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# Kempe Equivalence of Colorings

Bojan Mohar

**Abstract.** Several basic theorems about the chromatic number of graphs can be extended to results in which, in addition to the existence of a $k$-coloring, it is also shown that all $k$-colorings of the graph in question are Kempe equivalent. Here, it is also proved that for a planar graph with chromatic number less than $k$, all $k$-colorings are Kempe equivalent.

## 1. Introduction

Let $G$ be a graph and $k \geq 1$ an integer. A vertex set $U \subseteq V(G)$ is *independent* if no two vertices of $U$ are adjacent in $G$. A *$k$-coloring* of $G$ is a partition of $V(G)$ in $k$ independent sets $U_1, \ldots, U_k$, called *color classes*. If $v \in U_i$ ($i \in \{1, \ldots, k\}$), then $v$ is said to have *color $i$*. Every $k$-coloring can be identified with a mapping $c : V(G) \to \{1, \ldots, k\}$ where $c(v)$ is the color of $v$. The *chromatic number* of $G$ is denoted by $\chi(G)$.

Let $a, b \in \{1, \ldots, k\}$ be distinct colors. Denote by $G(a, b)$ the subgraph of $G$ induced on vertices of color $a$ or $b$. Every connected component $K$ of $G(a, b)$ is called a *K-component* (short for *Kempe component*). By switching the colors $a$ and $b$ on $K$, a new coloring is obtained. This operation is called a *K-change* (short for *Kempe change*). Two $k$-colorings $c_1, c_2$ are K-*equivalent* (or $\mathrm{K}^k$-*equivalent*), in symbols $c_1 \sim_k c_2$, if $c_2$ can be obtained from $c_1$ by a sequence of K-changes, possibly involving more than one pair of colors in successive K-changes.

Let $\mathcal{C}_k = \mathcal{C}_k(G)$ be the set of all $k$-colorings of $G$. The equivalence classes $\mathcal{C}_k / \sim_k$ are called the $\mathrm{K}^k$-*classes* (or just K-*classes*). The number of $\mathrm{K}^k$-classes of $G$ is denoted by $\mathrm{Kc}(G, k)$.

K-changes have been introduced by Kempe in his false proof of the four color theorem. They have proved to be an utmost useful tool in graph coloring theory. It remains one of the basic and most powerful tools. The results of this paper show that some basic theorems about graph colorings can usually be turned into

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stronger K-equivalence results where it can be proved that all $k$-colorings are K-equivalent. These results have been one of our motivations to study K-equivalence of colorings. Several such results have been published by Meyniel and Las Vergnas [9, 7] who proved, in particular, that all 5-colorings of a planar graph (respectively, a $K_5$-minor free graph) are K-equivalent. Fisk [5] proved that all 4-colorings of an Eulerian triangulation of the plane are K-equivalent. We extend these results by showing that in every planar graph $G$ with chromatic number less than $k$, all $k$-colorings are K-equivalent (see Corollary 4.5).

The second motivation to study K-equivalence is the possibility to generate colorings either by using K-changes as a heuristic argument [1, 12], or with the goal of obtaining a random coloring by applying random walks and rapidly mixing Markov chains [16]. For instance, Vigoda [16] proved that the Markov chain, whose state space is $\mathcal{C}_k(G)$ and whose transitions correspond to K-changes, quickly converges to the stationary distribution if $k \geq \frac{11}{6}\Delta(G)$. On the other hand [8], there are bipartite graphs for which the Markov chain needs exponentially many steps to come close to the stationary distribution if $k = O(\Delta/\log\Delta)$. Later, Hayes and Vigoda [6] proved rapid mixing for $k > (1+\varepsilon)\Delta(G)$ for all $\varepsilon > 0$ assuming that $G$ has girth more than 9 and $\Delta = \Omega(\log n)$. Dyer et al. [3] studied the same phenomenon on random graphs with expected average degree $d$, where $d$ is a constant. Kempe change method has been successfully applied in some experiments [13] leading to new theoretical results. In relation to this, let us mention that K-changes appear in theoretical physics in the study of the Glauber dynamics for the hard-core lattice gas model at zero temperature. The related Wang-Swendsen-Kotecký dynamics [17, 18] uses K-changes to move from state to state. The question whether the associated Markov chain is ergodic is the same as asking if all colorings are K-equivalent. We refer to a survey by Sokal [14] for further details.

Finally, let us observe that Claude Berge, to whom we dedicate this paper, considered Kempe changes in some of his late papers, e.g., [2].

## 2. Basic results

The following result shows that the study of $\mathrm{K}^k$-equivalence may be interesting also when $k$ is much larger than the chromatic number of the graph. It also shows that it is possible that $\mathrm{Kc}(G, k-1) = 1$ and $\mathrm{Kc}(G, k) > 1$.

**Proposition 2.1.**

(a) *Let $G$ be a bipartite graph and $k \geq 2$ an integer. Then* $\mathrm{Kc}(G, k) = 1$.

(b) *For any integers $l \geq 3$ and $k > l$, there exists a graph $G$ with chromatic number $l$ such that* $\mathrm{Kc}(G, l) = 1$ *and* $\mathrm{Kc}(G, k) > 1$.

*Proof.* (a) Clearly, any two 2-colorings are $\mathrm{K}^2$-equivalent. Hence, it suffices to prove that every $k$-coloring of $G$ is K-equivalent to a 2-coloring. This is easy to see and is left to the reader.

(b) Let $G$ be the categorical product $K_l \times K_k$. Its vertices are pairs $(i, j)$, $1 \le i \le l$, $1 \le j \le k$, and vertices $(i, j)$ and $(i', j')$ are adjacent if and only if $i \ne i'$ and $j \ne j'$. Let $c$ be the $l$-coloring of $G$ where $c((i, j)) = i$, and let $c'$ be the $k$-coloring of $G$ where $c'((i, j)) = j$. It is easy to see that $c$ is the unique $l$-coloring of $G$, so $\chi(G) = l$ and $\mathrm{Kc}(G, l) = 1$. On the other hand, $c'$ is not K-equivalent to any other $k$-coloring since all its 2-colored subgraphs $G(a, b)$ are connected. In particular, it is not K-equivalent to $c$, so $\mathrm{Kc}(G, k) > 1$. □

There are other graphs with the same properties as in Proposition 2.1(b). They can be obtained from $K_l \times K_k$ by replacing every vertex $(i, j)$ by an independent set $U(i, j)$ (of any size) and, for any two adjacent vertices $(i, a)$ and $(i', b)$ of $K_l \times K_k$ adding edges between $U(i, a)$ and $U(i', b)$ so that the subgraph induced on $\cup_{i=1}^{l}(U(i, a) \cup U(i, b))$ is connected. This construction describes the $l$-colorable graphs with a $k$-coloring which is not K-equivalent to any other $k$-coloring. More generally, it would be interesting to characterize $l$-colorable graphs ($l < k$) with a $k$-coloring ($k$ large) which is not K-equivalent to any $(k-1)$-coloring. This problem was considered by Las Vergnas and Meyniel [7] who conjectured that such graphs contain the complete graph $K_k$ as a minor.

**Lemma 2.2.** *Suppose that $c_0$ is a $(k-1)$-coloring of a graph $G$ and that $U$ is an independent vertex set of $G$. Then $c_0$ is* K*-equivalent in $\mathcal{C}_k(G)$ to a $k$-coloring of $G$, one of whose color classes is $U$.*

*Proof.* Let $U = \{u_1, \ldots, u_r\}$. For $i = 1, \ldots, r$, let $c_i$ be the $k$-coloring of $G$ that is obtained from $c_{i-1}$ by recoloring the vertex $u_i$ with color $k$. It is clear that the vertex $u_i$ forms a K-component in $c_{i-1}$ for colors $k$ and $c_{i-1}(u_i) = c_0(u_i)$. Therefore, $c_i$ is K-equivalent to $c_{i-1}$. This shows that $c_r$ is a coloring that is K-equivalent to $c_0$, and one of its color classes is $U$. □

**Corollary 2.3.** *Let $k$ be an integer. Suppose that $G$ is a graph such that every $k$-coloring of $G$ is* K*-equivalent to some $(k-1)$-coloring. If $U$ is an independent vertex set of $G$, then* $\mathrm{Kc}(G, k) \le \mathrm{Kc}(G - U, k - 1)$.

*Proof.* For $i = 1, 2$, let $c_i$ be a $k$-coloring of $G$. By assumption, $c_i$ is K-equivalent to a $(k-1)$-coloring $c_i'$. By Lemma 2.2, $c_i'$ is K-equivalent to a $k$-coloring $c_i''$, one of whose color classes is $U$. The restrictions of $c_1''$ and $c_2''$ to $G - U$ are $(k-1)$-colorings of $G - U$. If they are $\mathrm{K}^{k-1}$-equivalent, then $c_1'' \sim_k c_2''$, and hence $c_1 \sim_k c_2$. This completes the proof. □

A graph $G$ is *d-degenerate* if every subgraph of $G$ contains a vertex of degree $\le d$. Las Vergnas and Meyniel [7, Proposition 2.1] proved the following result, whose proof we include for completeness.

**Proposition 2.4.** *If $G$ is a $d$-degenerate graph and $k > d$ is an integer, then* $\mathrm{Kc}(G, k) = 1$.

*Proof.* The proof is by induction on $|V(G)|$. The statement is clear if $G = K_1$. Otherwise, let $v$ be a vertex of degree $\le d$, and let $G' = G - v$. Let $c_1$ and $c$ be arbitrary $k$-colorings of $G$. By $c'_1$ and $c'$ we denote their restrictions to $G'$. By the induction hypothesis, $c'_1$ is K-equivalent to $c'$. There is a sequence of K-changes, $c'_1 \sim_k c'_2 \sim_k \cdots \sim_k c'_r = c'$.

For $i = 2, \ldots, r$, let $c_i$ be an extension of $c'_i$ to $G$ which is obtained as follows. There are two colors, say $a_i$ and $b_i$, that are involved in the K-change yielding $c'_i$ from $c'_{i-1}$. We assume that $a_i \neq c_{i-1}(v)$. Now we distinguish three cases. If $c_{i-1}(v) \neq b_i$, then we let $c_i(v) = c_{i-1}(v)$, and the same K-change as performed on $G'$ shows that $c_i \sim_k c_{i-1}$. If $c_{i-1}(v) = b_i$ and $v$ has precisely one neighbor $u$ with $c'_{i-1}(u) = a_i$, then the K-components for $a_i$ and $b_i$ of the coloring $c_{i-1}$ are the same as in $G'$, except that the component containing $u$ is extended by one vertex, namely $v$. Now we set $c_i(v) = c_{i-1}(v)$ if the K-change yielding $c'_i$ does not involve $u$, and we set $c_i(v) = a_i$ if it does. In both cases, $c_i$ is K-equivalent with $c_{i-1}$. Finally, suppose that $c_{i-1}(v) = b_i$ and $v$ has more than one neighbor whose color in $c_{i-1}$ is $a_i$. Then there is a color $b'_i \neq b_i$ that is not contained among the neighbors of $v$ in $c_{i-1}$. Now we first make a K-change replacing $c_{i-1}(v)$ with $b'_i$, and then another change as described above (since now the color of $v$ is different from $a_i$ and $b_i$). This gives the coloring $c_i$ which is K-equivalent with $c_{i-1}$ also in this case.

Finally, repeating the above changes, we see that $c_1$ is K-equivalent to $c_r$, an extension of $c'_r$. Note that $c_r$ and $c$ are the same except that they possibly disagree on $v$. Therefore, if $c_r \neq c$, another K-change can replace the color $c_r(v)$ by $c(v)$ since none of these colors appears in the neighborhood of $v$. This shows that $c_r \sim_k c$, and the proof is complete. $\square$

An immediate corollary of Proposition 2.4 is:

**Corollary 2.5.** *Let $\Delta$ be the maximum degree of a graph $G$ and let $k \geq \Delta + 1$ be an integer. Then* $\mathrm{Kc}(G, k) = 1$*. If $G$ is connected and contains a vertex of degree $< \Delta$, then also* $\mathrm{Kc}(G, \Delta) = 1$.

We conjecture that the last statement of Corollary 2.5 can be extended to include all connected $\Delta$-regular graphs with the exception of odd cycles and complete graphs.

The following proposition is left as an exercise.

**Proposition 2.6.** *Let $G$ be a graph of order $n$, let $\alpha$ be the cardinality of a largest independent vertex set in $G$, and let $k \geq n - \alpha + 1$ be an integer. Then every $k$-coloring of $G$ is* $\mathrm{K}^k$*-equivalent to the $k$-coloring in which a fixed maximum independent set is a color class and every other color class is a single vertex. In particular,* $\mathrm{Kc}(G, k) = 1$.

## 3. Edge-colorings

Coloring the edges of a graph $G$ is the same as coloring the vertices of its line graph $L(G)$. Vizing's Theorem states that the edges of a graph with maximum degree $\Delta$ can be colored with $\Delta + 1$ colors, i.e., $\chi'(G) = \chi(L(G)) \leq \Delta + 1$.

We prove:

**Theorem 3.1.** *Let $\Delta$ be the maximum degree of a graph $G$. If $k \geq \chi'(G) + 2$ is an integer, then* $\mathrm{Kc}(L(G), k) = 1$.

*Proof.* The proof is by induction on $\chi'(G)$. The case when $\chi'(G) \leq 2$ follows by Proposition 2.1(a), so assume that $\chi'(G) \geq 3$.

Let $c$ be an arbitrary $k$-edge-coloring of $G$. First, we claim that $c$ is K-equivalent to a $(k-1)$-edge-coloring. To prove this, we may assume that $c$ has $m > 0$ edges of color $k$ and that every $k$-edge-coloring which is K-equivalent to $c$ has at least $m$ edges of color $k$. The standard "fan" arguments of Vizing show that there is a sequence of K-changes which transforms $c$ into an edge-coloring with $m - 1$ edges of color $k$, a contradiction. (Cf., e.g., [4] for details.) However, in order to make the proof self-contained, we repeat those arguments.

Let $c$ and $m > 0$ be as above. We say that a color $a$ is *missing at* a vertex $v$ of $G$ if no edge incident with $v$ is colored $a$. Since $k \geq \Delta + 2$, at least two colors are missing at each vertex, and at least one of them is different from $k$. Clearly, if the same color $a$ is missing at adjacent vertices $u$ and $v$, then the change of the color of the edge $uv$ to $a$ represents a K-change.

Let $e = uv$ be an edge of color $k$. Suppose that color $a_1$ is missing at $u$ and that $a_0$ is missing at $v$. If $a_0 = a_1$, then changing the color of $e$ to $a_0$ is a K-change, yielding a coloring with $m - 1$ edges of color $k$, a contradiction. So, there is an edge $vv_1$ of color $a_1$. There is a color $a_2 \neq k$ which is missing at $v_1$. If $a_2$ is missing at $v$, we recolor $vv_1$ with $a_2$ and, as mentioned above, are henceforth able to get rid of the color $k$ at $e$. Therefore, there is an edge $vv_2$ of color $a_2$.

Consider the K-component $K \subseteq G(a_0, a_2)$ at the vertex $v_1$. After the corresponding K-change, $a_0$ becomes missing at $v_1$. If $a_0$ is still missing at $v$, then we get a contradiction as above. Therefore, the corresponding K-change has changed the color $a_2$ at $v$ to $a_0$, so $vv_2 \in E(K)$.

We shall now repeat the above procedure and henceforth have distinct edges $vv_1, \ldots, vv_r$ whose colors are $a_1, \ldots, a_r$ (respectively), and such that color $a_i$ is missing at $v_{i-1}$ for $i = 2, \ldots, r$. Moreover, the K-component in $G(a_0, a_i)$ at $v_{i-1}$ is a path from $v_{i-1}$ to $v$, whose last edge is $v_i v$. Having this situation, there is a color $a_{r+1} \neq k$ that is missing at $v_r$. If $a_{r+1} \notin \{a_0, \ldots, a_r\}$, then we consider the K-component $K \subseteq G(a_0, a_{r+1})$ at $v_r$. If this is a path ending at $v$, its last edge, call it $v_{r+1}v$, has color $a_{r+1}$, and we proceed with the next step. If $K$ does not contain $v$, then after the K-change at $K$, $a_0$ is missing at $v_r$ and at $v$. Now, we recolor $vv_r$ with $a_0$, then we recolor $vv_{r-1}$ with $a_r$, $vv_{r-2}$ with $a_{r-1}$, $\ldots$, $vv_1$ with $a_2$. Finally, recolor $e$ with $a_1$. All these recolorings were K-changes, so we have a coloring with $m - 1$ edges of color $k$, a contradiction.

From now on, we may assume that $a_{r+1} = a_j$, where $0 \leq j < r$. Let us consider $K \subseteq G(a_0, a_j)$ at $v_r$. The K-change at $K$ makes $a_0$ missing at $v_r$. Since the component of $G(a_0, a_j)$ containing $vv_j$ is a path from $v_{j-1}$ to $v_j$ and $v$, $K$ does not contain $vv_j$. Therefore, $a_0$ is still missing at $v$. Now we conclude as above. This completes the proof of the claim.

By repeating the above arguments again if necessary, we conclude that $c$ is K-equivalent to a $(\Delta+1)$-edge-coloring.

Fix an edge-coloring $c_0$ with the color partition $E(G) = M_1 \cup \cdots \cup M_r$, $r = \chi'(G)$. It suffices to prove that any $(\Delta+1)$-edge-coloring $c$ of $G$ is K-equivalent with $c_0$ in $\mathcal{C}_{\Delta+2}(L(G))$. By Lemma 2.2, $c$ is K-equivalent to a $(\Delta+2)$-coloring $c'$ whose first color class is $M_r$. Now, the proof is complete by applying induction on the graph $G - M_r$. □

It would be interesting to extend Theorem 3.1 to include $(\Delta+1)$-colorings as well. It is quite plausible that $\mathrm{Kc}(L(G), \chi'(G)+1)$, $\mathrm{Kc}(L(G), \Delta+2)$, or even $\mathrm{Kc}(L(G), \Delta+1)$ are always 1. Let us remark, however, that there are graphs for which $\mathrm{Kc}(L(G), \chi'(G)) > 1$. Such examples are given after Theorem 3.3 below. We emphasize, specifically, the following interesting special case of the above speculations:

**Conjecture 3.2.** *If $G$ is a graph with $\Delta(G) \le 3$, then all its 4-edge-colorings are K-equivalent.*

The most challenging example, the Petersen graph, has been checked using computer by Drago Bokal (and it satisfies the conjecture).

We can say more if $G$ is bipartite.

**Theorem 3.3.** *Let $\Delta$ be the maximum degree of a bipartite graph $G$. If $k \ge \Delta + 1$ is an integer, then* $\mathrm{Kc}(L(G), k) = 1$.

*Proof.* The proof is the same as for Theorem 3.1 except that we need to show that every $k$-edge-coloring of $G$ is $\mathrm{K}^k$-equivalent to a $\Delta$-edge-coloring. This is a standard exercise and is left to the reader. □

The complete bipartite graph $K_{p,p}$ (where $p$ is a prime) has a $p$-edge-coloring in which any two color classes form a Hamiltonian cycle. This example shows that Theorem 3.3 cannot be extended to $\Delta$-colorings, not even for complete bipartite graphs.

**Problem 3.4.** *For which cubic bipartite graphs is* $\mathrm{Kc}(L(G), 3) = 1$*?*

A special case of this problem, when $G$ is planar and 3-connected has been solved by Fisk. Let $G$ be a 3-connected cubic planar bipartite graph. Its dual graph $T$ is a 3-colorable triangulation of the plane. Fisk [5] proved (see Theorem 4.1 below) that any two 4-colorings of $T$ are K-equivalent. If $c_1, c_2$ are 3-edge-colorings of $G$, they determine 4-colorings $c_1^*, c_2^*$ (respectively) of $T$. It is easy to see that a K-change on 4-colorings of $T$ corresponds to a sequence of one or more K-changes among the corresponding 3-edge-colorings in $G$. This implies that $c_1$ and $c_2$ are K-equivalent, and hence $\mathrm{Kc}(L(G), 3) = 1$.

Let us observe that planarity is essential for the above examples since the graph $K_{3,3}$ has non-equivalent edge-colorings, $\mathrm{Kc}(L(K_{3,3}), 3) = 2$.

## 4. Planar graphs

In [11], the author described an infinite class of "almost Eulerian" triangulations of the plane that have a special 4-coloring which is not K-equivalent to any other 4-coloring (and other 4-colorings exist). This shows that there are planar triangulations for which $\mathrm{Kc}(G,4) \geq 2$. By taking 3-sums of such graphs, we get planar triangulations with arbitrarily many equivalence classes of 4-colorings.

Meyniel [9] proved that $\mathrm{Kc}(G,5) = 1$ for every planar graph (and also $\mathrm{Kc}(G,k) = 1$ if $k \geq 6$, which follows by 5-degeneracy of planar graphs). In this section we prove a similar result for 4-colorings in the case when $G$ is 3-colorable (cf. Theorem 4.4). A special case of this result, when $G$ is a 3-colorable triangulation of the plane was proved by Fisk [5].

**Theorem 4.1 (Fisk** [5]**).** *Let $G$ be a 3-colorable triangulation of the plane. Then $\mathrm{Kc}(G,4) = 1$.*

In order to extend Theorem 4.1, we shall need two auxiliary results.

**Lemma 4.2.** *Suppose that $G$ is a subgraph of a graph $\tilde{G}$. Let $\tilde{c}_1, \tilde{c}_2$ be $r$-colorings of $\tilde{G}$. Denote by $c_i$ the restriction of $\tilde{c}_i$ to $G$, $i = 1,2$. If $\tilde{c}_1$ and $\tilde{c}_2$ are $\mathrm{K}^r$-equivalent, then $c_1$ and $c_2$ are $\mathrm{K}^r$-equivalent colorings of $G$.*

*Proof.* Any K-component in $\tilde{G}$ gives rise to one or more K-components in $G$, with respect to the induced coloring of $G$. This implies the lemma. □

A *near-triangulation* of the plane is a plane graph such that all its faces except the outer face are triangles.

**Proposition 4.3.** *Suppose that $G$ is a planar graph with a facial cycle $C$. If $c_1, c_2$ are 4-colorings of $G$, then there is a near-triangulation $T$ of the plane with the outer cycle $C$ such that $T \cap G = C$ and there are 4-colorings $c_1', c_2'$ of $G$ which are K-equivalent to $c_1$ and $c_2$, respectively, such that they both can be extended to 4-colorings of $G \cup T$. Moreover, if the restriction of $c_1$ to $C$ is a 3-coloring, then $c_1' = c_1$, and $c_1$ can be extended to a 3-coloring of $T$.*

*Proof.* Let $C = v_1v_2 \dots v_kv_1$. The proof is by induction on $k$. If $k = 3$, then $T = C$, $c_1' = c_1$, and $c_2' = c_2$. Suppose now that $k \geq 4$. If there are indices $i, j$ $(1 \leq i < j \leq k)$ such that $v_i$ and $v_j$ are not consecutive vertices of $C$ and such that $c_1(v_i) \neq c_1(v_j)$ and $c_2(v_i) \neq c_2(v_j)$, then we add the edge $v_iv_j$ inside $C$ and apply induction on $C_1 = v_iv_{i+1} \dots v_jv_i$ and $C_2 = v_jv_{j+1} \dots v_kv_1 \dots v_iv_j$.

More precisely, let $G_1 = G + v_iv_j$. By the induction hypothesis for the cycle $C_1$, there are sequences of K-changes in $G_1$ (and hence also in $G$, by Lemma 4.2) transforming $c_1$ into $c_{11}$, and transforming $c_2$ into $c_{12}$, respectively, and there is a near-triangulation $T_1$ with outer cycle $C_1$ such that $c_{11}$ and $c_{12}$ can be extended to colorings $\bar{c}_{11}$ and $\bar{c}_{12}$ of $G_1 \cup T_1$. Next, apply the induction hypothesis to $G_1 \cup T_1$ for the facial cycle $C_2$ and colorings $\bar{c}_{11}$ and $\bar{c}_{12}$. Let $T_2$ be the corresponding near-triangulation, and $c_{11}', c_{12}'$ the corresponding colorings of $G_1 \cup T_1$ that can be extended to $G_1 \cup T_1 \cup T_2$. By Lemma 4.2, the K-changes which produce $c_{11}'$ and

$c'_{12}$ from $\bar{c}_{11}$ and $\bar{c}_{12}$, respectively, can be made in $G$. All together, the restriction $c'_l$ of the coloring $c'_{1l}$ to $G$ is K-equivalent to $c_l$ in $G$ ($l = 1, 2$). Clearly, $c'_l$ has an extension to $G \cup T$, where $T = T_1 \cup T_2$. Therefore, $T$ can be taken as the required near-triangulation for $C$.

If the restriction of $c_1$ to $C$ is a 3-coloring, then $c_{11} = c_1$ and the restriction of $c_1$ to $C_1$ can be extended to a 3-coloring of $T_1$. Similarly, $c'_{11} = c_1$, and $c_1$ can be extended to a 3-coloring of $T$. This proves the "moreover" part of the proposition.

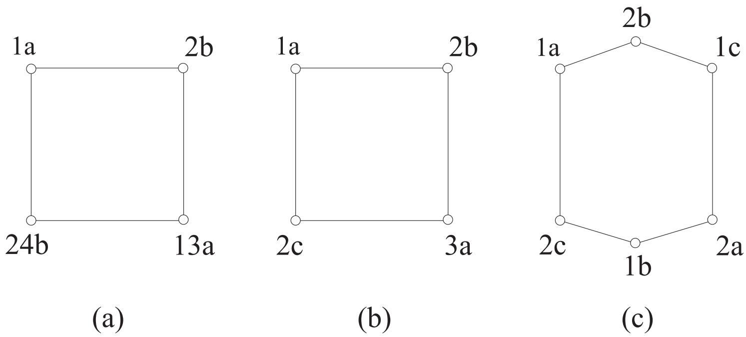


FIGURE 1. The special cases

Next, we show that vertices $v_i, v_j$ exist unless one of the cases in Figure 1 occurs (where $c_1$ is represented by colors 1–4 and $c_2$ by colors a–d), up to permutations of colors, dihedral symmetries of $C$ and up to changing the roles of $c_1$ and $c_2$. This is easy to see if $k = 4$. The details are left to the reader. If $k \geq 5$, we argue as follows. Suppose that $v_i, v_j$ do not exist. We may assume that $c_1(v_1) = c_1(v_3) = 1$. Then $c_1(v_1) \neq c_1(v_4)$, so $c_2(v_1) = c_2(v_4) = a$ may be assumed. Suppose that $c_1(v_2) = 2$ and $c_2(v_2) = b$. Since $b = c_2(v_2) \neq c_2(v_4) = a$, we have $c_1(v_4) = c_1(v_2) = 2$. Now, $c_1(v_2) \neq c_1(v_5)$, so $c_2(v_5) = b$. Next, $c_2(v_5) \neq c_2(v_3)$ implies that $c_1(v_5) = c_1(v_3) = 1$. It follows, in particular, that $k \geq 6$. Similar conclusions as before imply that $c_1(v_6) = 2$ and $c_2(v_3) = c_2(v_6) = c$. If $k = 6$, this is the exceptional case of Figure 1(c). If $k \geq 7$, then we see that either $v_2, v_7$ or $v_4, v_7$ is the required pair $v_i, v_j$.

Let us consider the exceptional case shown in Figure 1(c). Let $v_1$ be the vertex with $c_1(v_1) = 1$ and $c_2(v_1) = $ a (upper-left). Try first a K-change of colors 1 and 3 at $v_1$. If this change gives rise to the same exception, there is a (1,3)-colored path $P$ joining $v_1$ and $v_3$. Now, a K-change of colors 2 and 4 at $v_2$ changes $c_1$ into a coloring which does not fit Figure 1(c). None of these K-changes affects $c_2$, and we are done unless we do not want to change $c_1$ because of the "moreover" part. In that case we are allowed to use the fourth color in the extension of $c_2$, and we take $T$ to be the near-triangulation with one interior point joined to all vertices on $C$.

Consider now the case of Figure 1(b). By symmetry, we may assume that $c_1$ is not allowed to be changed according to the "moreover" part of the proposition. Let $v_1$ be the vertex in the upper-left corner. By a K-change of colors a and d at $v_1$, or of b and c at $v_2$, we replace $c_2$ either by a 4-coloring which uses on $C$ all four or only two of the colors. In each case, we can triangulate $C$ by adding two adjacent vertices $p, q$ such that $p$ is adjacent to $v_1, v_2, v_4$, and $q$ is adjacent to $v_2, v_3, v_4$.

The final case is the one shown in Figure 1(a). If we want $c_2' = c_2$, then we let $T$ be the near-triangulation consisting of $C$ and a vertex of degree 4 inside. Then $c_2$ extends to a 3-coloring of $T$, and $c_1$ also extends to a 4-coloring with the exception of the case when all vertices on $C$ have distinct colors. In the latter case, we can either K-change colors 1 and 3 at $v_1$, or change 2 and 4 at $v_2$, without affecting the colors at $v_3$ and $v_4$. The new coloring $c_1'$ extends to $T$.

Suppose, finally, that we want $c_1' = c_1$. In this case, $c_1$ is a 3-coloring on $C$. Up to symmetries, we may assume either $c_1(v_3) = 1$ and $c_1(v_4) = 2$, or $c_1(v_3) = 3$ and $c_1(v_4) = 2$. In the first case we can take the same near-triangulation $T$ as above (one interior vertex of degree 4). In the latter case, we take two interior vertices $u_1, u_2$ in $T$, where $u_1$ is adjacent to $u_2, v_4, v_1, v_2$ and $u_2$ is adjacent to $u_1, v_2, v_3, v_4$. This completes the proof. □

**Theorem 4.4.** *Let $G$ be a 3-colorable planar graph. Then* $\mathrm{Kc}(G, 4) = 1$.

*Proof.* In order to be able to assume that $G$ is 2-connected, we apply induction on the number of blocks of $G$. If $G = G_1 \cup G_2$, where $G_1 \cap G_2$ is either empty or a cutvertex $v$, we apply induction hypotheses on $G_1$ and $G_2$ but making sure that we never use a K-change on a K-component containing $v$. This is possible since a K-change using component $K$ of $G_i(a, b)$ is the same as making K-changes on all components of $G_i(a, b)$ distinct from $K$.

From now on, we assume that $G$ is 2-connected. Let $c_1$ be a 3-coloring of $G$. It suffices to see that every 4-coloring of $G$ is K-equivalent to $c_1$.

Let $c_2$ be a 4-coloring of $G$. Let $C_1, \ldots, C_m$ be the facial cycles of $G$. Let $G_0 = G$, $c_1^0 = c_1$ and $c_2^0 = c_2$. For $i = 1, \ldots, m$, we apply Proposition 4.3 to the graph $G_{i-1}$, its facial cycle $C_i$ and the colorings $c_1^{i-1}$ and $c_2^{i-1}$. We conclude that there is a near-triangulation $T_i$ with outer cycle $C_i$ such that $G_{i-1} \cap T_i = C_i$. Let $G_i = G_{i-1} \cup T_i$. By Proposition 4.3, $c_1^{i-1}$ can be extended to a 3-coloring $c_1^i$ of $G_i$, and $c_2^{i-1}$ is K-equivalent in $G_{i-1}$ to a 4-coloring that has an extension $c_2^i$ to $G_i$.

The final graph $G_m$ is a triangulation with the 3-coloring $c_1^m$. By Theorem 4.1, $c_2^m$ is K-equivalent to $c_1^m$. By successively applying Lemma 4.2 to $G_{m-1} \subseteq G_m$, etc. up until $G_0 \subseteq G_1$, we conclude that $c_2^{m-1} \sim_k c_1^{m-1}$ in $G_{m-1}$, etc., until finally concluding that $c_1 = c_1^0 \sim_k c_2^0 = c_2$ in $G_0 = G$. □

It is worth mentioning that there exist 3-colorable planar graphs $G$ with $\mathrm{Kc}(G, 3) \geq 2$. An infinite family of such examples can be obtained as follows. In [11], a family of planar triangulations $T$ is constructed for which a special 4-coloring exists for which no nontrivial K-change exists. Let $T^*$ be the dual cubic graph, and let $L$ be its line graph. It is well-known that every 4-coloring of a triangulation gives rise to a 3-edge-coloring of its dual. The special 4-coloring of $T$ therefore determines a (vertex) 3-coloring of $L$. The property of the 4-coloring of $T$ implies that the 3-coloring of $L$ is not K-equivalent with any other 3-coloring of $L$. Since $T$ admits other 4-colorings, also $L$ admits other 3-colorings, hence $\mathrm{Kc}(L, 3) \geq 2$.

Theorem 4.4 combined with Proposition 2.1(a) and the aforementioned result of Meyniel [9] yield:

**Corollary 4.5.** *Let $G$ be a planar graph and $k > \chi(G)$ and integer. Then*

$$\mathrm{Kc}(G, k) = 1.$$

A planar graph $G$ may have 4-colorings which are not K-equivalent. However, if $G$ is "almost 3-colorable", this is not likely to happen.

**Problem 4.6.** *Suppose that $G$ is a 4-critical planar graph. Is it possible that $G$ has two 4-colorings that are not K-equivalent to each other?*

## 5. Some further open problems

In the preceding sections, we have exposed several open problems about K-classes of graph colorings. Two further questions are presented below.

Meyniel [10] proved that a graph, in which every odd cycle of length 5 or more has at least two chords, is perfect. He conjectured that for every such graph $G$ and every integer $k \geq \chi(G)$, $\mathrm{Kc}(G, k) = 1$. He proved that every $k$-coloring of $G$ is K-equivalent to some $\chi(G)$-coloring. The last property does not hold for arbitrary perfect graphs.

Let $G$ be a triangulation of some orientable surface, and let $c$ be a 4-coloring of $G$. Let $t_1^+$ be the number of facial 3-cycles whose coloring (in the clockwise order around the face) is 234, and let $t_1^-$ be the number of facial 3-cycles whose coloring is 432. Let $d(c) = |t_1^+ - t_1^-|$. The number $d(c)$ turns out to be invariant on permutations of the colors, and it is called the *degree* of the coloring $c$ [5]. The degree, in particular its parity has been studied by Tutte who also observed that this is Kempe invariant, i.e., all colorings within the same K-class have the same parity.

One can also define two colorings to be *close* if they have at least one color class in common. Two colorings are *similar* if there is a sequence of colorings, starting with one and ending with the other, such that any two consecutive colorings in this sequence are close. The parity of the degree is constant on close colorings. It is not difficult to find triangulations of the plane with two similarity classes of 4-colorings. However, in all such examples known, colorings in different similarity classes have different parity of the degree. Tutte [15] asked if it is possible to have non-similar 4-colorings of a planar triangulation whose degrees have the same parity.

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Bojan Mohar
Department of Mathematics, University of Ljubljana
1000 Ljubljana, Slovenia
e-mail: `bojan.mohar@uni-lj.si`