a cross face makes at least one transition, it makes more than one transition. So one cross face makes no transitions, since we can have at most 6 transitions. Let the four cross faces be F_1, F_2, F_3, F_4 and let F_1 be the one, which does not make any transition. Because of the symmetry, we can assume that $F_1 = C_1$.

There are four cross faces and six intersections between them. This implies that they must all pairwise intersect and in particular, all cycles F_2, F_3, F_4 intersect F_1. All transitions of cross faces are transitions of F_i to C_1 and from C_1, $i = 2, 3, 4$. In particular, the cycle F_2 makes a transition to the cycle C_1 at some star Y_j and a transitions from C_1 at the star Y_{i+1}. But then F_2 is not induced, which is a contradiction. $\square$

This completes the proof of Theorem 4.1.

5 Goldberg snarks

In 1981 Goldberg discovered another infinite family of snarks [1]. Let G_k be a cubic graph with vertices a_i, b_i, c_i, d_i, g_i, h_i, e_i, f_i for $i = 0, \ldots, k-1$, and edges $a_i a_{i+1}$, $a_i b_i$, $b_i c_i$, $b_i d_i$, $c_i e_i$, $c_i g_i$, $d_i f_i$, $d_i h_i$, $e_i f_i$, $g_i h_i$, $e_i f_{i+1}$, $h_i g_{i+1}$, for $i = 0, \ldots k-1$, where indices are modulo k. If $k \geq 5$ is odd, then G_k is known as the *Goldberg snark*. Accordingly, we refer to all graphs G_k as *Goldberg graphs*. The graph G_5 is shown in Figure 7.

Theorem 5.1 *No Goldberg graph has a polyhedral embedding in an orientable surface. On the other hand, every Goldberg graph G_k, $k \geq 3$, has a polyhedral embedding in the non-orientable surface of Euler genus k.*

Proof. Suppose that the graph G_k has a polyhedral embedding in an orientable surface. For every $i = 0, \ldots, k-1$ we have have two 5-cycles $B_i = b_i d_i h_i g_i c_i b_i$ and $C_i = b_i d_i f_i e_i c_i b_i$. By Lemma 2.3 both are facial cycles. This is a contradiction, since B_i and C_i intersect in two edges $c_i b_i$ and $b_i d_i$.

An embedding in a non-orientable surface has the following facial cycles:

(a) $A = a_0 a_1 \ldots a_{k-1} a_0$ and $B = f_0 e_0 f_1 e_1 \ldots f_{k-1} e_{k-1} f_0$,

(b) $C_i = b_i d_i f_i e_i c_i b_i$, $i = 0, \ldots, k-1$,

(c) $D_i = g_i h_i g_{i+1} h_{i+1} d_{i+1} f_{i+1} e_i c_i g_i$, $i = 0, \ldots, k-1$,

(d) $E_i = a_i a_{i+1} b_{i+1} c_{i+1} g_{i+1} h_i d_i b_i a_i$, $i = 0, \ldots, k-1$.

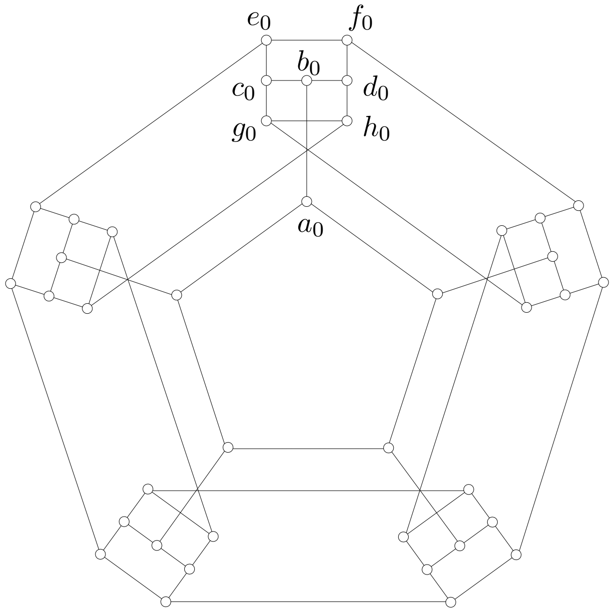

Figure 7: The Goldberg snark G_5.

It is easy to see that this determines a non-orientable polyhedral embedding. The Euler genus of the underlying surface of the embedding is calculated from Euler's formula $2-\epsilon(G_k) = |V(G_k)| - |E(G_k)| + |F(G_k)| = 8k - \frac{3}{2}8k + 3k + 2 = 2 - k$. $\square$

Goldberg graphs have more than one polyhedral embedding, not all of the same genus. They can be described as follows.

Consider the subgraph T_i induced on vertices a_i, b_i, c_i, d_i, e_i, f_i, g_i and h_i. Let us look at how facial cycles can traverse it. There are (at least) two possibilities.

There is a facial 5-cycle $C^i = b_i d_i h_i g_i c_i b_i$ and there are facial cycles that contain paths $P_1^i = a_{i-1}a_ia_{i+1}$, $P_2^i = g_{i-1}h_ig_ih_{i+1}$, $P_3^i = g_{i-1}h_id_if_ie_if_{i-1}$, $P_4^i = h_{i+1}g_ic_id_ie_if_ie_{i+1}$, $P_5^i = e_{i+1}f_id_ib_ia_ia_{i+1}$ and $P_6^i = f_{i-1}e_ic_ib_ia_ia_{i-1}$, where P_1^i and P_2^i can possibly be part of the same facial cycle. In such case, we say that T_i is of *type 1*.

The second possibility is the following. There is a facial 5-cycle $D^i = b_ic_ie_if_id_ib_i$ and there are facial cycles that contain paths $R_1^i = a_{i-1}a_ia_{i+1}$, $R_2^i = f_{i-1}e_if_ie_{i+1}$, $R_3^i = a_{i-1}a_ib_id_ih_ig_{i-1}$, $R_4^i = a_{i+1}a_ib_ic_ig_ih_{i+1}$, $R_5^i = f_{i-1}e_ic_ig_ih_ig_{i-1}$ and $R_6^i = e_{i+1}f_id_ib_ih_ig_ih_{i+1}$, where R_1^i and R_2^i can possibly be part of the same facial cycle. We say that T_i is of *type 2*.

We now choose arbitrary the types of all subgraphs T_i and join facial segments described above into facial cycles as follows. There is an automorphism of the graph G_k, which sends all cycles C^i into cycles D^i, so we can assume that the subgraph T_i is of type 1. If not, we join facial segments symmetrically according to this automorphism.

If subgraphs T_i and T_{i+1} are both of type 1, we join facial segments P_1^i and P_1^{i+1}, P_2^i and P_2^{i+1}, P_4^i and P_3^{i+1} and facial segments P_5^i and P_6^{i+1}.

If the subgraph T_i is of type 1 and T_{i+1} of type 2, we join facial segments P_1^i, R_3^{i+1} and P_2^i, facial segments R_1^{i+1}, P_5^i and R_2^{i+1} and facial segments P_4^i and R_5^{i+1}.

If all subgraphs T_i are of type 1 (or all are of type 2), then the embedding is the one described in the proof of Theorem 5.1. If there are two consecutive subgraphs T_i and T_{i+1} of different types, we say that there is a *transition at i*. It is easy to see that the embedding is polyhedral if we have at least 6 transitions. It is also easy to see that the number of facial cycles of the embedding is $3k$. In this manner we have obtained a large number of (combinatorially) different polyhedral embeddings of the graph G_k in a surface of Euler genus $k - 2$.

This shows that Goldberg snarks admit polyhedral embeddings in distinct non-orientable surfaces (of Euler genus k and $k-2$) and that they admit combinatorially different polyhedral embeddings in the same non-orientable surface (of Euler genus $k - 2$). This fact by itself is of certain interest, see [5, Section 5].

Corollary 5.2 *For every positive integer k there exists a cubic graph of class 2, which polyhedrally embeds in the non-orientable surface $\mathbb{N}_k$ of Euler genus k. For every $k > 0$ and $k \neq 2$ there exists a snark, which polyhedrally embeds in $\mathbb{N}_k$.*

Proof. The Petersen graph P has a polyhedral embedding in $\mathbb{N}_1$. By Theorem 5.1 the Goldberg snark G_{2k+1} has a polyhedral embedding in $\mathbb{N}_{2k+1}$ for every $k \geq 1$. The graph G_3 is not a snark since it contains a 3-cycle $C = a_0a_1a_2a_0$. If we contract C to a vertex, we obtain a snark G_3', which polyhedrally embeds in $\mathbb{N}_3$ (cf. Theorem 3.1). For $k > 1$ we have a snark $H_{2k+2} = G_{2k+1} \cdot P$, which polyhedrally embeds in $\mathbb{N}_{2k+2}$, and $H_4 = G_3' \cdot P$, which polyhedrally embeds in $\mathbb{N}_4$ (cf. Theorem 3.2). The dot product $H_2 = P \cdot J_3$ polyhedrally embeds in $\mathbb{N}_2$. The graph H_2 is not 3-edge-colorable, but is not a snark, since the girth of H_2 is 4. □

There are two non-isomorphic dot products of two copies of the Petersen graph P, which are known as Blanuša's snarks. But since the dual of P in the projective plane is K_6, we cannot use Theorem 3.2 to obtain polyhedral embeddings in the Klein bottle for either of them. Indeed, it can be shown that they do not have such embeddings. It is possible that no snark exists which polyhedrally embeds in the Klein bottle.

Problem 5.3 *Is there a snark that has a polyhedral embedding in the Klein bottle?*

6 Computer search

All snarks and all cyclically 4-edge-connected cubic graphs of class 2 having up to 30 vertices are known. They were generated by computer and are available at [9]. We have used this database to verify Conjecture 1.2 on them (using computer). Combined with Theorem 3.1 this computation shows:

Theorem 6.1 *Every cubic graph with at most 30 vertices that has a polyhedral embedding in an orientable surface is 3-edge-colorable.*

References

[1] M. K. Goldberg, Construction of class 2 graphs with maximum vertex degree 3, J. Comb. Theory Ser. B 31 (1981), 282–291.

[2] B. Grünbaum, Conjecture 6, in Recent Progress in Combinatorics, Ed. W. T. Tutte, Academic Press, New York, 1969, 343.

[3] D. A. Holton, J. Sheehan, The Petersen Graph, Austral. Math. Soc. Lecture Series 7, Cambridge University Press, 1993.

[4] R. Isaacs, Infinite families of nontrivial trivalent graphs which are not Tait colorable, Amer. Math. Monthly 82 (1975), 221–239.

[5] B. Mohar, C. Thomassen, Graphs on Surfaces, The Johns Hopkins University Press, Baltimore and London, 2001.

[6] N. Robertson, R. P. Vitray, Representativity of surface embeddings, in: Paths, Flows, and VLSI-Layout, B. Korte, L. Lovász, H. J. Prömel, and A. Schrijver Eds., Springer-Verlag, Berlin, 1990, 293–328.

[7] G. Szekeres, Polyhedral decompositions of cubic graphs, Bull. Austral. Math. Soc. 8 (1973), 367–387.

[8] G. Szekeres, Non-colourable trivalent graphs, Combinatorial Mathematics (Proc. Third Austral. Conf., Univ. Queensland, St. Lucia, 1974), Lecture Notes in Math., Vol. 452, Springer, Berlin, 1975, 227–233.

[9] http://www.cs.uwa.edu.au/~gordon/remote/cubics/index.html