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Some Recent Progress and Applications in Graph Minor Theory

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Abstract. In the core of the seminal Graph Minor Theory of Robertson and Seymour lies a powerful theorem capturing the "rough" structure of graphs excluding a fixed minor. This result was used to prove Wagner's Conjecture that finite graphs are well-quasi-ordered under the graph minor relation. Recently, a number of beautiful results that use this structural result have appeared. Some of these along with some other recent advances on graph minors are surveyed.

Key words. Graph minor theory, Tree-width, Tree-decomposition, Path-decomposition, Complete graph minor, Excluded minor, Complete bipartite minor, Connectivity, Hadwiger Conjecture, Grid minor, Vortex structure, Near embedding, Graphs on surfaces

1. Introduction

Graphs in this paper are finite and may have loops and multiple edges. An exception is a short section on infinite graphs. A graph H is a *minor* of a graph K if H can be obtained from a subgraph of K by contracting edges. A graph H is a *topological minor* of K if K contains a subgraph which is isomorphic to a graph that can be obtained from H by subdividing some edges. In such a case, we also say that K contains a *subdivision* of H.

The deepest and by many considered as the most important work in graph theory is the Graph Minor Theory developed by Robertson and Seymour. It took more than 21 years to publish this seminal work in a series of 20+ long papers. All together there are 23 papers, Graph Minors I–XXIII. The first 20 of these have been written back in the 1980's and have already appeared, the last one being published in 2004. They include overall theory of graph minors. Two additional papers, Graph Minors

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XXI and XXII, provide some missing details of proofs in Graph Minors XIII. The last one, Graph Minors XXIII, gives an extension which has been found later and settles a conjecture of Nash-Williams about immersions of (finite) graphs.

Graph Minors project resulted in many theoretical advances, but it also has algorithmic applications, and some of the methods have been successfully used in practical computation.

Two most important and actually best known results concerning graph minor theory are presented below. The first one was conjectured by Wagner and was known as Wagner's Conjecture. It implies that graphs are well-quasi ordered with respect to the graph minor relation.

Theorem 1.1 ([150]). *For every infinite sequence $G_1, G_2, \dots$ of graphs, there exist distinct integers $i < j$ such that G_i is a minor of G_j.*

Graph minors are intimately related to the *k-Disjoint Paths Problem*: given a graph G and k pairs $(s_1, t_1), \dots, (s_k, t_k)$ of vertices of G, decide whether there are k mutually vertex disjoint paths of G, the ith path linking s_i and t_i for $i = 1, \dots, k$. If k is part of the input of the problem, then this is one of the well-known NP-complete problems [79], and it remains NP-complete even if G is restricted to be planar. However, for any fixed number of pairs, the situation changes.

Theorem 1.2 ([143]). *For every fixed integer k, there is a polynomial time algorithm to resolve the k-Disjoint Paths Problem. Actually, the time complexity is $O(n^3)$, where n is the order of the input graph G.*

This result is, in a sense, surprising since the corresponding problem for digraphs is NP-complete even when we consider the fixed value $k = 2$ (Fortune, Hopcroft and Wylie [54]).

The Disjoint Paths Problem is easily seen to be polynomially equivalent to the problem of deciding if a fixed graph H with k edges is a topological minor in G. Consequently, it is also polynomially equivalent to the *H-Minor Problem* of deciding if a fixed graph H is a minor in G.

Theorem 1.3 ([143]). *For every fixed graph H, there exists an $O(n^3)$ algorithm for deciding if a given graph of order n contains H as a minor.*

Bruce Reed (private communication) announced an improvement – an $O(n^2)$ algorithm for solving the k-Disjoint Paths Problem and the H-Minor Problem.

But the heart of Graph Minor Theory is a decomposition theorem capturing the structure of all graphs excluding a fixed minor. At a high level, the theorem says that every such graph can be decomposed into a collection of graphs each of which can be "almost" embedded into a surface of bounded genus, combined in a tree-like structure. What we are most interested in is this structural theorem and its applications. In this article, we shall focus on this point. Let us remind that there are some existing texts that cover this topic. A good introduction is Diestel's textbook [37].

Excluded minors are surveyed by Thomas in [172]. Graph minors and tree-width are studied by Reed [126]. This survey may also be viewed as an up-to-date information on these surveys.

There are three main purposes for this paper. First, the techniques and tools developed in graph minor theory are presented. In particular, tree-width and tangles are now well-understood, and their use led to several beautiful conjectures and results. We shall present some of them in this survey. Also, graph minors and embeddings of graphs in surfaces are closely related. A stimulating survey article by Thomassen [185] addresses how they are related to each other. We would focus on tools from graph minors, and their applications to coloring on a fixed surface. This was already overvieweded by Mohar [119], but there is some progress in recent years.

Second, we would like to present main structural theorems from Graph Minor Theory. There are several "spin off" formulations of these structural theorems. One of them, the main result of Graph Minors XVII [147] which is less known, turns out to be more useful in several instances. So we would like to present it here, and address how to use it.

Third, we would like to present recent progress on the applications of graph minor structure theorem. Several results concerning connectivity, toughness and their applications to Hadwiger's conjecture have been obtained recently by using these structure results.

These three are main topics of this paper. But before proceeding with details, let us begin with some motivation for Graph Minor Theory. The starting point of graph minors is definitely the Kuratowski-Wagner theorem.

Theorem 1.4. *A graph G is planar if and only if G does not contain K_5 or $K_{3,3}$ as a minor.*

This theorem tells us that for planar graphs, there are only two forbidden minors. It is a natural question to ask if a similar result holds for other surfaces: can one characterize graphs embeddable in a fixed surface Σ by a finite list of forbidden minors? Until now, the only surface besides the plane (or the sphere), for which such a list is known, is the projective plane. There are precisely 35 forbidden minors for projective planar graphs [4, 66], see also [122]. It remains a challenging open problem to produce the list for the torus and the Klein bottle. The number of minimal forbidden minors for these two surfaces is already enormous, several thousands of them have been found by computer searches.

Nevertheless, the existence of a finite list of obstructions was proved for non-orientable surfaces by Archdeacon and Huneke [5]. Their proof is constructive. A non-constructive proof for general surfaces was obtained by Robertson and Seymour in [138]. The first constructive proof by Mohar [118] has appeared more than ten years afterwards.

A class $\mathcal{M}$ of graphs is said to be *minor-closed* if for every $G \in \mathcal{M}$, all minors of G are also in $\mathcal{M}$. Examples of minor-closed classes are the collection of all planar graphs, and more generally, all graphs that can be embedded in a fixed surface. Every minor-closed class $\mathcal{M}$ can be described by specifying the set of all minor-minimal graphs that are not in $\mathcal{M}$ — these graphs are called the *forbidden minors* for $\mathcal{M}$.

Having the above facts, Klaus Wagner formulated a fundamental conjecture, which extends this finite basis property of graphs on a fixed surface to arbitrary minor-closed classes of graphs. This conjecture is known as *Wagner's Conjecture* and is equivalent to Theorem 1.1. This problem was the main motivation for developing Graph Minors Theory.

Theorem 1.5 (Robertson and Seymour [150]). *For every minor-closed family of graphs, the set of forbidden minors is finite.*

There are other fundamental problems that were motivated by Theorem 1.1, namely structural result for K_k-minor-free graphs, and Hadwiger's conjecture. Theorem 1.4 is important because it gives a good characterization for planar graphs, but we can think of it as a structural theorem excluding K_5 and $K_{3,3}$-minors. What about just excluding K_5-minors? In 1937, Wagner [193] gave a characterization of graphs without K_5-minors. To state the theorem, we need some definitions.

Let G_1 and G_2 be graphs with disjoint vertex sets, let $k \geq 1$ be an integer, and for $i = 1, 2$, let $X_i \subseteq G_i$ be a k-clique in G_i, i.e., a set of k mutually adjacent vertices. For $i = 1, 2$, let G'_i be obtained from G_i by deleting a (possibly empty) set of edges with both ends in X_i. Let G be the graph obtained from G'_1 and G'_2 by identifying X_1 and X_2. Then we say that G is a *clique-sum* of *order* k, or simply a k-*sum* of G_1 and G_2. Let V_8 be the graph obtained from the 8-cycle C_8 by joining each pair of diagonally opposite vertices by an edge (see Figure 1). Now we can present Wagner's characterization of graphs without K_5-minors.

Theorem 1.6 (Wagner [193]). *A graph has no K_5-minors if and only if it can be obtained from planar graphs and subgraphs of V_8 by means of clique-sums of orders at most three.*

Theorem 1.6 implies that the Four Color Theorem is equivalent to the statement that every graph without K_5-minors can be colored with four colors (Wagner's

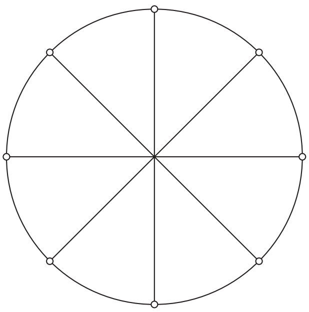


Fig. 1. Wagner's graph V_8

Equivalence Theorem). This result prompted Hadwiger [69] to make his famous conjecture: every graphs without K_k-minor is $(k-1)$-colorable. This conjecture is considered by many as the deepest open problems in graph theory. To attack this conjecture, we would want to know more about the structure of graphs with excluded K_k-minors. Robertson and Seymour proved such a result. Their structure theorem is not strong enough to prove Hadwiger's conjecture, but it yields some algorithmic applications to Hadwiger's conjecture. See Section 5.

Wagner has also described the structure of graphs excluding $K_{3,3}$. These are precisely graphs that are obtained from planar graphs and copies of the complete graph K_5 by means of clique-sums of orders at most two.

There are many other results concerning the structure of graphs that do not contain certain graph as a minor. These excluded graphs include V_8 [129], the 3-cube [112], the octahedron [113], the octahedron plus an edge [114], graphs with single crossing [155], and K_6^- [81]. Three additional cases, $K_7 - E(C_7)$, $K_7 - E(P_7)$, and $K_6 - E(P_3)$, have been recently presented by Maharry [115]. See also [36], [176], [74], and [119].

Such characterizations are useful. We often need to exclude certain minors when they are obvious obstructions to some desired property, and knowledge of the structure forced by their exclusion may enable one to establish that property for the remaining graphs.

The paper is organized as follows. In Section 2 we overview some extremal problems related to graph minors. In Section 3, we discuss tools and techniques developed in Graph Minor Theory. Section 4 contains an outline of structure theorems of Graph Minor Theory. As far as we see, there are three main versions of the Excluded Minor Theorem, which describe local and global rough structure of graphs which do not contain a fixed graph H as a minor. Each version of this result has some applications which are discussed in the sequel. Section 5 is devoted to minors in large graphs, while in Section 6 we treat Hadwiger's Conjecture and some other problems related to graph coloring. In Section 7, we survey some algorithmic aspect of Graph Minor Theory and applications to graph colorings. In Section 8, we summarize some open problems and prospects of future research.

The selection of topics in this survey is biased upon authors' own interests and expertise and is not intended to be comprehensive. There are some other important recent achievements in the area of graph minors, and we apologise to their authors if they did not get the treatment they would deserve.

2. Extremal Problems

2.1. Extremal Functions for Complete Graph Minors

Wagner and Mader studied extremal problems concerning maximum possible number of edges in K_k-minor-free graphs. Wagner [194] proved that a sufficiently large chromatic number (which depends only on k) guarantees K_k as a minor, and Mader [107] showed that a sufficiently large average degree will do the same. Later, Kostochka [96, 97] and Thomason [179] independently proved that $\Theta(k\sqrt{\log k})$ is the correct order of the average degree forcing K_k as a minor.

Theorem 2.1 (Kostochka [96, 97] and Thomason [179]). *There exist constants $c_1 \geq c_2 > 0$ such that every graph with average degree at least $c_1 k\sqrt{\log k}$ contains K_k as a minor. On the other hand, for every $k \geq 3$, there are graphs with average degree at least $c_2 k\sqrt{\log k}$ which do not contain K_k as a minor.*

Recently, Thomason [180] found the asymptotically best possible value of this "extremal" function. He proved that it is equal to $(\alpha + o(1))k\sqrt{\log k}$, where $\alpha = 0.319...$ is an explicit constant determined by an equation, and the log function is the natural logarithm.

These results show that if the minimum degree of a given graph G is $o(k\sqrt{\log k})$, then G does not necessarily contain a K_k-minor. This does not improve even if we add a connectivity condition. Only the connectivity of order $\Theta(k\sqrt{\log k})$ forces the presence of K_k-minors.

However, as Thomason [180] pointed out, extremal graphs are exactly vertex disjoint unions of suitable dense random graphs. Such graphs cannot have too many vertices. We will return to this point later in Section 5.2.

Precise extremal numbers of edges for K_k-minors are known only for $k \leq 9$. For up to K_7-minors, these were obtained by Mader [107]. For the K_8-minor, this is due to Jørgensen [76]. Recently, the K_9-minor case was settled by Song and Thomas [172].

Similarly to Theorem 2.1, large average degree forces K_k as a topological minor [16, 95]. Here the extremal function is of order $\Theta(k^2)$.

Theorem 2.2. *There exist constants $c_1 \geq c_2 > 0$ such that every graph with average degree at least $c_1 k^2$ contains K_k as a topological minor. On the other hand, for every $k \geq 3$, there are graphs with average degree at least $c_2 k^2$ which do not contain K_k as a topological minor.*

2.2. Minors in Graphs of Large Girth

It was proved by Thomassen [183] that graphs of minimum degree at least three and with large girth contain large clique minors. The required bound on the girth forcing a K_k-minor was linear in k. This was recently improved by Diestel and Rempel [39] to $6\log_2 k+3$. The bound on the girth was improved further to $4\log_2 k+27$ by Kühn and Osthus [100]. The leading factor 4 is conjectured to be best possible. Kühn and Osthus used this result to prove that Hadwiger's conjecture (see Section 6.1) for the case when k is large is true for C_4-free graph. In fact, they proved that Hadwiger's conjecture for the case $k = f(s)$ (where the value $f(s)$ is rather large compared to s) is true for $K_{s,s}$-free graphs, see [102]. Here $K_{s,s}$ is excluded as a subgraph and not as a minor.

Using similar technique, Kühn and Osthus also proved that Hajós' conjecture (which involves topological complete graph minors) is true for graphs of girth at least 186. See [101].

Mader [109] proved that for every graph H of maximum degree $k \geq 3$, there is an integer $g(H)$ such that every graph of minimum degree k and girth at least $g(H)$ contains a subdivision of H.

2.3. Linkage Problem

A graph L is said to be *k-linked* if it has at least $2k$ vertices and for any ordered k-tuples $(s_1, \dots, s_k)$ and $(t_1, \dots, t_k)$ of $2k$ distinct vertices of L, there exist pairwise disjoint paths $P_1, \dots, P_k$ such that for $i = 1, \dots, k$, the path P_i connects s_i and t_i. Such collection of paths is called a *linkage* from $(s_1, \dots, s_k)$ to $(t_1, \dots, t_k)$.

Graph minors are intimately related to linkage problems in graphs via Theorem 1.2 and the methods used in its proof. Robertson and Seymour [143] used the idea that $2k$-connectivity and a complete graph minor of order at least $3k$ are enough to make the graph k-linked. This gave a lot of inspiration in the linkage problems, and created many beautiful results.

Clearly every k-linked graph is k-connected. The converse is not true. This brings up the natural question of how much connectivity, as a function of k, is necessary to ensure that a graph is k-linked. Larman and Mani [103] and Jung [77] were first to show that there is a function $f(k)$ such that every $f(k)$-connected graph is k-linked. They reduced this to showing that the existence of a topological complete graph minor of order $3k$ in G and $2k$-connectivity of G suffice to make the graph G k-linked. This result was combined with an earlier result of Mader that sufficiently high average degree forces a large topological complete graph minor [107].

Robertson and Seymour [154] proved in their graph minors series (Graph Minors XIII [143]) that $2k$-connectivity and existence of a K_{3k}-minor suffice to make a graph k-linked. This, together with Theorem 2.1 shows that $f(k) = O(k\sqrt{\log k})$. However, the order of $k\sqrt{\log k}$ cannot be improved by this approach. Bollobás and Thomason [15] noticed that the same effect can be achieved by replacing the K_{3k}-minor with a sufficiently dense (noncomplete) minor, whose existence requires only $ck|V(G)|$ edges for a constant c, and they consequently proved that every $22k$-connected graph is k-linked.

Later Thomas and Wollan eliminated the middle steps in the arguments of Robertson and Seymour and Bollobás and Thomason by showing directly that in a minor-minimal graph satisfying a suitable weakened version of the hypotheses, the neighborhood of every vertex has minimum degree at least $8k$. This relaxed condition enabled them to show that the neighborhood of a vertex of minimum degree has a k-linked subgraph, say L. Then, they are trying to find $2k$ disjoint paths from terminals to L. This idea together with an inductive argument implies that every $16k$-connected graph is k-linked. This approach was adapted by Kawarabayashi, Kostochka and Yu [85], who proved that every $12k$-connected graph is k-linked. Finally, Thomas and Wollan [177] gave currently best known bound.

Theorem 2.3 ([177]). *Every $2k$-connected graph G with at least $5k|V(G)|$ edges is k-linked.*

Theorem 2.3 implies, in particular, that every $10k$-connected graph is k-linked.

For small values of k and some other linkage problem, Robertson and Seymour's idea can be adapted. We refer the reader to Chen et al. [23].

The complete characterization for 2-linked graphs has been obtained by Thomassen [182], Seymour [168] and Shiloach [171], respectively. The first unsettled case is about 3-linked graphs. It was proved by Thomas and Wollan [178] that every 10-connected graph is 3-linked. Actually, they gave the best possible extremal function for the number of edges forcing 3-linkedness.

The technique used in [39, 101] is also applied to some linkage problems. For example, Kawarabayashi [80] proved that every $2k$-connected graph of girth at least 11 is k-linked. The connectivity bound $2k$ is best possible as proved by Mader [109].

2.4. Erdős-Pósa Property

A graph H is said to have the *Erdős-Pósa Property* if for every integer k there is an integer $f(k, H)$ such that every graph G either contains k vertex-disjoint subgraphs, each containing an H-minor, or a set C of at most $f(k, H)$ vertices such that $G - C$ has no H-minor. The term Erdős-Pósa Property arose because in [49], Erdős and Pósa proved that the cycle C_3 has this property.

Robertson and Seymour [135] proved that the Erdős-Pósa property holds for a graph H if and only if H is planar, see also Thomassen [184] and [37]. Hence in general, the Erdős-Pósa Property does not hold, not even for K_5-minors.

But Kawarabayashi and Mohar [92] characterized such graphs. Their result says that either G has k disjoint K_5-minors, or else G has a vertex set T of order at most $f(k)$ such that $G - T$ can be embedded into a surface of genus at most $k - 1$, up to 3-separations.

But if we restrict our attention to graphs that are "highly" connected or have large minimum degree, then the situation changes. The main result in [10] implies the following general result.

Theorem 2.4. *Suppose that G is $16a$-connected without a subdivision of $K_{a,sk}$. There exists a constant $f(s, k, a)$ such that either there are s disjoint copies of $K_{a,k}$-minor in G, or G contains a set F of at most $f(s, k, a)$ vertices such that $G - F$ has no minor isomorphic to $K_{a,k}$.*

When H has genus $g \geq 1$, a graph G embedded in a surface Σ of genus g cannot contain two disjoint H-minors. On the other hand, if G is very densely embedded in Σ, then deleting a bounded number of vertices will not destroy all H-minors. Another approach was suggested by Robin Thomas (private communication) for nonplanar graphs. We say that G has *half-integral packing* of k H-minors if there are k H-minors in G such that each vertex of G is used by at most two of them.

Conjecture 2.5 (Thomas). *The Erdős-Pósa Property holds for half-integral packing of H-minors, i.e., for any fixed k, for any graph H and every graph G, either G has a half-integral packing of k H-minors, or G has a vertex set T of order at most $f(k)$ such that $G - T$ is H-minor-free.*

The case of K_5-minors is settled in [92].

3. Tools and Techniques from Graph Minor Theory

Graph Minor Theory is important and even spectacular not only because it gives unexpected solutions to long-standing open problems, but also because its techniques, especially the various ways in which minors are handled and constructed, will influence the development of graph theory for many years to come. In this section, we shall survey this point. We will deal with four tools and techniques: tree-width, tangles, distance on a surface, and society minors. All of them are now well-understood, and intensively used in graph theory.

3.1. Tree-Width and Tree-Decompositions

Tree-width was introduced by Halin in [70], but it went unnoticed until it was rediscovered by Robertson and Seymour [133] and, independently, by Arnborg and Proskurowski [6].

A *tree decomposition* of a graph G is a pair (T, Y), where T is a tree and Y is a family $\{Y_t \mid t \in V(T)\}$ of vertex sets $Y_t \subseteq V(G)$, such that the following two properties hold:

(W1) $\bigcup_{t \in V(T)} Y_t = V(G)$, and every edge of G has both ends in some Y_t.
(W2) If $t, t', t'' \in V(T)$ and t' lies on the path in T between t and t'', then $Y_t \cap Y_{t''} \subseteq Y_{t'}$.

The pair (T, Y) is a *path decomposition* if T is a path. The *width* of a tree decomposition (T, Y) is $\max_{t \in V(T)}(|Y_t| - 1)$. The *tree-width* $tw(G)$ of G is defined as the minimum width taken over all tree decompositions of G. Similarly, the *path-width* of G is the minimum width taken over all path decompositions of G.

Let (T, Y) be a tree decomposition of a graph G. For an edge $tt' \in E(T)$, let $Z_{tt'} = Y_t \cap Y_{t'}$. We define the *adhesion* of a tree decomposition (T, Y) as $\max |Z_{tt'}|$ taken over all edges $tt' \in E(T)$.

It was shown in [125] that if a graph G has a tree decomposition of width at most w, then G has a tree decomposition of width at most w that further satisfies:

(W3) For every two vertices t, t' of T and every positive integer k, either there are k disjoint paths in G between Y_t and $Y_{t'}$, or there is a vertex t'' of T on the path between t and t' such that $|Y_{t''}| < k$.
(W4) If t, t' are distinct vertices of T, then $Y_t \neq Y_{t'}$.
(W5) If $t_1t_2 \in E(T)$ and B is the component of $T - t_1t_2$, which contains t_1, then the set $V_1 = \bigcup_{t \in V(B)} Y_t \setminus Y_{t_2}$ is nonempty.

Let $t_1t_2 \in E(T)$ and let V_1 be the vertex set defined in (W5). Define similarly the set V_2. Then also $V_2 \neq \emptyset$ and hence $Z_{t_1t_2}$ is a separating set of G which separates V_1 and V_2 in G.

One of the most important results about graphs, whose tree-width is large, is existence of a large grid minor or, equivalently, a large wall. Let us recall that an r-*wall* is a graph which is isomorphic to a subdivision of the graph W_r with vertex set $V(W_r) = \{(i, j) \mid 1 \leq i \leq r,\ 1 \leq j \leq r\}$ in which two vertices (i, j) and (i', j') are adjacent if and only if one of the following possibilities holds:

(1) $i' = i$ and $j' \in \{j-1, j+1\}$.
(2) $j' = j$ and $i' = i + (-1)^{i+j}$.

We can also define an $(a \times b)$-wall in a natural way, so that the r-wall is the same as the $(r \times r)$-wall. It is easy to see that if G has an $(a \times b)$-wall, then it has an $(\lfloor \frac{1}{2}a \rfloor \times b)$-grid minor, and conversely, if G has an $(a \times b)$-grid minor, then it has an $(a \times b)$-wall. Let us recall that the $(a \times b)$-grid is the Cartesian product of paths $P_a \square P_b$. The (4×5)-grid and the (8×5)-wall are shown in Figure 2.

The main result of Graph Minors V [135] says the following.

Theorem 3.1. *For every positive integer r, there exists a constant $f(r)$ such that if a graph G has tree-width at least $f(r)$, then G contains an r-wall as a (topological) minor.*

The best currently known upper bound for $f(r)$ is given in [160], see also [40, 126]. It is 20^{64r^5}, and 20^{2r^5} for the $(r \times r)$-grid minor. The best known lower bound on $f(r)$ is of order $\Theta(r^2 \log r)$, see [160].

The tree-width can be viewed as a measure of "global connectivity." Let us consider a k-wall W. Since W has no vertices of degree 4 or more, it contains no 4-connected subgraphs. On the other hand, if X is a set of at most three vertices of W, then $W - X$ is either connected or has two components, one of which is a vertex, except for the case when X is adjacent to vertices of degree 1 or 2 on the "boundary" of W (which we neglect at this moment). Similarly, if X is a set of at most k vertices, then the largest component of $W - X$ contains all but at most k^2 vertices of W since otherwise, at least $k+1$ of these vertices would be either in the same row or in the same column, which is clearly impossible. Hence no separation of order at most k "globally" separates W into two large parts. By the above theorem, large tree-width implies that there is a large wall, which is highly "globally connected." Conversely, if a given graph contains a large wall, it cannot have small tree-width. This explains why the tree-width can be viewed as a measure of "global connectivity."

An analogue of tree-width for digraphs was introduced in [75], but not many results are known until now.

The tree-width was used, among others, in the following areas:

1. Graph minor theory [126, 135, 143, 146, 147].
2. Structural graph theory [135, 125, 160, 127, 40, 13].

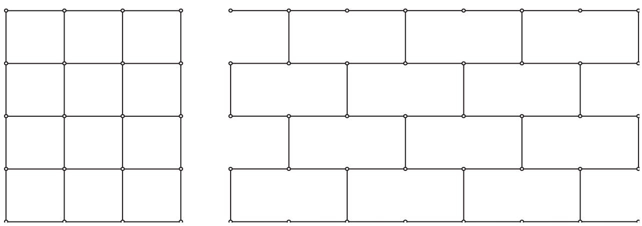


Fig. 2. The (4×5)-grid and the (8×5)-wall

3. Algorithmic applications due to the fact that many NP-hard problems can be solved in polynomial time, even linear time, when input is restricted to graphs of bounded tree-width, see [6, 9].
4. Several practical applications.

For further applications of tree-width, we refer the reader to the survey of Reed [126].

3.2. Tangles

Let G be a graph. A pair (A, B) of subgraphs of G is called a *weak separation* of G if $G = A \cup B$. If, moreover, $V(A) \setminus V(B) \neq \emptyset$ and $V(B) \setminus V(A) \neq \emptyset$, then (A, B) is a *separation*. The number of vertices in the intersection of A and B, $|A \cap B|$, is the *order* of the (weak) separation. Separations of order k are also called *k-separations*.

A special concept introduced by Robertson and Seymour in [140] is that of a tangle. It provides a kind of duality to the notion of the tree-width. Formally, a *tangle* of *order t* in a graph G is a set $\mathcal{T}$ of weak separations of G, each of order less than t, such that:

(i) For every weak separation (A, B) of G of order less than t, the family $\mathcal{T}$ contains either (A, B) or (B, A).
(ii) If $(A_i, B_i) \in \mathcal{T}$ for $i = 1, 2, 3$, then $A_1 \cup A_2 \cup A_3 \neq G$.
(iii) If $(A, B) \in \mathcal{T}$, then $V(A) \neq V(G)$.

The *tangle number $tn(G)$* of a graph G, which is defined as the maximum order of a tangle in G, and its tree-width $tw(G)$ are closely related. In [140] (see also [126] and [160]) it is proved that $tw(G) + 1 \geq tn(G) \geq \frac{2}{3}(tw(G) + 1)$. Hence, every graph with tree-width at least $3t/2$ has a tangle of order at least t.

The most important tangles are related to walls. Suppose that G contains a $3t$-wall W. If (A, B) is a weak separation of order less than t, then $A \cap B$ intersects at most $t - 1$ "rows" of W, and at most $t - 1$ "columns". (The "rows" in a wall W are defined as paths in W whose vertices (i, j) from the definition of a wall all have the same first coordinate. Similarly, the vertices in "columns" of W have all their second coordinates taking two values, $2j - 1$ and $2j$.) Consequently, more than $2t$ rows and more than $\lfloor t/2 \rfloor$ columns (which obviously form a connected subgraph) are contained in the same part of the separation, either in $B \setminus A$ or in $A \setminus B$. Observe that they form a $\lceil t/2 \rceil$-wall in that part. Now we define the tangle $\mathcal{T}_W$ of order t corresponding to the wall W by putting (A, B) in $\mathcal{T}_W$ if and only if $B \setminus A$ contains more than $2t$ of the rows of W. It is easy to verify that $\mathcal{T}_W$ is a tangle of order t.

Let us observe that, if G has large tree-width, then G also contains a large wall W [135], so it has a tangle $\mathcal{T}_W$ related to that wall which is of large order.

How can one use tangles? The following discussion shows how useful they are. Robertson and Seymour's excluded minor theorem (Version 1 in Section 4.1) says that every graph without a fixed graph H as a minor has a "tree-structure" such that each node in this tree is a subgraph which is "nearly" embeddable on a surface of bounded genus. But how can one prove this? Ideally, we would like to focus on one of the nodes in this tree. The first approach for this would be induction on the number of vertices. If there is no separation dividing the graph into two substantial pieces, then we should be able to focus on one node of the tree-structure provided that the theorem is to be true, while if there is a small separation, and this separation divides a given graph G into two substantial pieces, then we may think that G may be expressed as a clique-sum of two smaller graphs, and would like to apply induction to these smaller graphs. But there is one problem here. It is possible that both smaller graphs are H-minor-free, but G is not. To avoid this situation, we need the concept of the tangle. We can assume that the tree is as refined as possible in the sense that no node of the tree can be split into smaller nodes, and so for every separation of small order, most of nodes in the tree will be on one side. Therefore, if we fix one node, every small separation has "small" side and "big" side. This node defines tangle, which is an assignment of big and small sides of any small order separation. Conversely, it can be shown (Graph Minors X [140]) that any tangle of large order will be accommodated in one node of the tree structure. Hence, by using tangles of high order one can analyse the global structure of a given graph. One can focus on one node of the tree-structure by using this tangle.

This example shows that when using induction, we would focus on "one" side. This side is actually highly "globally" connected since every separation of small order divides into a "big" and a "small" side. So the tangle has large order, and hence there is a large grid minor by Theorem 3.1 combined with the fact that the tangle is a dual concept to the tree-width. This grid minor is always contained in the big side. This enables us to "control" all the bridges of this grid minor. This sketched idea is actually important in the proof of the Graph Minor Theorem.

3.3. Distance on a Surface – metric and Face-width

Robertson and Seymour [134] introduced the notion of "representativity" or "face-width." This concept is now widely used in topological graph theory. We shall survey some of the most important results in this area, but before that, let us give the definitions and sketch the corresponding results of Robertson and Seymour.

A *surface* Σ is a compact connected 2-manifold without boundary. We assume familiarity with basic notions of surface topology, like genus and Euler's formula. We define the *Euler genus* of a surface S as $2-\chi(S)$, where $\chi(S)$ is the Euler characteristic of S. An *arc* in Σ is a subset homeomorphic to [0, 1]. An *O-arc* is a subset of Σ homeomorphic to a circle. Let G be a graph that is embedded in Σ. To simplify notation we do not distinguish between a vertex of G and the point of Σ used in the embedding to represent the vertex, and we do not distinguish between an edge and the arc on the surface representing it. We also consider G as the union of the points corresponding to its vertices and edges. A *region* or *face* of G in Σ is a connected component of $\Sigma \setminus (E(G) \cup V(G))$. Every region is an open set. We use the notation $R(G)$ for the set of regions of G. The embedding is said to be a *2-cell embedding* if every region is homeomorphic to a disk.

If $\Delta \subseteq \Sigma$, then $\overline{\Delta}$ denotes the *closure* of Δ, and the boundary of Δ is $\partial\Delta = \overline{\Delta} \cap \overline{\Sigma \setminus \Delta}$. An edge e (or a vertex v) is incident with a region r if $e \subseteq \partial r$ ($v \in \partial r$).

A subset of Σ meeting the embedded graph only in vertices of G is said to be *G-normal*. If an O-arc is G-normal then we call it a *noose*. We say that a disk $D \subset \Sigma$ is *bounded* by a noose N if $N = \partial D$. A graph G, which is 2-cell embedded in a surface Σ, has *face-width* at least θ if every noose, which intersects G in fewer than

θ vertices is contractible (null-homotopic) in Σ. Alternatively, the *face-width* of G is equal to the minimum number of facial walks whose union contains a cycle which is non-contractible in Σ. See [122] for further details.

We shall use the notion of a radial graph. Informally, the radial graph of a graph G, which is 2-cell embedded in Σ, is the bipartite graph R_G obtained by selecting a vertex in every region r of G and connecting it to every vertex of G incident to that region. However, a region may be "incident more than once" with the same vertex, so one needs a more formal definition. A *radial graph* R_G (also called *vertex-face incidence graph* [122]) of a 2-cell embedded graph G is the graph embedded in the same surface Σ, whose vertex set is the union of $V = V(G)$ and $F = \{v_r \mid r \in R(G)\}$ such that the following are satisfied:

(1) Each region $r \in R(G)$ contains the corresponding vertex $v_r \in F$.
(2) R_G is a bipartite graph with the bipartition (V, F).
(3) If $e \in E(R_G)$ is joining vertices $v \in V$ and $v_r \in F$, then the edge e is embedded in $r \cup \{v\}$. In particular, vertex v and face r are incident.
(4) If e, f are distinct edges of R_G with the same ends $v \in V$, and $v_r \in F$, then $e \cup f$ does not bound a closed disk contained in $r \cup \{v\}$.
(5) R_G is maximal subject to (1)–(4).

Let G be a graph embedded in a surface Σ. A tangle $\mathcal{T}$ of order θ is said to be *respectful* if for every noose N in Σ with $|N \cap V(G)| < \theta$, there is a closed disk $\Delta \subseteq \Sigma$ with $\partial\Delta = N$ such that the weak separation $(G \cap \Delta, G \cap \overline{\Sigma \setminus \Delta})$ is in $\mathcal{T}$.

The following is one of the main results in [141].

Theorem 3.2 ((4.1) in [141]). *Let Σ be a surface, which is not homeomorphic to the sphere. Let $\theta \geq 1$, and let G be a graph 2-cell embedded in Σ with face-width at least θ. Then there is a respectful tangle in G of order θ.*

Let us define a "distance" function d for pairs of elements in $V(R_G) \cup E(R_G) \cup R(R_G)$ as follows:

1. If $a = b$, then $d(a, b) = 0$.
2. If $a \neq b$, and a and b are both in the interior of a contractible closed walk of the radial graph of length $< 2\theta$, then $d(a, b)$ is half the minimum length of such a walk.
3. Otherwise, $d(a, b) = \theta$.

According to Section 9 of [141] (see also [142]), the existence of a respectful tangle makes it possible to define the function d as metric. The main result of Graph Minors XI [141] roughly states the following.

Theorem 3.3. *The function d introduced above defines a metric on a fixed surface such that for any point c of the surface and for every $\theta' < \theta$, the set of points within distance θ' from c is simply connected.*

In Graph Minors XII [142], the following theorem is proved.

Theorem 3.4. *Let k be an integer and G be a graph that is embedded in a fixed surface with large face-width. Let* $\{v_1, u_1\}, \ldots, \{v_k, u_k\}$ *be pairs of vertices of G. If all distances* $d(v_i, v_j), d(u_i, u_j)$ $(1 \leq i < j \leq k)$ *and* $d(u_i, v_j)$ $(1 \leq i, j \leq k)$ *are large enough, where d is the above distance function of Robertson and Seymour, then G contains k disjoint paths* $P_1, \ldots, P_k$ *such that* P_i *is joining* v_i *and* u_i *for* $i = 1, \ldots, k$.

Theorem 3.4 implies the following result, which became one of the cornerstones in topological graph theory.

Theorem 3.5. *Let H be a graph embedded in a surface of Euler genus g. Then there is an integer* $f(H, g)$ *satisfying the following. If a graph G is embedded in a surface* Σ *of Euler genus at most g with face-width at least* $f(H, g)$*, then G contains H as a surface minor.*

Let us recall that a *surface minor* of G means an embedded graph which is obtained from G by deleting edges and vertices and performing contractions "on the surface" so that the embedding is locally preserved. See, e.g., [122] for more details.

Let us observe that this result was already proved in [137], but the proof in [142] gives several applications used in the proof of the Graph Minor Theorem.

Theorem 3.5 and its proof by Robertson and Seymour does not give explicit bounds on the values $f(H, g)$. Quantitative versions for some special cases are known, cf. [122].

Theorem 3.5 can be viewed as a generalization of the following planarity counterpart.

Theorem 3.6 (Robertson and Seymour [135]). *Let H be a plane graph. Then there is an integer r such that H is a surface minor of an r-wall.*

One specific special case of Theorem 3.5, where H is a cycle embedded in Σ such that it separates Σ into two parts of positive genera (*surface-separating cycle*), has received special attention. The following conjectures basically claim that $f(H, g) = 3$ in all admissible cases.

Conjecture 3.7 (Barnette, 1982). *Every triangulation of a surface of genus* $g \geq 2$ *contains a noncontractible surface-separating cycle.*

Conjecture 3.8 (Zha, 1991). *Every graph embedded in a surface of genus* $g \geq 2$ *with face-width at least three contains a noncontractible surface-separating cycle.*

It follows from Theorem 3.5 that large face-width forces existence of noncontractible surface-separating cycles (where "large" may depend on the surface). Zha and Zhao [196] and Brunet, Mohar, and Richter [17] proved that face-width 6 (and even 5 for nonorientable surfaces) is sufficient. Ellingham and Zha [47] proved a weakening of Conjecture 3.8 for the double torus assuming that the face-width is at least 4. Sulanke [173] reported that Conjecture 3.7 holds for the orientable surface of genus 2 and for nonorientable surfaces of genus 2, 3, or 4.

Suppose that the embedding of the graph H in Theorem 3.5 is a minimum genus embedding. If H is a surface minor of another embedded graph G (in the same surface), then also the embedding of G is a minimum genus embedding. Therefore, a consequence of Theorem 3.5 is that large face-width of an embedding implies that this is a minimum genus embedding. Robertson and Vitray [165] have shown that the face-width, which is linear in terms of the genus, implies genus minimality. Suppose now that H is uniquely embeddable in Σ and that its embedding has face-width at least three. (Such graphs are easy to find.) If G is a 3-connected graph embedded in Σ such that H is a surface minor of G, then also the embedding of G in Σ is unique. Consequently, sufficiently large face-width of a 3-connected graph implies uniqueness of the embedding. Both of these results are treated in Seymour and Thomas [170] and Mohar [117] who proved that face-width of order $O(\log g/\log\log g)$ implies uniqueness and Euler genus minimality for embeddings of 3-connected graphs in surfaces of genus at most g.

There are numerous other results where Theorem 3.5 is used in one or another form. Examples include [35, 121, 187] and the "approximate" flow-coloring duality on general surfaces by Devos et al. [33].

3.4. Society Minors

The concept of a society and the corresponding society minors are studied in Graph Minors IX [139]. The motivation is to introduce the vortex structure, see Section 4.2. The main theorem in [139] roughly says that if a cyclic order is imposed on a subset S of vertices of a given graph G, then either

(i) G can be drawn into a disk with no crossings except for one "small" area, which is a "vortex" of bounded adhesion in a sense of Robertson and Seymour's main structural theorem, or
(ii) there is a large number of vertex disjoint paths, each of which connects two vertices in S, and they form a structure similar to one of those shown in Figure 3 below.

Formally, we define a *society* in a graph G as a vertex set $F \subseteq V(G)$ together with a fixed *cyclic order* of F. Then for any two society vertices $a, b \in F$, we denote by $F(a, b)$ the set of all society vertices which are after a but before b in the given cyclic order of F. We also write $F[a, b] = F(a, b) \cup \{a, b\}$. For every $a, b \in F$, $F = F[a, b] \cup F(b, a)$ is a partition of F.

The pair (G, F), where F is a society in G, is called a *vortex*. With respect to graph minors, vortices with the following property arise naturally. Suppose that (G, F) is a vortex such that for any two society vertices a, b, there are no p vertex-disjoint paths in G connecting $F[a, b]$ and $F(b, a)$. Then we say that the vortex (G, F) has *adhesion* at most p. It can be shown, see Theorem 3.9 below, that a vortex with adhesion p admits a special (cyclic) decomposition, very similar to a path-decomposition, whose adhesion is at most p.

The following result was proved in [139, Theorem 8.1]. See also [154, Theorem 11.1].

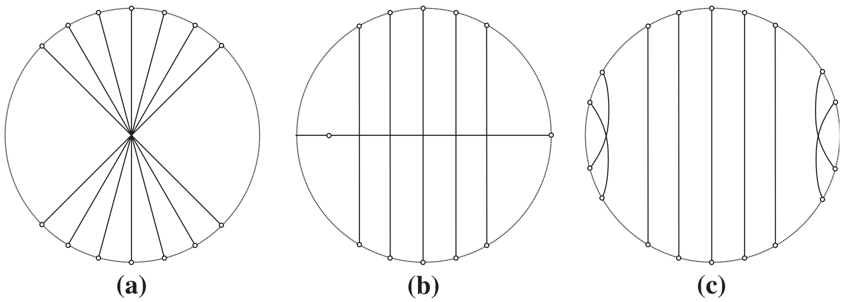


Fig. 3. Disjoint paths in a vortex: (a) crosscap, (b) leap, (c) doublecross

Theorem 3.9. *Let (G, F) be a vortex with society F and with adhesion at most p. Then for each vertex $v \in F$, there exists a subgraph G_v of G such that the following conditions are satisfied:*

(1) *The subgraphs G_v are mutually edge-disjoint, their union is G, and $v \in V(G_v)$ for every $v \in F$.*
(2) *If $u, v \in F$, then each vertex in $G_v \cap G_u$ is either contained in $\bigcap_{w \in F[u,v]} G_w$ or in $\bigcap_{w \in F[v,u]} G_w$.*
(3) *If $u, v \in F$ and $u \neq v$, then $|G_v \cap G_u| \leq p$.*

Theorem 3.9 gives a rough structure characterization of vortices of bounded adhesion in the sense that every vortex (G, F) satisfying (1)–(3) has adhesion at most $2p + 1$.

The structure of a vortex specified by subgraphs G_v, $v \in F$, satisfying (1)–(3) in Theorem 3.9 is called a *vortex decomposition*. If $t = \max\{|G_v|\,;\ v \in F\}$, then we say that the vortex decomposition has *width* $t - 1$. The minimum width of all vortex decompositions of (G, F) is called the *width* of the vortex (G, F).

Let (G, F) be a vortex, and suppose that there are p disjoint paths P_i in G with endpoints a_i, b_i for $i = 1, \ldots, p$. Some specific arrangements of endpoints of these paths along F have special significance, cf. Figure 3 and [139, p. 59].

(i) If $(a_1, \ldots, a_p, b_1, \ldots, b_p)$ appear in F in this clockwise order, then we say that $P_1, \ldots, P_p$ form a *crosscap* of *order* p.
(ii) If $(a_1, \ldots, a_p, b_{p-1}, b_{p-2}, \ldots, b_1, b_p)$ appear in F in this clockwise order, then we say that $P_1, \ldots, P_p$ form a *leap* of *order* p.
(iii) If $(a_1, \ldots, a_p, b_{p-1}, b_p, b_{p-2}, b_{p-3}, \ldots, b_3, b_1, b_2)$ appear in this clockwise order in F, then we say that $P_1, \ldots, P_p$ form a *doublecross* of *order* p.

The importance of these special cases lies in the fact, given below as Theorem 3.10, which shows that their exclusion yields a graph which is essentially embedded in a disk, together with a vortex of bounded adhesion. Some parts of the graph may not adhere to this statement, but they can be precisely described by means of separations of order at most three. The next definition describes how these separations are attached to the disk.

If a graph G_0 can be written as $G_1 \cup G_2$, where $G_1 \cap G_2 = \{v_1, \ldots, v_t\} \subset V(G_0)$, $1 \leq t \leq 3$, $V(G_2) \setminus V(G_1) \neq \emptyset$, then we replace G_0 by the graph G' obtained from G_1 by adding the edge v_1v_2 if $t = 2$ and by adding the triangle $T = v_1v_2v_3$ if $t = 3$. In the latter case we say that T is the *reduction triangle*. We say that G' is obtained from G_0 by an *elementary reduction*. Every graph G that can be obtained from G_0 by a sequence of elementary reductions is a *reduction* of G_0. We say that the graph G_0 can be *embedded* in a surface Σ *up to* 3-*separations* if there is a reduction G of G_0 such that G has an embedding in Σ in which every reduction triangle bounds a face of length 3 in Σ.

The following result is the most important contribution of Graph Minors IX [139, Theorem 7.1].

Theorem 3.10. *Suppose that (G, F) is a vortex and p is an integer. Then either G contains a crosscap, a leap or a doublecross of order p, or G can be written as $G = G' \cup H$, where $F' = G' \cap H \subseteq V(G)$, such that the following holds:*

(i) *H can be embedded up to 3-separations in a cylinder with outer face C_1 and inner face C_2, all vertices of the society F appear on C_1 in the given cyclic order, and the inner face C_2 contains all vertices of F'.*

(ii) *If F' is equipped with the cyclic order inherited from C_2, then (G', F') is a vortex of adhesion at most $3p + 9$.*

Theorem 3.10 is a basic tool for deriving the excluded minor structure presented in the next section. It was also heavily used in the proof of Jørgensen's conjecture for K_6-minors in large graphs by DeVos et al. [34]. In Section 5, we shall give a sketch of the proof and address how to use this society theorem.

4. The Excluded Minor Theorem

The theory of graph minors builds on the associated "rough structure." The idea is to describe the structure of graphs that do not contain a given graph H as a minor, the *H-minor-free* graphs. The precise structure can be given only in some very specific cases. In others we are satisfied with rough description only. The natural requirement is that the involved structure describes a class of graphs which is closed under taking minors.

In describing the rough structure of all graphs which do not contain H as a minor, one has to consider four inevitable ingredients:

(i) *Clique-sums and tree-decompositions*: Suppose that H is a k-connected graph. If G_1 and G_2 are graphs, neither of which contains H as a minor, then their clique-sum of order less than k gives another H-minor-free graph. By generalizing this procedure to more than two graphs, we can build large H-minor-free graphs whose structure can be described by means of a tree-decomposition. Such a tree-structure became apparent already in the early graph minor theorems, for instance in Wagner's characterization of K_5-minor-free and $K_{3,3}$-minor-free graphs [193], see also Section 1.

(ii) *Graphs on a fixed surface*: If H cannot be embedded in a surface Σ, then all graphs that can be embedded in Σ are H-minor-free.

(iii) *Bounded extensions and apex vertices*: Suppose that for every k-vertex set $U \subseteq V(H)$, the graph $H - U$ cannot be embedded in a surface Σ. If G is a graph containing a set of at most k vertices whose deletion yields a graph embeddable in Σ, then G is H-minor-free. The vertices removed from G are sometimes called "apex vertices" since there is no restriction to what their neighbors in G are – they can even be adjacent to all other vertices.

(iv) *Vortices*: Let us consider the following example. Take a large plane graph with outer facial walk $C = u_1 u_2 \dots u_r$ and let k be an integer. Then add all edges $u_i u_j$, where either $|i - j| \leq k$ or $|i - j| \geq r - k$ $(i, j = 1, \dots, r)$. If $k = 2$, it is easy to see that the resulting graph may contain a K_7-minor, but cannot contain a K_8-minor. See [167]. Since this graph cannot be described by the above structures (i)–(iii), we need another component. This is the motivation for the vortex structure which generalizes the above example and all minors obtained from it.

The above components (i)–(iv) are necessary for describing H-minor-free families of graphs. The Excluded Minor Theorem of Robertson and Seymour [146] says that these four components and their combinations are indeed sufficient to yield a rough structure of graphs with no H-minors. We shall describe three forms of this result.

4.1. Version 1: Clique-sum Structure

Let us first present the best known form of the Excluded Minor Theorem.

Let G be a graph, S be a surface, and k an integer. We say that G can be *k-nearly embedded* in S if G has a set A of at most k vertices such that $G - A$ can be written as $G_0 \cup G_1 \cup \dots \cup G_k$ satisfying the following conditions:

(i) G_0 is embedded in S.
(ii) The graphs $G_1, \dots, G_k$ are pairwise disjoint.
(iii) There are (not necessarily distinct) faces $F_1, \dots, F_k$ of G_0 in S, and there are pairwise disjoint disks $D_1, \dots, D_k$ in S, such that for $i = 1, \dots, k$, $D_i \subset \overline{F_i}$. If $U_i = D_i \cap G_0 \subseteq V(G_i)$ is cyclically ordered as imposed by the boundary of the disk D_i, then (G_i, U_i) is a vortex of width at most k.

The vertices in A are called the *apex vertices* of the *k-near embedding*. It may happen that $A = V(G)$, and $G - A$ is empty. In that case we say that the k-near embedding of G in Σ is *trivial*. Otherwise, G_0 is nonempty. The subgraph G_0 of G is said to be the *embedded subgraph* with respect to the k-near embedding and the decomposition $G_0, G_1, \dots, G_k$. The pairs (G_i, U_i), $i = 1, \dots, k$, are the *vortices* of the k-near embedding. The vortex (G_i, U_i) is said to be *attached* to the disk D_i.

Now we can state the following theorem, which is regarded as the Excluded Minor Theorem [146, Theorem 1.3].

Theorem 4.1. *For every graph H, there exists an integer $k \geq 0$ such that every graph that does not contain H as a minor can be obtained by clique sums of order at most k from graphs that can be k-nearly embedded in some surface, in which H cannot be embedded.*

If H is a planar graph, then H can be embedded in an arbitrary surface. In this case, the embedded part of the near-embedding is empty, and all it remains are at most k apex vertices. Theorem 4.1 then shows that every H-minor-free graph G can be obtained by clique-sums from graphs of order at most k, i.e., G has tree-width less than k. This special case of Theorem 4.1 was proved in Graph Minors V [135].

Corollary 4.2. *If H is a planar graph, then every H-minor-free graph has bounded tree-width.*

Theorem 4.1 is well-known, and has been used, for instance, by DeVos et al. [32] to prove, in particuar, that for a fixed graph H, every H-minor-free graph has a vertex partition into two parts V_1, V_2 such that both the graph induced by V_1 and the graph induced by V_2 have bounded tree-width. See Section 7 for more details.

4.2. Version 2: Nodes in the Tree-structure Capturing a Large Wall

Although Version 1 as given in the previous subsection is the best known form of the Excluded Minor Theorem, almost all of the proof of Graph Minor Theorem is devoted to the proof of a more elaborate version. In this version, one possible outcome is the bounded tree-width structure. Otherwise, the given H-minor-free graph G contains a large wall W. Then W (or a large part of it) is contained within a single node of the tree-structure explained in the previous subsection. Version 2 describes the structure based on this node. Roughly speaking, the whole graph G can be described as a graph G_0 embedded (up to 3-separations) in some surface S, in which H cannot be embedded, and our large wall W is contained in G_0 and is embedded inside a disk in S (cf. remark (1) after Theorem 4.3). The rest of the graph G is attached to a bounded number of faces and has a structure of vortices with bounded adhesion. (Of course, in this formulation, we cannot ask for bounded width.) In addition to this, there is a bounded set of apex vertices, whose behaviour cannot be controlled. Moreover, we may assume that G_0 has large face-width, see remark (2) after Theorem 4.3.

In the above described structure, there are two new ingredients which will be formally described in the sequel. The first one concerns the condition that the wall W is "captured" in the surface, $W \subseteq G_0$. Robertson and Seymour use the notion of a tangle to describe this condition, while we will state it by defining a k-near embedding of the pair (G_0, W). The clique-sums of orders at most 3 corresponding to the elementary reductions, occur on the surface part of G_0 and may eventually eliminate some edges of G_0 and hence also of W. However, as a minor, the edges removed while taking clique-sums of orders 2 or 3 can be recovered within the attached components. This is technically described by means of elementary reductions, which were introduced in the section on society minors. Here, we extend them to the setting of embedding a pair (G_0, W).

Let W be an r-wall of large order in a graph G_0. Suppose that $G_0 = G_1 \cup G_2$, where $G_1 \cap G_2 = \{v_1, \dots, v_t\} \subset V(G_0)$, $1 \le t \le 3$, $V(G_2) \setminus V(G_1) \ne \emptyset$. We also assume that $G_2 - \{v_1, \dots, v_t\}$ contains at most one vertex of degree 3 in W. If G_2 contains a vertex of degree 3 in W, then $t = 3$, and then we also assume that at least one of v_1, v_2, v_3 is of degree 2 in W. By performing the corresponding elementary reduction, we replace G_0 by the graph G', which is either equal to G_1 (if $t = 1$), is obtained from G_1 by adding the edge v_1v_2 (if $t = 2$), or by adding the reduction triangle $T = v_1v_2v_3$ (if $t = 3$). If $W \cap G_2$ is a path in G_2 connecting v_i and v_j, then we replace that path in W by the edge v_iv_j. If $W \cap G_2$ has vertex of degree 3 distinct from v_1, v_2, v_3, and v_1 has degree 2 in W, then we replace $W \cap G_2$ with the two edges v_1v_2 and v_1v_3. The resulting graph W' is a subgraph of G', and we say that the pair (G', W') was obtained from (G_0, W) by an *elementary reduction*. (It is easy to see that W' is also an r-wall in G' except that the vertices of degree 1 in W may disappear.) Every pair (G'', W'') that can be obtained from (G_0, W) by a sequence of elementary reductions is a *reduction* of (G_0, W).

If W is a wall in a graph G_0, we say that the pair (G_0, W) can be *embedded* in a surface Σ *up to* 3-*separations* if there is a reduction (G'', W'') of (G_0, W) such that G'' has an embedding in Σ in which every reduction triangle bounds a face of length 3 in Σ.

In order to be able to focus on the graph as a whole but really consider only one node in the tree decomposition of Theorem 4.1, we have to capture the clique-sums. Those clique-sums, which involve vortices, are hidden in (iii) below, where we no longer require that the vortex decomposition has bounded width, but we ask it to have bounded adhesion only. The clique-sums "on the surface" will be of orders at most three, and they are hidden in (i) below, where the embedding is replaced by an embedding up to 3-separations.

Let G be a graph, W an r-wall in G, let S be a surface, and k an integer. We say that (G, W) can be *k-almost embedded* in S if G has a set A of at most k vertices such that $G - A$ can be written as $G_0 \cup G_1 \cup \cdots \cup G_k$ satisfying the following conditions:

(i) W is a subgraph in G_0 and (G_0, W) has an embedding in S up to 3-separations.
(ii) The graphs $G_1, \ldots, G_k$ are pairwise disjoint.
(iii) There are (not necessarily distinct) faces $F_1, \ldots, F_k$ of G_0 in S, and there are pairwise disjoint disks $D_1, \ldots, D_k$ in S, such that for $i = 1, \ldots, k$, $D_i \subset \overline{F_i}$. If $U_i = D_i \cap G_0 \subseteq V(G_i)$ is cyclically ordered as imposed by the boundary of the disk D_i, then (G_i, U_i) is a vortex of adhesion at most k.

Now we can give another formulation of the Excluded Minor Theorem, stated as Theorem 3.1 in [146].

Theorem 4.3. *For every graph H, there is a positive integer k such that for every positive integer w, there exists a positive integer $r = r(H, w)$, tending to infinity with w for any fixed H, such that every G not containing H as a minor either has tree-width at most w, or contains an r-wall W such that (G, W) has a k-almost embedding in some surface Σ in which H cannot be embedded.*

Concerning the excluded minor structure described by Theorem 4.3 we may furthermore assume two additional properties:

(1) The r-wall W is *planarly embedded* in Σ, i.e., every cycle in W is contractible in Σ and there is a disk $D \subset \Sigma$ such that W and all 6-faces of the embedding of W in the plane are contained in D. To see this, observe that the cycle space of W is generated by the facial 6-cycles of its planar embedding. If all these cycles are

contractible in Σ, then an $(r/2)$-subwall of W is planarly embedded in Σ. If more than $171g$ of the facial 6-cycles of W are noncontractible in Σ, where g is the Euler genus of Σ, then there are $9g$ such cycles, $F_1, \dots, F_{9g}$, such that any two of them are at distance at least 3 in W. It can be shown that this implies, in particular, that $W - (F_1 \cup \cdots \cup F_{9g})$ is connected and hence no four cycles among $F_1, \dots, F_{9g}$ are homotopic. Consequently, $F_1, \dots, F_{9g}$ contains a subfamily of $3g$ cycles, no two of which are homotopic. This is not possible (cf., [122, Proposition 4.2.6]). Hence, at most $171g$ of the 6-cycles of W are noncontractible and W contains a large subwall that is planarly embedded, and we can take this subwall instead of W. The size r' of this smaller wall still satisfies the condition that $r' = r'(H, w) \to \infty$ as w increases.

(2) We may additionally assume that the face-width of the embedded subgraph G'' in Σ is as large as we want (in terms of H). To see this, suppose that there is a non-contractible closed curve C that intersects G'' only at vertices and $|C \cap V(G'')|$ is small. Then we delete all the vertices in $C \cap V(G'')$ from G'' and add them into the set of apex vertices. Then the genus of Σ goes down, and the number of apex vertices is still bounded. Continuing this procedure, we get the graph on a simpler surface whose face-width is as large as we wanted. See [154, 143, 127] for details. For the survey on the face-width of embeddings, we refer to [122].

4.3. Version 3: Tamed Vortices

The result in the previous subsection, Version 2, is already a general result and strong enough to describe the rough structure of graphs without H-minors. But it does not seem enough to prove both Theorems 1.1 and 1.2. In order to do so, Robertson and Seymour needed to have a strong connectivity property inside vortices. The main result in Graph Minors XVII addresses this question and assures that any graph that can be near embedded in some surface can also be "nicely" near embedded, in a sense of connectivity of vortices, in some surface, which is homeomorphic to or simpler than the first one, after removing a bounded number of vertices. Before stating the main result of Graph Minors XVII, we need some notation.

In Subsection 3.4 on society minors, we have introduced the definition of a vortex, defined its adhesion, and stated Theorem 3.9 which shows that vortices with bounded adhesion admit a nice cyclic vortex decomposition into graphs G_v, where v's are the society vertices. In this section we start building with a decomposition of a vortex. In order to stay close to the notion of path-decompositions, we change the cyclic structure into a linear order. Having a vortex of bounded adhesion, this is easy – just remove the vertices in the intersection of two graphs G_u, G_v corresponding to consecutive society vertices u, v. In the final structure theorem, these removed vertices can be put into the apex set and still describing a similar rough structure.

Let us add another requirement (4) to conditions (1)–(3) of Theorem 3.9 forming the description of a vortex decomposition:

(4) If v is a vertex in the society F and $v \in G_u$ for $u \in F$, then either $u = v$ or u is the successor of v in the society order.

The vortex decomposition as defined by (1)–(4) is *linked* if for any three consecutive society vertices u, v, w, $G_v - F$ contains a collection of disjoint paths linking

$V(G_u) \cap V(G_v)$ with $V(G_v) \cap V(G_w)$. In particular, any two consecutive graphs G_u, G_v in the vortex decomposition intersect in the same number of vertices distinct from u and v. The *linked adhesion* of a vortex is the minimum adhesion taken over all linked decompositions of the vortex. Let us observe that in a linked decomposition of adhesion q, there are q disjoint paths in $G - F$ passing through all graphs G_v, $v \in F$.

Now we can state the revised Excluded Minor Theorem as given in Graph Minors XVII.

Theorem 4.4 ([147]). *For every graph H, there is a positive integer k such that for every positive integer w, there exists a positive integer $r = r(H, w)$, tending to infinity with w for any fixed H, such that every G not containing H as a minor either has tree-width at most w, or contains an r-wall W such that (G, W) has an k-almost embedding in some surface Σ in which H cannot be embedded, under which each vortex has linked vortex decomposition of adhesion at most k.*

Theorem 4.4 is not stated in [147] precisely as given above. Because of a rather different language used in [147], an appendix in [10] addresses how Theorem 4.4 can be derived from the main results (9.8) and (13.4) in [147]. As we see here, the statement of Theorem 4.4 is exactly the same as Theorem 4.3, except for the requirement of linked adhesion of the vortices. Theorem 4.4 needs to have all vortices linked, while Theorem 4.3 does not.

5. Minors in Large Graphs

An important achievement in the recent advance concerning the existence of large complete graph minors is the result by Böhme et al. [10] which says that every $16k$-connected graph, which is large enough, has a K_k-minor. The proof of this result uses Theorem 4.4 given in the previous section. This result is then used to prove partial results on minimal-counterexamples to the well-known Hadwiger's conjecture. Kawarabayashi and Mohar [89] also made progress on the algorithmic aspect of Hadwiger's conjecture. We shall discuss these results in the next section.

Another important theorem has been obtained by Diestel and Thomas [41]. Their result extends Robertson and Seymour's structural result on the clique-sum (Version 1) to infinite graphs.

There are several structures which guarantee that a certain minor exists in a graph G if G is large enough. For instance, any 5-connected graph on at least 11 vertices contains the 3-cube as a minor [112]. Any 5-connected non-planar graph on at least 8 vertices contains a V_8-minor [129]. In addition, there are Ramsey-type results similar to the fact that any sufficiently large connected graph contains either a k-path or a k-star. Oporowski, Oxley and Thomas [125] proved that any large 4-connected graph must have a large minor from a set of four families of 4-connected graphs. They found a similar result for large 3-connected graphs. Recently, Kawarabayashi [83] proved a corresponding result for large 5-connected graphs. Ding [42] has characterized large graphs that do not contain a $K_{2,k}$ minor.

A corollary of his result is that every sufficiently large 5-connected graph contains a $K_{2,k}$-minor.

These graphs that are excluded are not so general in a sense. In particular, these results only need quite small connectivity. But what condition is necessary to obtain K_k-minors for general k (for large graphs)? We would definitely need some connectivity assumption, otherwise, we would have infinite family of graphs by gluing two graphs that do not have K_k-minors with only small number of vertices. So, what we would like to do is to focus on graphs with some moderate connectivity, and then find K_k-minors for general k for large graphs. To obtain this kind of results, we need the structure theorem of Robertson and Seymour. It turns out that there are many exciting corollaries, some of which were conjectured in the 1970s. We shall survey these results in this section. Topics are focused on infinite graphs, connectivity and minors, toughness and minors, and applications to the well-known Hadwiger's conjecture.

5.1. Structure Theorem for Infinite Graphs

When considering minors in large graphs, the test case is the "limiting" case when graphs are infinite. There is one essential difference, though. High minimum degree, connectivity and girth do not force large clique minors as in the case of finite graphs (cf. Section 2). The reason is that for every k there exist k-connected infinite planar graphs with girth more than k, and yet they do not contain K_5-minors.

Let us first overview results concerning structure of K_n-minor-free infinite graphs, where n is finite. It turns out that the result in [41] gives exactly the same structure as Robertson and Seymour's excluded minor theorem presented in Version 1. The only difference is that nearly embedded graphs can be infinite and that there could be infinitely many such basic pieces forming the tree-like structure.

Diestel and Thomas then used this result to derive a theorem describing the structure of all graphs without a $K_{\aleph_0}$-minor, where $K_{\aleph_0}$ is the complete graph of countable order. According to Diestel and Thomas [41], their result gives the precise and not only a "rough" structure.

Theorem 5.1. *A graph has no $K_{\aleph_0}$-minor if and only if it has a tree-decomposition of finite adhesion over plane graphs with at most one vortex.*

Diestel and Thomas gave two motivations for the above theorem. One is that $K_{\aleph_0}$ is in a sense the most general countable minor to exclude, and so this would be the first choice. Second, there is a challenging conjecture, which says that Wagner's conjecture is true for countable graphs [175]. This would extend the graph minor theorem. In general, it is known to be false [174], but it was conjectured in [175] that it could be extend to countable graphs. For a possible approach, perhaps the excluded minor theorem for $K_{\aleph_0}$ might be the first step towards a proof of this conjecture, similarly as Robertson and Seymour did it for the proof of Wagner's conjecture for finite graphs [150]. Let us remark that the structure of graphs without a subdivision of $K_{\aleph_0}$ is much simpler and easier to characterize [38, 158]. For additional information on excluded minor theorems in infinite graphs, we refer the reader to [157, 161].

5.2. Forcing Complete Graph Minors in Large Graphs

In this subsection, we shall present some recent results about existence of complete graph minors or complete bipartite graph minors in large graphs with moderately small connectivity.

By Theorem 2.1, graphs whose connectivity is $\Omega(k\sqrt{\log k})$ contain K_k-minors. However, as already mentioned in Section 2, it seems that K_k-minor-free graphs of high connectivity are close to random graphs of suitable edge density, and their size is bounded. This fact motivated Mader [110] (see [180, 181]) to ask the following.

Question (Mader). Suppose that G is a large ck-connected graph without K_k-minor, where c is some constant. What does G look like?

Motivated by this question and the results stated above, Böhme et al. [10] proved the following theorem, which in particular answers the question of Mader.

Theorem 5.2 ([10]). *For any integers a, s and k, there exists a constant $N(s,k,a)$ such that every $(3a+2)$-connected graph of minimum degree at least $\frac{31}{2}(a+1)-3$ and with at least $N(s,k,a)$ vertices either contains $K_{a,sk}$ as a topological minor or a minor isomorphic to s disjoint copies of $K_{a,k}$.*

By taking $s=1$ and $k=a$ in Theorem 5.2, we obtain $K_{a,a}$ and consequently also K_a as a minor. This is the first result showing that a linear function of connectivity guarantees the existence of K_a-minors. This settles a conjecture of Thomason [180, 124]. See also Theorem 5.5 below.

The extremal number of edges of K_a-minor-free graphs is known only for $a \le 9$. For up to K_7-minors, this is due to Mader [107]. For the K_8-minor, this is due to Jørgensen [76]. Recently, the K_9-minor case was settled by Song and Thomas [172].

Conjecture 8.13 (see later) speculates that connectivity condition (and the minimum degree condition) can be dropped all the way down to $2a+1$ and still force $K_{a,k}$ (nad hence also K_a) as minors in sufficiently large graphs. Connectivity $2a+1$ is necessary since there are arbitrarily large $2a$-connected graphs (of tree-width $3a-1$) none of which contains a $K_{a,2a+1}$-minor; see [13].

In a forthcoming paper [11], Böhme et al. extended the proof method in [10] and proved the following special case of Conjecture 8.13.

Theorem 5.3 ([11]). *For any positive integer k, there exists a constant $N=N(k)$ such that every 7-connected graph of order at least N contains $K_{3,k}$ as a minor.*

In another paper [87], Kawarabayashi and Mohar are developing further, and verify Conjecture 8.13 for $a=4$:

Theorem 5.4 ([87]). *For any positive integer k, there exists a constant $N=N(k)$ such that every 9-connected graph of order at least N contains $K_{4,k}$ as a minor.*

Theorems 5.3 and 5.4 are sharp in the sense that the 7-connectivity and 9-connectivity (respectively) conditions cannot be relaxed, as mentioned above.

When considering only graphs of bounded tree-width, one can improve slightly upon Theorem 5.2. In [10] it is proved that for any positive integers a, k, s and w, there exists a constant $N = N(a, k, s, w)$ such that every $(3a + 1)$-connected graph with minimum degree at least $\frac{27}{2}(a + 1)$, of tree-width at most w and of order at least N, either contains s disjoint $K_{a,k}$ minors or contains a subdivision of $K_{a,sk}$. Connectivity $3a + 1$ cannot be relaxed beyond $3a$ since for every positive integer a, there exist arbitrarily large $(3a - 1)$-connected graphs of minimum degree $4a - 2$ and tree-width $4a - 2$ that contain neither $K_{a,k}$-subdivision nor a minor isomorphic to a disjoint copies of $K_{a,k}$ for $k \geq 4a - 1$, see [10].

All theorems stated above use Theorem 4.4 in their proofs. The idea is to prove the theorem for tree-width bounded case first and then apply Theorem 4.4 by excluding the conjectured minor.

If we only consider just one K_a-minor, then the proof in [10] actually implies the following.

Theorem 5.5. *For every positive integer a, there exists a constant $N(a)$ such that every $2(a+1)$-connected graph of minimum degree at least $\frac{29(a+1)}{2}$ with at least $N(a)$ vertices contains a K_a-minor.*

Let us remark that, as observed in [13], the sequence of graphs $K_{a,k}$, where a is fixed and k tends to infinity, is essentially the only family of graphs for which a result like Theorem 5.2 holds. More precisely:

Theorem 5.6 ([13]). *Let c and $w \geq c$ be positive integers, and let H_k $(k \geq 1)$ be a sequence of graphs such that $\lim_{k\to\infty} |V(H_k)| = \infty$. Suppose that for any positive integer k there exists an integer $N(k)$ such that every c-connected graph of tree-width $\leq w$ and of order at least $N(k)$ contains H_k as a minor. Then H_k is a minor of $K_{c,N(k)}$ for $k \geq 1$.*

5.3. Large Graphs without K_6-minors

In this subsection, another example that uses techniques and tools from Graph Minor Theory to prove existence of minors in large graphs is given.

Robertson, Seymour and Thomas proved the following result when dealing with Hadwiger's Conjecture for K_6-minor-free case [159]. Let us recall that a graph G is an *apex graph* if it has a vertex v such that $G - v$ is planar.

Theorem 5.7. *Let G be a graph with no K_6-minor such that G is not 5-colorable, and subject to that, $|G|$ is as small as possible. Then G is an apex graph.*

This theorem implies that Hadwiger's Conjecture for K_6-minor-free case is equivalent to the Four Color Theorem. But unfortunately, this theorem does not give any structural characterization for graphs with no K_6-minor. Motivated by this fact, Jørgensen [76] made the following beautiful conjecture.

Conjecture 5.8 (Jørgensen). *Every* 6*-connected graph with no* K_6*-minor is apex.*

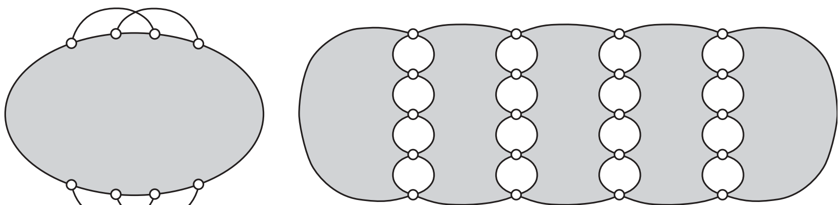
Mader [107] proved that the graph G mentioned in Theorem 5.7 is 6-connected. Hence the above conjecture implies Theorem 5.7. This conjecture is still open, but recently, DeVos et al. [34] proved the following remarkable result.

Theorem 5.9. *Jørgensen's conjecture is true for large graphs. More precisely, there exists a constant* N *such that every 6-connected graph with no* K_6*-minor and with at least* N *vertices is apex.*

The proof is lengthly and complicated, but let us briefly sketch the main ideas from [34] since it uses several tools described before.

The proof is divided into two parts. First, the bounded tree-width case is settled. Then one uses the grid theorem [160] (cf. Theorem 3.1) to confirm that there is a large wall H. Excluding K_6 as a minor enables us to use Theorem 4.3 which shows that we may assume that H is actually embedded in some disk. Look at all the vertices on the outer face of H. We can think of these vertices as a society S, and the outside bridge of H attached to S can be thought of as a vortex with society S. If this vortex contains either a crosscap or a double cross of large order, then we try to find a K_6-minor using this crosscap or double cross. If there are no leaps, then we can assume by Theorem 3.10 and Euler's formula that there is a large vortex of bounded width. In this case, one can show that the graph cannot be apex. Next, we try to find a K_6-minor by studying the vortex structure. In [34], some lemmas concerning the society that is independent of Robertson and Seymour's results are developed to handle this case. The last remaining case is when there is a leap of large order. In this case, G may be an apex graph. But if it is not, then some additional arguments show that there is a K_6-minor.

One may ask what about 5-connected graphs with no K_6-minor? As far as we know, there are five families of graphs that do not contain K_6-minors. These are planar graphs, apex graphs, double cross graphs, planar plus a triangle, graphs with hamburger structure and graphs with hose structure. For *double cross graphs* and the *hose structure* see Figure 4, in which shaded "blobs" represent planar graphs embedded in the shaded disk with specified vertices on the boundary. For consecutive "blobs" in the hose structure, the five vertices are identified, not necessarily



(a) **(b)**

Fig. 4. (a) Double cross and (b) hose structure graphs

in order as suggested by their closeness in the figure, but the three white vertices are identified with white and the two black with the black ones in the neighboring "blob." Graphs with *hamburger structure* are obtained from three 5-connected planar graphs G_i ($i = 1, 2, 3$), each of which having a specified vertex w_i of degree 5. Let $v_{i1}, \ldots, v_{i5}$ be the neighbors of w_i in the clockwise order around w_i. To get a graph with hamburger structure, take $G_1 - w_1$, $G_2 - w_2$, and $G_3 - w_3$ and identify for $j = 1, \ldots, 5$ their vertices v_{1j}, v_{2j}, v_{3j}. These examples give rise to infinitely many 5-connected graphs without K_6-minors and with different structure. At this moment it seems hopeless to characterize 5-connected graphs with no K_6-minor, even for large graphs.

5.4. Toughness and Unavoidable Minors in Large Graphs

Böhme, Mohar, and Reed [14] showed that Theorem 5.2 can be strengthened by modifying the connectivity assumptions. Recall that a connected graph G is *t-tough* if for every separating vertex set S, the subgraph $G - S$ of G has at most $|S|/t$ connected components.

If d and k are positive integers, then P_k^d denotes the dth power of the path on k vertices $v_1, \ldots, v_k$, i.e., distinct vertices v_i and v_j of P_k^d are adjacent if and only if $|j - i| \leq d$.

Theorem 5.10 (Böhme, Mohar, and Reed [14]). *For every positive integer d, there is a number $T = T(d)$ such that for every positive integer k there is a constant $N = N(d, k)$ such that every T-tough graph of order at least N contains the dth power P_k^d of the k-path as a minor.*

It can be shown that P_k^d is $\frac{d}{2}$-tough if $k \geq d + 2$. Therefore, whenever for a fixed toughness T a sequence of graphs $G_1, G_2, \ldots, G_k, \ldots$ has the property that every large enough T-tough graph contains a G_k-minor, then G_k is a minor in the $(2T)$th power of a large path. This remark shows that graphs P_k^d (and their minors) are essentially the only family of graphs for which a statement analogous to Theorem 5.10 holds.

6. Graph Minors and Coloring Problems

6.1. Hadwiger's Conjecture

Much of research related to graph minors is also motivated by Hadwiger's Conjecture from 1943 which suggests a far reaching generalization of the Four Color Theorem [2, 3, 130] and is one of the most interesting open problems in graph theory. In this subsection, we shall point out how Theorem 5.5 can be used to make some progress on Hadwiger's conjecture.

Conjecture 6.1 (Hadwiger [69]). *For every $k \geq 1$, every graph with chromatic number at least k contains the complete graph K_k as a minor.*

For $k = 1, 2, 3$, this is easy to prove, and for $k = 4$, Hadwiger himself [69] and Dirac [44] proved it. For $k = 5$, however, it becomes extremely difficult. In 1937, Wagner [193] proved that the case $k = 5$ is equivalent to the Four Color Theorem; cf. Theorem 1.6. So, assuming the Four Color Theorem [2, 3, 130], the case $k = 5$ in Hadwiger's Conjecture holds. Robertson, Seymour and Thomas [159] proved that a minimal counterexample to the case $k = 6$ is a graph G that has a vertex v such that $G - v$ is planar. Hence, assuming the Four Color Theorem, the case $k = 6$ of Hadwiger's Conjecture holds. This result is one of the deepest in this area. So far, the conjecture is open for every $k \geq 7$. For the case $k = 7$, Kawarabayashi and Toft [93] proved that any 7-chromatic graph has K_7 or $K_{4,4}$ as a minor, and recently, Kawarabayashi [84] proved that any 7-chromatic graph has K_7 or $K_{3,5}$ as a minor.

It is not even known if there exists an absolute constant c such that any ck-chromatic graph has K_k as a minor. Theorem 2.1 shows that there exists a constant c such that any $ck\sqrt{\log k}$-chromatic graph has K_k as a minor. So it would be of great interest to prove that a linear function of the chromatic number is sufficient to force a K_k-minor. Reed and Seymour [128] proved the fractional version of this conjecture.

Perhaps Theorem 5.5 may be the first step to prove that conjecture since by Mader's result [108], any minimal counterexample to Hadwiger's conjecture has a "highly" connected subgraph, and Kawarabayashi [82] proved that any minimal counterexample to Hadwiger's conjecture is $\frac{2k}{27}$-connected. So if this graph had the order larger than $N(k)$ in Theorem 5.5, there would be a constant c such that every ck-chromatic graph has K_k as a minor. However, it is not clear whether this graph is large or not. Theorem 5.5 only implies that a minimum counterexample to the conjecture has "small" order. It also implies that there exist absolute constants c_1 and c_2 with $c_1 \geq c_2$ such that there are only finitely many c_1k-connected c_2k-color-critical graphs without K_k as a minor. This fact is related to Thomassen's result [187] which says that there are only finitely many 6-color-critical graphs on a fixed surface. Notice that the set of graphs embeddable on a fixed surface is closed under taking minors. More generally, Mohar [120] conjectured the following.

Conjecture 6.2. *There are only finitely many* 3*-connected* k*-color-critical graphs without* K_k *as a minor.*

Note that the above conjecture without the condition on 3-connectivity would be equivalent to Hadwiger's Conjecture since if we have one such graph, then we would have infinitely many by applying the Hajós' construction. Hadwiger's conjecture suggests that there are no k-color-critical graphs without K_k as a minor. Since every 4-color-critical planar graph joined with the complete graph K_{k-5} gives rise to a $(k-1)$-color-critical graph without K_k-minors, the number k of colors is necessary.

Let G be a graph satisfying the following conditions:

(i) G is k-chromatic.
(ii) G is minimal with respect to the minor-relation in the class of all k-chromatic graphs.

Any graph satisfying (i) and (ii) is said to be *k-contraction-critical*. Such graphs were first defined and studied by Dirac [45, 46]. Theorem 5.5 together with the main result of [82] implies that there exists a constant c such that there are only finitely many ck-contraction-critical graphs without K_k-minors.

Kawarabayashi and Mohar also studied the list-coloring version of Hadwiger's conjecture, and made some progress in [88–90] using Theorem 5.2.

Let G be a graph and t a positive number. A *list-assignment* is a function L which assigns to every vertex $v \in V(G)$ a set $L(v)$ of natural numbers, which are called *admissible colors* for that vertex. An *L-coloring* is an assignment of admissible colors to all vertices of G, i.e., a function $c : V(G) \to \mathbb{N}$ such that $c(v) \in L(v)$ for every $v \in V(G)$, and for every edge uv we have $c(u) \neq c(v)$. If $|L(v)| \geq t$ for every $v \in V(G)$, then L is a *t-list-assignment*. The graph is *t-choosable* if it admits an L-coloring for every t-list-assignment L. The L-colorings are also called *list-colorings*. This subject was first introduced in the second half of the 1970s, in two papers by Vizing [191] and independently by Erdős, Rubin and Taylor [51].

When relaxing the Hadwiger Conjecture to allow ck colors, the following conjecture from [90] involving list colorings may also be true:

Conjecture 6.3 ([90]). *There is a constant c such that every graph without K_k-minors is ck-choosable.*

Conjecture 6.1 does not hold for list colorings. For example, there exist planar graphs (without K_5 minors) which are not 4-choosable. However, Conjecture 6.3 is formulated in such a way that it may also be true for $c = 1$. The main result in [90] says the following.

Theorem 6.4. *For any integer k, there is a constant $f(k)$ such that for every graph G without K_k as a minor and for every $15.5k$-list-assignment L, there is a vertex partition $\{V_i \mid i \in \mathbb{N}\}$ of $V(G)$ such that for every i, $V_i \subseteq \{v \in V(G) \mid i \in L(v)\}$ holds, and such that every component of the subgraph of G induced on V_i has at most $f(k)$ vertices.*

If $f(k)$ would be equal to 1, we would get a coloring of G. Hence, Theorem 6.4 gives a relaxation of coloring, and may be viewed as the first step to attack Conjecture 6.3. Another result of similar flavor appeared in [88]:

Theorem 6.5. *For any integer k, there is a constant $f(k)$ such that every graph without K_k minors is either $15.5k$-choosable, or it contains a subgraph of order at most $f(k)$ that is not $(9.5k-6)$-choosable.*

6.2. Algorithmic Aspect of Hadwiger's Conjecture

Although, it is still open if there exists a constant c such that any ck-chromatic graph contains K_k as a minor, it is known that that from an algorithmic point of view, we can test this in polynomial time. The following result was proved by Kawarabayashi and Mohar [89].

Theorem 6.6. *For every fixed k, there is an algorithm with running time $O(n^3)$ for deciding either that*

(1) *a given graph G of order n is 27k-colorable, or*
(2) *G contains K_k-minor, or*
(3) *G contains a minor H of bounded size which does not contain a K_k-minor and has no 27k-colorings.*

Let us remark the following:

(a) If (3) holds, then H is a counterexample to Hadwiger's conjecture. In fact, this would be a counterexample to the weaker conjecture that any $27k$-chromatic graph has K_k as a minor. The conclusion of Theorem 6.6(3) that such a minor has bounded number of vertices is an interesting theoretical outcome of the algorithm.
(b) If (1) holds, we can actually find a coloring of G using at most $27k$ colors. If (3) holds, we can exhibit the minor H by means of a subgraph $\tilde{H}$ of G whose contraction yields H.
(c) We need the result of [10], Theorem 5.2, but we do not need any deep results from Graph Minors series. (The proof of Theorem 5.2 given in [10] depends on the Excluded Minor Theorem, though.)

Recently, Kawarabayashi and Mohar [91] proved that Theorem 5.2 can be improved as follows: For any k, there exists a constant $N(k)$ such that every $2k$-connected graph with minimum degree at least $9k$ and with at least $N(k)$ vertices has a K_k-minor. Also, if the tree-width is large (in a sense that we can apply Robertson and Seymour's result in [147] to G, see a detailed description in [10, 91]), then the minimum degree condition can be improved to $\frac{15a}{2}$. Also, Kawarabayashi in [83] gave an improvement on [82]. Together with these results, the algorithm implies that the chromatic number in Theorem 6.6 can be improved from $27k$ to $12k$.

Furthermore, Robertson and Seymour (private communication) have the following unpublished result, which would give rise to a polynomial-time algorithm for k-coloring K_k-minor free graphs if the Hadwiger Conjecture is true.

Theorem 6.7 (Robertson and Seymour [156]). *For every fixed k, there is a polynomial-time algorithm for deciding either that*

(1) *a given graph G is k-colorable, or*
(2) *G contains K_{k+1}-minor, or*
(3) *G contains a minor H without K_{k+1}-minors, of order at most $N(k)$, and with no k-coloring.*

Neil Robertson (private communication) pointed out that in order to prove the above theorem, Robertson and Seymour used the following lemma, which is of independent interest, and is perhaps the strongest result in this direction.

Lemma 6.8 ([156]). *Let $k \geq 4$ be an integer. For any graph G with no K_{k+1}-minor, one of the followings holds:*

(1) *There exists an integer* $f(k)$ *such that* G *has tree-width at most* $f(k)$.
(2) G *contains a vertex of degree at most* k.
(3) G *contains a vertex* v *of degree* $k+1$ *whose neighbors include three mutually nonadjacent vertices.*
(4) G *has a separation* (A, B) *of order at most* k *with* $V(A) \neq V(G)$ *such that* A *can be contracted to a clique on* $A \cap B$ *such that each vertex of* $A \cap B$ *is contained in the different node of this clique minor.*
(5) G *has a vertex set* X, $|X| \leq k-4$, *such that* $G - X$ *is planar.*

Note that (2), (3) and (4) cannot happen in minimal counterexamples to Hadwiger's conjecture, and (5) is no longer counterexample, assuming the Four Color Theorem. The proof is complicated and uses the graph minor structure theory (cf., e.g., [146, 147]) heavily (but does not use the well-quasi-ordering result). Paul Seymour also pointed out that outcome (3) can be eliminated on the expense of a considerably longer proof.

6.3. List coloring Graphs in Minor-closed Families

There has been some progress on algorithms for list-coloring minor-closed class of graphs. Before we explain this, let us point out the difficulty of computing the list-chromatic number. It seems that computing the list-chromatic number is much harder than computing the chromatic number. Let us briefly survey this.

Graph coloring is arguably the most popular subject in graph theory. Also, it is one of the central problems in combinatorial optimization, since it is one of the hardest problems to approximate. In general, the chromatic number is inapproximable in polynomial time within factor $n^{1-\epsilon}$ for any $\epsilon > 0$, unless $coRP = NP$, cf. Feige and Kilian [52] and Håstad [71]. Even for 3-colorable graphs, the best known polynomial approximation algorithm achieves a factor of $O(n^{3/14} \log^{O(1)} n)$ in [8]. An interesting generalization of the classical problem of properly coloring the vertices of a graph is that of list-colorings which we have already met in Section 6.1.

The problem of computing the list-chromatic number of a given graph is difficult, even for small graphs with a simple structure. One example is that the complete bipartite graph $K_{5,8}$ is 3-choosable, but a proof given in [111] is lengthly and nontrivial. More formally, it is shown in [68] that the problem of deciding if the list-chromatic number is k for $k \geq 3$ is Π_2^p-complete. Hence if the complexity classes NP and $coNP$ are different, as is commonly believed, the problem is strictly harder than the NP-complete problem of deciding if the chromatic number is k (if $k \geq 3$).

Although there are many negative results as stated above, there are some positive results, which are mainly related to the Four Color Theorem and coloring planar graphs. One celebrated example is Thomassen's result on planar graphs [188]. It says that every planar graph is 5-choosable, and its proof is short and gives rise to a linear time algorithm to 5-list-color planar graphs. In contrast with the Four Color Theorem, there are planar graphs that are not 4-choosable [192]. A natural question is: can we extend the result of Thomassen to more general minor-closed families of graphs? Motivated by this question, Kawarabayashi and Mohar proved the following [88].

Theorem 6.9. *Let $\mathcal{M}$ be a minor-closed family of graphs such that $K_k \notin \mathcal{M}$. Then there is a polynomial time algorithm for list-coloring graphs in $\mathcal{M}$ with $O(k)$ colors. If a coloring is not found, the algorithm finds a small certificate that the graph is not $\Theta(k)$-choosable. The time complexity is $O(n^3)$.*

The main idea in the proof of Theorem 6.9 involves precoloring extension.

In [88], the algorithmic counterpart of Theorem 6.5 is also derived. In particular, it is proved that for every fixed k, there is a constant $f(k)$ and an algorithm with running time $O(n^3)$ for deciding either that G is $15.5k$-choosable, or G contains a K_k-minor, or G contains a subgraph H of bounded size which does not contain a K_k-minor and is not $(9.5k - 6)$-choosable. In the case of the last outcome, H is a counterexample to the list-coloring version of Hadwiger's conjecture. In fact, this would be a counterexample to the weaker conjecture that every graph whose list-chromatic number is at least $9.5k - 6$ has K_k as a minor. The conclusion that such a subgraph has bounded number of vertices is an interesting theoretical consequence of the algorithm.

6.4. Arboricity of Graphs in Minor-closed Families

Algorithms on the arboricity of graphs in minor-closed families are also given in [88]. An *arboreal k-coloring* of G is a partition of the vertices of G into at most k classes, each of which induces an acyclic subgraph of G (a forest). The *arboricity* (sometimes also called *vertex-arboricity*) of G, denoted by $a(G)$, is the the minimum number k for which G has an arboreal k-coloring. The problem of finding arboreal colorings of graphs has applications in the domain of design for testability in VLSI circuits, see [67, 166] for details. The problem of computing $a(G)$ for a given graph G is known to be NP-hard. However, a simple upper bound on $a(G)$ is also known in the literature. Suppose G is d-degenerate, i.e, every induced subgraph of G has a vertex of degree at most d. Then it is easy to see that $a(G)$ is at most $1 + \lfloor d/2 \rfloor$.

Arboricity has been extensively examined for planar graphs. There is a linear-time algorithm to give an arboreal 3-coloring of planar graphs. This was also extended to K_5-minor-free graphs and $K_{3,3}$-minor-free graphs. See [20–22] for details. These algorithms produce arboreal colorings which use at most $a(g) + 1$ colors. We are interested in extending these result to general minor-closed families of graphs, for which we may assume that they are without K_k-minors. By Theorem 2.1, K_k-minor-free graphs are $O(k\sqrt{\log k})$-degenerate. This implies that their arboricity is also of order $O(k\sqrt{\log k})$. Any improvements of this bound would also yield some information on Hadwiger's Conjecture since $\chi(G) \leq 2a(G)$.

As a related conjecture to Hadwiger's Conjecture, Chartrand, Geller and Hedetniemi [19], and Woodall [195] proposed the following (m, n)-Contraction-Conjecture:

Conjecture 6.10 (Chartrand, Geller and Hedetniemi [19], and Woodall [195]). *For integers $1 \leq n \leq m$, every graph G without a K_{m+1}-minor and without a $K_{\lfloor \frac{m+2}{2} \rfloor, \lceil \frac{m+2}{2} \rceil}$-minor, has a partition of $V(G)$ into $m - n + 1$ parts, each part inducing a subgraph without a K_{n+1}-minor and without a $K_{\lfloor \frac{n+2}{2} \rfloor, \lceil \frac{n+2}{2} \rceil}$-minor.*

This conjecture is true for $m \leq 4$ as proved by Chartrand, Geller and Hedetniemi [19] except for the case $(m, n) = (4, 1)$, which is equivalent to the Four Colour Theorem. In these cases, the conjecture is best possible in the sense that there are graphs whose vertex set cannot be partitioned into fewer sets with the desired property. That there exist planar graphs of arboricity 3 was first shown by Chartrand and Kronk [18]. Several interesting applications are obtained in [43].

Although it is still open if there exists a constant c such that any graph without K_k as a minor has arboricity at most ck, from an algorithmic point of view this can be decided in polynomial time up to certain extent. Among others, the following result appeared in [88].

Theorem 6.11 ([88]). *For every fixed k, there is an algorithm with running time $O(n^3)$ which for a given graph G of order n outputs one of the following conclusions:*

(1) *G has arboricity at most $8k$,*
(2) *G contains a K_k-minor, or*
(3) *G contains a minor H of bounded size which is K_k-minor-free and has arboricity more than $2k$.*

Let us observe that if the output is (3), then H is a counterexample to Conjecture 6.10.

7. Applications of the Excluded Minor Theorem in Combinatorial Optimization

DeVos et al. [32] used the Excluded Minor Theorem to prove that for every integer $k \geq 1$ and every fixed graph H, every H-minor-free graph has a vertex partition into parts $V_1, \ldots, V_k$ and edge partition into parts $E_1, \ldots, E_k$ such that for every $i \in \{1, \ldots, k\}$, the graphs $G - V_i$ and $G - E_i$ have bounded tree-width. A special case of this result restricted to planar graphs was proved by Baker [7] who used it to devise efficient approximation algorithms (and approximation schemes) for some hard approximation algorithms on planar graphs. Baker used the planar separator theorem of Lipton and Tarjan [104]. Her work was extended by Alon, Seymour, and Thomas [1] who proved a separator theorem for graphs excluding any fixed minor.

Eppstein [48] extended Baker's ideas to graphs in arbitrary proper minor-closed classes of graphs. Baker's result on planar graphs can be dualized, and this was essentially settled by Klein [94] who proved that every planar graph G has an edge partition into parts $E_1, \ldots, E_k$ such that for every $i \in \{1, \ldots, k\}$, the contracted graph G/E_i has bounded tree-width. Demaine et al. [31] extended this result to graphs embedded on general surfaces and derived polynomial time approximation schemes for several optimization problems on such graphs.

Demaine et al. [28] have been working on the direct use of graph minor theorem for algorithmic applications. They obtained subexponential fixed-parameter algorithm for dominating set, vertex cover and set cover in any class of graphs excluding a fixed graph H as a minor. Specifically, the running time is $2^{O(\sqrt{k})}n^h$, where h is a constant depending only on H. For further applications, see the survey [29] or the paper [30].

8. Some Future Directions and Open Problems

8.1. Shorter Proof

It would be quite important to have simpler and more accessible proofs of the main graph minor results, hopefully also with more explicit bounds. Many mathematicians have tried, but only partial success has been reported.

The proofs of Robertson and Seymour up to Graph Minors X have already been greatly simplified, in part by the authors themselves. In particular, there is now a well-understood and short proof of Kuratowski theorem for general surfaces, whose original proof is contained in Graph Minors VIII [138] and is based on the preceeding graph minor papers. The simplified proof consists of the following three components.

(i) Graphs of large tree-width contain large grid minor, cf. Theorem 3.1.
(ii) Minimal forbidden minors for a fixed surface cannot contain large grid minors, and hence have bounded tree-width.
(iii) Minimal forbidden minors for a fixed surface of bounded tree-width cannot be too large.

Part (i) has an accessible proof by Diestel et al. [40]. It takes only ten pages and is considerably shorter than the proof in [134]. Thomassen gave a very short and simple proof for (ii), see [186]. Geelen, Gerards, and Whittle [55] found a shorter direct proof for the well-quasi-ordering of graphs of bounded tree-width, which in particular implies (iii). Moreover, Mohar [119] gave a constructive proof for (iii), which only takes three pages. Combining the papers [40, 186, 119], we now have a nice proof of Kuratowski theorem for general surfaces. This whole proof is included in the book [122]. Another, slightly different proof of the same result has been obtained by Seymour [169].

But so far, no one came up with shorter proofs of Theorems 4.1, 4.3 and 4.4.

Another improvement of Robertson-Seymour's theory is possible by making proofs constructive. Parts of graph minor proofs are existential and nonconstructive and do not give any clue on how large an excluded minor could be. Therefore, constructive proofs would be of great importance for our deeper understanding and for possible applications.

8.2. Extensions to Matroid Minors

Geelen, Gerards, and Whittle are undertaking a program of research aimed at extending the results and techniques of the Graph Minor Project of Robertson and Seymour to matroids. In particular, they are trying to find the structure of minor-closed classes of matroids representable over a fixed finite field. This requires a peculiar synthesis of graphs, topology, connectivity, and algebra. They expect the structure theory will help in proving Rota's conjecture that for every field there are only finitely many excluded minors for representability over that field, and Robertson and Seymour's Well-Quasi-Ordering conjecture that for any finite field any infinite list of matroids representable over that field contains two members such that one is a minor of the other. They also expect the theory will help to find an efficient algorithm for recognizing a fixed minor-closed property over a fixed finite field. But there is a warning. Their project already produced more than ten papers. Some of them already appeared, see [55–63]. According to the authors, there is still a long way to go. Their surveys [64, 65] give additional informations and insights.

8.3. Topological Problems

There are many minor-closed families of graphs that arise in the study of topological problems. Illustrative examples are likelessly embeddable graphs. By a *linkless embedding*, we mean an embedding of a graph in the 3-space $\mathbb{R}^3$ in such a way that for every cycle C of G, there exists a closed disk $D \subseteq \mathbb{R}^3$ with $D \cap G = \partial D = C$.

Robertson, Seymour and Thomas [162–164] proved that G is linklessly embeddable if and only if G does not have any graph in the Petersen family as a minor. By the *Petersen family* we mean graphs obtained from K_6 by a series of $Y\Delta$ and ΔY-changes. See Figure 5 where drawings of these graphs on the projective plane are shown; note that the third one, $K_{4,4} - e$, cannot be embedded in the projective plane.

Similarly, graphs which can be knotlessly embedded in $\mathbb{R}^3$ (every cycle of G considered as a knot in $\mathbb{R}^3$ is a trivial knot). The family of all such graphs is also minor-closed and thus there is a finite collections of forbidden minors. Conway [27] proved that K_7 is one of them. Several other forbidden minors for knotlessly embeddable graphs are known, see [27]. By using $Y\Delta$ operations, several other forbidden minimal minors have been found [53].

Two invariants introduced by Colin de Verdière [24–26] and by Lovász and Schrijver [105, 106], respectively, are closely related to linklessly embeddable graphs and seem to have some ties with knotlessly embeddable graphs.

8.4. Open Problems

Finally, let us expose some open questions about graph minors. The main open problem in this area is definitely Hadwiger's Conjecture.

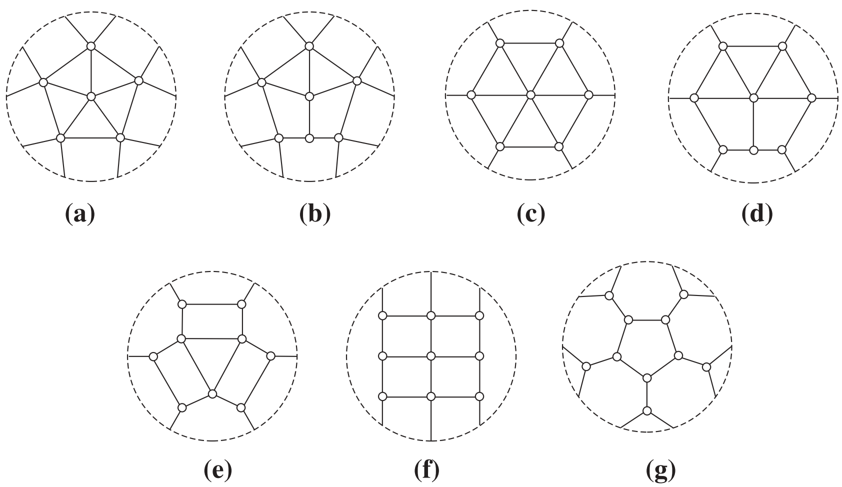


Fig. 5. Petersen's family

Conjecture 8.1 (Hadwiger [69]). *For every $k \geq 1$, every graph with chromatic number at least k contains the complete graph K_k as a minor.*

Conjecture 8.1 is open for all values $k \geq 7$. The case $k = 7$ states the following.

Conjecture 8.2. *Every 7-chromatic graph G has a K_7-minor.*

Mader [107] proved that a minimal counterexample to Conjecture 8.2 is 7-connected. Hence the following implies Conjecture 8.2:

Conjecture 8.3. *Every 7-connected graph with no K_7-minor has two vertices u, v such that $G - v - u$ is planar.*

This conjecture is a strengthening of Jørgensen's Conjecture stated below.

So far, the case $k = 7$ seems hopeless, but perhaps one can prove the following along the similar line of the proof by DeVos et al. [34].

Conjecture 8.4. *There are only finitely many minor-minimal counterexamples to Hadwiger's Conjecture for the K_7-minor-free case.*

More generally, the following would be the first step toward a solution of Hadwiger's Conjecture.

Conjecture 8.5. *For every fixed k, there are only finitely many minor-minimal counterexamples to Conjecture 8.1.*

We do not even know the complete characterization for graphs without K_6-minors. Motivated by this fact, there is a beautiful conjecture by Jørgensen [76].

Conjecture 8.6. *Every 6-connected graph G with no K_6-minors contains a vertex v such that $G - v$ is planar.*

As mentioned before, this conjecture was solved for large graphs by DeVos et al. [34], but their method never gives a solution to this conjecture. Even the following conjecture seems still open.

Conjecture 8.7. *Every 7-connected graph has a K_6-minor.*

For the complete characterization of K_6-minor-free graphs, one may also need to solve the following.

Problem 8.8. Characterize 5-connected graphs with no K_6-minor.

This conjecture seems much harder than Jørgensen's conjecture, and even for large graphs, this seems out of reach. The arguments used in [34] suggest that the following three conjectures, all proposed by Seymour and Thomas (private communication), may be possibly attacked by using Graph Minor techniques, tools and results.

Conjecture 8.9. *Every* $(k+1)$*-connected sufficiently large graph has a* K_k*-minor.*

Conjecture 8.10. *Any* k*-connected sufficiently large graph with no* K_k*-minors has a set* X *of exactly* $k-5$ *vertices such that* $G-X$ *is planar.*

Conjecture 8.11. *Any* k*-connected sufficiently large graph with at least* $k|V(G)|$ *edges has a* K_k*-minor.*

The following weakening of Conjecture 8.3 would be the first step in this direction.

Conjecture 8.12. *Every sufficiently large* 7*-connected graph* G *with no* K_7*-minors has two vertices* u, v *such that* $G-u-v$ *is planar.*

As far as Theorems 6.4 and 5.5 are concerned, the following related conjecture can be found in [10].

Conjecture 8.13. *For any* a, k*, there exists a constant* $f(a,k)$ *such that every* $(2a+1)$*-connected graph with at least* $f(a,k)$ *vertices has a* $K_{a,k}$*-minor.*

This conjecture is known to be true for $a = 3$ [11] and $a = 4$ [87], see Section 5.2. As far as we know, the cases $a \geq 5$ are open.

Conjecture 8.14. *For every positive integers* a *and* k*, there exists a constant* $f(a,k)$ *such that every* $3a$*-connected graph with at least* $f(a,k)$ *vertices contains either* $K_{a,k}$*-subdivision or a minor isomorphic to a disjoint copies of* $K_{a,k}$*.*

If true, the connectivity bound is best possible as we have pointed out in the discussion after Theorem 5.4. Böhme et al. [10] proved a weakening of Conjecture 8.14 with connectivity $3a+1$ and restricted to graphs of bounded tree-width.

There are many weaker versions of Hadwiger's Conjecture. All of them seem to be out of reach right now. Let us state some of them here.

Conjecture 8.15. *There is a constant* c *such that for every integer* k*, every graph without* K_k*-minors is* ck*-colorable.*

The authors of this survey made a stronger conjecture in [90]:

Conjecture 8.16 ([90]). *There is a constant c such that for every integer k, every graph without K_k-minors is ck-choosable.*

Conjecture 8.17 (Chartrand, Geller and Hedetniemi [19], and Woodall [195]). *For integers $1 \leq n \leq m$, every graph G without a K_{m+1}-minor and without a $K_{\lfloor \frac{m+2}{2} \rfloor, \lceil \frac{m+2}{2} \rceil}$-minor, has a partition of G into $m-n+1$ parts, each part inducing a subgraph without a K_{n+1}-minor and without a $K_{\lfloor \frac{n+2}{2} \rfloor, \lceil \frac{n+2}{2} \rceil}$-minor.*

Conjecture 8.18 (Mohar [120]). *There are only finitely many 3-connected k-color-critical graphs without K_k as a minor.*

Concerning infinite graphs, the following is perhaps the most important open problem.

Conjecture 8.19 (Seymour's self-minor conjecture). *Every countably infinite graph is a proper minor of itself.*

This would imply Wagner's conjecture.

Let us look at linkage problems. The most important conjecture in this area is probably the following.

Conjecture 8.20. *For $k \geq 4$, every $(3k-2)$-connected graph is k-linked.*

This conjecture was made by Thomassen [182]. He actually conjectured the following later.

Conjecture 8.21. *Every $(2k+2)$-connected sufficiently large graph is k-linked.*

K_{3k-1} minus k independent edges shows that the condition "sufficiently large" is necessary. The complete characterization for 2-linked graphs are already obtained by Thomassen [182], Seymour [168] and Shiloach [171], respectively. The first unsettled case is 3-linked graphs. Let us state the main open problem for 3-linked graphs here.

Conjecture 8.22. *Every 8-connected graph is 3-linked.*

If true, connectivity 8 would be best possible as the following example shows. Let G_0 be a 5-connected planar graph with a face $s_1s_2t_1t_2$ of length 4. (Such graphs exist.) Now add two vertices s_3 and t_3 connected to all vertices of G_0. The resulting graph is 7-connected, but is not 3-linked as the special vertices s_i, t_i $(i = 1, 2, 3)$ show.

It was proved by Thomas and Wollan [178] that any 10-connected graph is 3-linked. Actually, they gave a best possible extremal function for the number of edges.

Similar line of the proof by DeVos et al. [34] may give a solution to the following conjecture.

Conjecture 8.23. *Every sufficiently large* 8*-connected graph is* 3*-linked.*

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