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Coloring Vertices and Faces of Locally Planar Graphs

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Abstract. If G is an embedded graph, a *vertex-face r-coloring* is a mapping that assigns a color from the set $\{1, \ldots, r\}$ to every vertex and every face of G such that different colors are assigned whenever two elements are either adjacent or incident. Let $\chi_{vf}(G)$ denote the minimum r such that G has a vertex-face r-coloring. Ringel conjectured that if G is planar, then $\chi_{vf}(G) \leq 6$. A graph G drawn on a surface S is said to be 1-embedded in S if every edge crosses at most one other edge. Borodin proved that if G is 1-embedded in the plane, then $\chi(G) \leq 6$. This result implies Ringel's conjecture. Ringel also stated a Heawood style theorem for 1-embedded graphs. We prove a slight strengthening of this result. If G is 1-embedded in S, let $w(G)$ denote the *edge-width* of G, *i.e.* the length of a shortest non-contractible cycle in G. We show that if G is 1-embedded in S and $w(G)$ is large enough, then the list chromatic number $\mathrm{ch}(G)$ is at most 8.

Key words. Vertex face coloring, 1-embedded, Locally planar

1. Background

Let G be an embedded graph. Suppose you wish to color the vertices and faces of G so that two elements get different colors whenever they are adjacent or incident. If G is planar, then you could use four colors on the vertices and an additional four colors on the faces. It is natural to wonder if fewer colors might suffice. In 1966, Ringel [14] showed that seven colors suffice and conjectured that six colors would also suffice. This was verified by Borodin [4]. Recent papers on vertex-face coloring planar graphs include those of Borodin, Kostochka, Raspaud, and Sopena [5], Lam and Zhang [12], and Wang and Liu [19].

Suppose that a graph G is embedded in some surface. Let $F(G)$ denote the set of faces. An assignment $c : V(G) \cup F(G) \to \{1, 2, \ldots, r\}$ is called a *vertex-face r-coloring* if $c(x) \neq c(y)$ whenever x, y are adjacent vertices, adjacent faces, or an

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incident vertex and face. This inspires the definition of G_{vf}, the *vertex-face graph* of an embedded graph. More precisely, $V(G_{vf}) = V(G) \cup F(G)$ and $E(G_{vf})$ collects all of the pairwise incidences and adjacencies of the vertices and faces of the embedded graph G. The *vertex-face chromatic number* of G, denoted by $\chi_{vf}(G)$, is the minimum r such that G has a vertex-face r-coloring. Clearly, $\chi_{vf}(G) = \chi(G_{vf})$.

We shall mainly consider list colorings, a notion that generalizes usual colorings. Suppose that for every vertex v of G, a nonempty set $L(v)$ is given. The set $L(v)$ is called the *list* of v, or the *set of allowable colors* for v. A *list coloring* assigns to each vertex a color from its list in such a way that adjacent vertices receive distinct colors. The graph is *list r-colorable* or *r-choosable* if for every selection of lists $L(v)$ $(v \in V(G))$, each of which contains at least r allowable colors, there exists a list coloring of G. The minimum r for which G is r-choosable is called the *choice number* or the *list chromatic number* of G and is denoted by $\mathrm{ch}(G)$. For vertex-face list colorings of an embedded graph G, we define $\mathrm{ch}_{vf}(G) = \mathrm{ch}(G_{vf})$.

A graph drawn on a surface S so that each edge crosses at most one other edge is said to be 1-*embedded* in S. If G is embedded in S, then the natural construction of superimposing the dual of G onto the embedding of G and adding the vertex-face incidences gives a 1-embedding of G_{vf} in S. Borodin [4] actually showed that every graph that is 1-embeddable in the plane can be 6-colored, thus proving a strengthening of Ringel's conjecture mentioned above.

It is straightforward to draw a 1-embedding of K_6 in the plane. Thus Borodin's result is best possible. However, it is easy to see that $K_6 \neq G_{vf}$ for any plane graph G. If $G = K_3 \square K_2$ (the triangular prism) is embedded in the plane, then G_{vf} has eleven vertices. No three of these vertices are independent so $\chi(G_{vf}) \geq 6$. On the other hand, suppose that we have a list of 6 admissible colors for every vertex and face of G. Clearly, there exists a list coloring of the faces of G. Any such coloring yields lists of three allowable colors on every vertex of G. Since $K_3 \square K_2$ is list 3-colorable, $\mathrm{ch}_{vf}(G) = 6$. It is worth mentioning that if one first 3-colors the vertices of G, then this will not extend to a 6-coloring of G_{vf}.

The idea of coloring first the faces and then list coloring the vertices suffices to show that cubic planar graphs are vertex-face 6-choosable. Dualizing, we immediately see that planar triangulations are also vertex-face 6-choosable. Similar arguments work on arbitrary surfaces. For example since the toroidal dual of K_7 is 3-list colorable, $\chi_{vf}(K_7) = 7$.

It is not surprising that the value of $\chi_{vf}(G)$ can depend on the embedding of G. For instance, if K_5 is embedded on the torus so that all faces are quadrilaterals, then $\chi_{vf}(K_5) = 5$. Hutchinson notes that an alternative embedding of K_5 in which one face is a pentagon has $\chi_{vf}(K_5) = 7$ [6]. Figure 1 exhibits a toroidal embedding of C_7^2 for which $\chi_{vf} = 7$. We know of no graph G that embeds on the torus or Klein's bottle that has $\chi_{vf}(G) > 7$.

2. Surfaces of Higher Genus

Although it would be natural to consider vertex-face colorings of graphs embedded on surfaces of higher genus, it appears that not much work on this has been done.

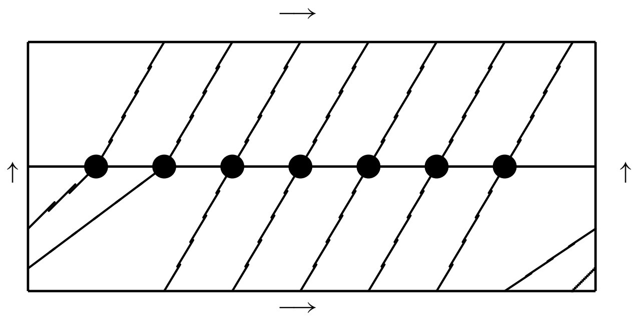

Fig. 1.

Ringel found an analogue of Heawood's Theorem for the chromatic number of 1-embeddable graphs. His result is that if G is 1-embeddable in the orientable surface S_g of genus $g \geq 1$, then $\chi(G) \leq \lfloor \frac{9+\sqrt{64g+17}}{2} \rfloor$ [15]. Ringel asserts that the proof of this inequality will appear elsewhere, but a search of MathSciNet reveals no such paper. We now fill that gap and extend Ringel's result not only to nonorientable surfaces but also to list colorings. Korzhik has a proof of this upper bound among other results in the unpublished paper [9] and has shown Ringel's inequality is sharp for infinitely many surfaces [8]. We begin with an upper bound on the number of edges in a 1-embeddable graph. This is a slight improvement of a result of Schumacher [17].

Theorem 1. *Let G be a graph with V vertices and E edges. If G is 1-embedded in a surface of Euler genus g and has t vertices of odd degree, then* $E \leq 4V - 8 + 4g - \frac{1}{6}t$.

Proof. Suppose G is a graph that is maximally 1-embedded on S, a surface of Euler genus g. Here, maximally means that it is not possible to insert additional edges without having some edge crossing two others. We also allow multiple edges subject to the provision that there is no face with just two boundary edges. Let G_0 be the graph obtained from G by removing every pair of crossing edges. G_0 is naturally embeded on S. Let V_0 $(= V)$, E_0, and F_0 denote the number of vertices, edges, and faces in G_0 and suppose F_i denotes the number of faces with exactly i boundary edges in G_0. Since G is maximal, the embedding of G_0 is a 2-cell embedding and every face in G_0 is either a triangle or a quadrilateral. Thus $F_0 = F_3 + F_4$.

Euler's Formula for G_0 embedded on S is $V_0 - E_0 + F_0 = 2 - g$. Counting edge-face incidences yields $2E_0 = 3F_3 + 4F_4 = 4F_0 - F_3$. This gives

$$8 - 4g = 4(V_0 - E_0 + F_0) = 4V_0 - (E_0 + 2F_4) - 3E_0 + 6F_0 - 2F_3.$$

Rearranging this equation and noting that $E = E_0 + 2F_4$, we get

$$E = 4V - 8 + 4g - 3E_0 + 6F_0 - 2F_3 = 4V - 8 + 4g - \frac{1}{2}F_3. \qquad (1)$$

Suppose now that G is not edge maximal as assumed above. Suppose also that s edges have been added to get a maximally 1-embedded graph G' containing G. If $s \geq t/6$, then (1) implies the statement of the theorem. Otherwise, G' contains at least $t - 2s \geq 0$ odd degree vertices. Consequently, G_0 has at least one triangle incident with each of these vertices, so $F_3 \geq \frac{t-2s}{3}$. Using (1) we get:

$$E \leq 4V - 8 + 4g - \tfrac{t-2s}{6} - s \leq 4V - 8 + 4g - \tfrac{t}{6}.$$

□

Theorem 1 yields a generalization of Ringel's bound [15] to arbitrary surfaces, its strengthening in the sense of Dirac's extension of Heawood's theorem (see [3]), and its extension to list colorings.

Corollary 1. *Let* $R(g) = \lfloor \frac{1}{2}(9 + \sqrt{32g + 17}) \rfloor$. *If* G *is* 1*-embedded in a surface of Euler genus* g*, then* $\mathrm{ch}(G) \leq R(g)$*. Moreover, if* $g = 2$ *or* $g \geq 4$*, then* $\mathrm{ch}(G) = R(g)$ *if and only if* G *contains the complete graph of order* $R(g)$ *as a subgraph.*

Proof. If $g = 0$, the corollary is just Borodin's Theorem [4]. When $g = 1$, the average degree of G is less than $R(1) = 8$ by Theorem 1. Thus every graph on the projective plane contains a vertex of degree less than $R(1)$, and $\mathrm{ch}(G) \leq R(1)$.

Suppose that G is a 1-embedded graph that is list critical for list-r-colorings (*i.e.*, it is r-choosable, but there is a list assignment of $r - 1$ colors to each vertex such that there is no list coloring of G, and every proper subgraph of G is $(r - 1)$-choosable). By Theorem 1, $E \leq 4V + 4g - 8$. If G is a complete graph K_r, then by Theorem 1, $\binom{r}{2} \leq 4r - 8 + 4g$. This implies that $r \leq R(g)$ and $\mathrm{ch}(G) \leq R(g)$.

Suppose now that $G \neq K_r$. It is easy to see that every graph on $r + 1$ vertices that does not contain K_r is r-choosable. Therefore, $V \geq r + 2$. Kostochka and Stiebitz [11] proved that every list critical graph distinct from the complete graph satisfies

$$2E \geq (r - 1)V + r - 3.$$

This inequality combined with Theorem 1 implies

$$(r - 9)V + r - 8g + 13 \leq 0. \tag{2}$$

Since $r = R(g) \geq 9$ for $g \geq 2$, inequality (2) and the condition that $V \geq r + 2$ imply that $(r - 9)(r + 2) + r - 8g + 13 \leq 0$. Solving this quadratic inequality shows that $r \leq R_1(g) = \lfloor \frac{1}{2}(6 + \sqrt{56 + 32g}) \rfloor$. If $g \geq 11$, then it is easy to see that

$$\tfrac{1}{2}(6 + \sqrt{56 + 32g}) + 1 \leq \tfrac{1}{2}(9 + \sqrt{17 + 32g})$$

which in turn implies that $R_1(g) \leq R(g) - 1$. For $g = 2$ and for $4 \leq g \leq 10$, the same conclusion can be drawn (by simply calculating the values $R_1(g)$ and $R(g)$). □

The analogue of Dirac's theorem in the preceding corollary is not true for $g = 0$ since $\mathrm{ch}((K_3 \square K_2)_{vf}) = 6$. Whether it is true for $g = 1$ and 3 remains open.

Ringel notes that for the torus or Klein's bottle, the inequality of Corollary 1.1 becomes $\chi(G) \leq 9$. He exhibits a 1-embedding of K_9 on both these surfaces by placing the nine vertices in a 3×3 grid and drawing all vertical, horizontal and diagonal edges [15]. Thus Corollary 1.1 is best possible when $g = 2$. In contrast, it is easy to see that there is no toroidal graph G with $G_{vf} = K_9$. Consequently, we immediately get the following corollary. Note that Schumacher has obtained the same result for usual coloring [16].

Corollary 2. *If G is a graph embedded in the torus or the Klein bottle, then* $\mathrm{ch}(G_{vf}) \leq 8$.

Ringel also showed that Corollary 1.1 is best possible for an orientable surface with $g = 82$ [15]. In contrast, unlike the Heawood bound which is optimal for all surfaces except for the Klein bottle, there are infinitely many cases where Ringel's bound is not sharp.

Theorem 2. *Let $g = \frac{1}{8}r(r-1)-r-2$, where r is a positive integer that is divisible by* 8, *and let S be a surface of Euler genus g. Then $r = R(g)$ and K_r cannot be* 1*-embedded in S. Consequently, every graph G that is* 1*-embedded in S has* $\mathrm{ch}(G) \leq R(g) - 1$.

Proof. A routine calculation shows that $r = \frac{1}{2}(9 + \sqrt{17 + 32g}) = R(g)$. Since r is even, all vertices of K_r have odd degree. By repeating the first (easy) part of the proof of Corollary 1.1 for $G = K_r$, and applying the stronger version of the inequality of Theorem 1 with $t = r$, we get a contradiction. This implies that K_r is not 1-embeddable in S. By Corollary 1.1, this implies that the choice number of 1-embedded graphs in S satisfies the stronger bound. □

3. Locally Planar Embeddings

Given a 1-embedded graph G, the *edge-width* of G, denoted by $w(G)$, is the length of a shortest non-contractible cycle in G. This definition generalizes the notion of the width of an embedded graph introduced in [2]. The notion of width has gained a prominent place in topological graph theory [13]. Thomassen has shown that if G is embedded on S_g and $w(G)$ is large enough, then $\chi(G) \leq 5$ [18]. A specific theorem of this type due to Albertson and Hutchinson [1] is that if G is embedded in a surface S of Euler genus $g > 0$ and $w(G) \geq 64(2^g - 1)$, then $\chi(G) \leq 5$.

If G is embedded in a surface of Euler genus g, let G_D denote the dual of G. If we control both $w(G)$ and $w^*(G) = w(G_D)$, we can use five colors on the vertices and five different colors on the faces. Formally, if $w(G) \geq 64(2^g - 1)$ and $w^*(G) \geq 64(2^g - 1)$, then $\chi_{vf}(G) \leq 10$. It is not surprising that fewer colors will suffice.

Theorem 3. *Suppose that G is* 1*-embedded in a surface of Euler genus g. If $w(G) \geq 104g - 204$, then* $\mathrm{ch}(G) \leq 8$.

Proof. We know that all graphs that can be 1-embedded in the plane or the projective plane are 8-choosable. Also, 1-embedded graphs in the torus or Klein bottle are 8-choosable if they do not contain K_9. However, since the edge-width of K_9 embedded in any surface is 3, while $w(G) \geq 4$, K_9 cannot be a subgraph in G.

Suppose now that $g \geq 3$ and that, contrary to the desired conclusion, $\mathrm{ch}(G) \geq 9$. Then G contains a list-9-critical subgraph, say G', with V' vertices and E' edges. If $G' = K_9$, then $w(G) \leq w(G') = 3 < 104g - 204$. Otherwise, by Gallai's Theorem for list colorings, see [10], $E' \geq 4V' + \frac{V'}{26}$. From Theorem 1 we know that $E' \leq 4V' - 8 + 4g$. Combining inequalities yields $V' \leq 104g - 208 < w(G)$. Consequently, G' cannot contain a non-contractible cycle and is therefore 1-embedded in the plane. Our earlier results now imply that $\mathrm{ch}(G') \leq 8$. □

4. Open Questions

We summarize some open problems related to vertex-face colorings of embedded graphs.

Question 1. *If G is planar, is* $\mathrm{ch}_{vf}(G) \leq 6$ (*or* 7)*?*

Schumacher has shown that if G is 1-embedded on the projective plane, then $\chi(G) \leq 7$, and that there exists G embedded on the projective plane such that $\chi_{vf}(G) = 7$ [16].

Question 2. *If G is embedded* {*resp.* 1*-embedded*} *on the projective plane, what is a best possible upper bound for* $\mathrm{ch}(G_{vf})$ {*resp.* $\mathrm{ch}(G)$} *?*

Question 3. *If G embeds on either the torus or Klein's bottle, is* $\chi_{vf}(G) \leq 7$*?*

We do not know of any locally planar 1-embedded graph that requires more than six colors.

Question 4. *Is there a surface such that for every w, there exists a* 1*-embedded graph G with* $w(G) \geq w$ *and* $\chi(G) \geq 8$ (*resp.* ≥ 7)*?*

If the answer to the preceding question is affirmative, we still have the following:

Question 5. *Is there a surface such that for every w, there exists a graph G embedded in S with* $w(G) \geq w$ *and* $\chi_{vf}(G) \geq 8$(*resp.* ≥ 7)*?*

The last three questions are also open if list coloring replaces standard coloring.

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