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A RELAXED HADWIGER'S CONJECTURE FOR LIST COLORINGS

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A relaxed Hadwiger's Conjecture for list colorings

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Abstract

Hadwiger's Conjecture claims that any graph without K_k as a minor is $(k-1)$-colorable. It has been proved for $k \leq 6$, and is still open for every $k \geq 7$. It is not even known if there exists an absolute constant c such that any ck-chromatic graph has K_k as a minor. Motivated by this problem, we show that there exists a computable constant $f(k)$ such that any graph G without K_k as a minor admits a vertex partition $V_1, \ldots, V_{\lceil 15.5k \rceil}$ such that each component in the subgraph induced on V_i $(i \geq 1)$ has at most $f(k)$ vertices. This result is also extended to list colorings for which we allow monochromatic components of order at most $f(k)$. When $f(k) = 1$, this is a coloring of G. Hence this is a relaxation of coloring and this is the first result in this direction.

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1 Introduction

In this paper, all graphs are finite and simple. We follow standard graph theory terminology and notation as used, for example, in [4]. A graph H is a *minor* of a graph K if H can be obtained from a subgraph of K by contracting edges.

Our research is motivated by Hadwiger's Conjecture from 1943 which suggests a far-reaching generalization of the Four Color Theorem and is one of the most challenging open problems in graph theory.

Conjecture 1.1 (Hadwiger [6]) *For every $k \geq 1$, every graph with chromatic number at least k contains the complete graph K_k as a minor.*

For $k = 1, 2, 3$, this is easy to prove, and for $k = 4$, Hadwiger himself [6] and Dirac [5] proved it. For $k = 5$, however, it becomes extremely difficult. In 1937, Wagner [17] proved that the case $k = 5$ is equivalent to the Four Color Theorem. So, assuming the Four Color Theorem [1, 2, 13], the case $k = 5$ of Hadwiger's Conjecture holds. Robertson, Seymour and Thomas [12] proved that a minimal counterexample to the case $k = 6$ is a graph G that has a vertex v such that $G - v$ is planar. By the Four Color Theorem, this implies Hadwiger's Conjecture for $k = 6$. This result is the deepest in this research area. So far, the conjecture is open for every $k \geq 7$. For the case $k = 7$, Kawarabayashi and Toft [9] proved that any 7-chromatic graph has K_7 or $K_{4,4}$ as a minor, and recently, Kawarabayashi [7] proved that any 7-chromatic graph has K_7 or $K_{3,5}$ as a minor.

It is even not known if there exists an absolute constant c such that any ck-chromatic graph has K_k as a minor. So far, it is known that there exists a constant c such that any $ck\sqrt{\log k}$-chromatic graph has K_k as a minor. This follows from results in [10, 11, 14, 15]. This result was proved 25 years ago, but no one can improve the superlinear order $k\sqrt{\log k}$ of the bound on the chromatic number. So it would be of great interest to prove that a linear function of the chromatic number is sufficient to force K_k as a minor. From an algorithmic point of view, we can "decide" this problem in polynomial time. This was proved in [8]. We refer to [16] for further information on the Hadwiger Conjecture.

Motivated by these facts, we shall prove the following relaxation.

Theorem 1.2 *There exists a computable constant $f(k)$ such that every graph G without K_k as a minor admits a vertex partition $V_1, \ldots, V_{\lceil 15.5k \rceil}$ such that every component in the subgraph of G induced on V_i has at most $f(k)$ vertices.*

By saying that $f(k)$ is *computable*, we mean that $f(k)$ can be expressed as a specific value, depending on k. The reader interested in this expression should consult [3].

When $f(k) = 1$, we get a coloring of G. Hence, Theorem 1.2 gives a relaxation of coloring, and this is the first result in this direction. In fact, since it is still not known if there exists a constant c such that any ck-chromatic graph has K_k as a minor, this may be viewed as the first step to attack this conjecture.

We also extend Theorem 1.2 to list colorings. First we recall some definitions. Let G be a graph and t a positive number. A *list-assignment* is a function L which assigns to every vertex $v \in V(G)$ a set $L(v)$ of natural numbers, which are called *admissible colors* for that vertex. An *L-coloring* is an assignment of admissible colors to all vertices of G, i.e., a function $c : V(G) \to \mathbb{N}$ such that $c(v) \in L(v)$ for every $v \in V(G)$, and for every edge uv we have $c(u) \neq c(v)$. If $|L(v)| \geq t$ for every $v \in V(G)$, then L is a *t-list-assignment*. The graph is *t-choosable* if it admits an L-coloring for every t-list-assignment L.

When relaxing the Hadwiger Conjecture to allow ck colors, the following conjecture involving list colorings may also be true:

Conjecture 1.3 *There is a constant c such that every graph without K_k minors is ck-choosable.*

Conjecture 1.1 does not hold for list colorings. For example, there exist planar graphs (without K_5 minors) which are not 4-choosable. However, Conjecture 1.3 is formulated in such a way that it may also be true for $c = 1$. We believe that Conjecture 1.3 holds with $c = \frac{3}{2}$.

In this paper we also extend Theorem 1.2 to the setting of list colorings.

Theorem 1.4 *Let k be an integer. There is a computable constant $f(k)$ such that for every graph G without K_k as a minor and for every $15.5k$-list-assignment L, there is a vertex partition $\{V_i \mid i \in \mathbb{N}\}$ of $V(G)$ such that for every i, $V_i \subseteq \{v \in V(G) \mid i \in L(v)\}$, and every component of the subgraph of G induced on V_i has at most $f(k)$ vertices.*

In fact, Theorem 1.4 is proved in Section 2 in a slightly more general form, where a small set of vertices is "precolored". See Theorem 2.1. Of course, Theorem 1.2 follows directly from Theorem 1.4 by taking $L(v) = \{1, 2, \ldots, \lceil 15.5k \rceil\}$ for every vertex $v \in V(G)$.

In the proof of Theorem 1.4, we will use a corollary of the following result from [3].

Theorem 1.5 *For any integers k, s and t, there exists a computable constant $N_0(k,s,t)$ such that every $(3k+2)$-connected graph of minimum degree at least $15.5k$ and with at least $N_0(k,s,t)$ vertices either contains $K_{k,st}$ as a topological minor or a minor isomorphic to s disjoint copies of $K_{k,t}$.*

Let A and B be induced subgraphs of G such that $G = A \cup B$. If $V(A) \setminus V(B) \neq \emptyset$ and $V(B) \setminus V(A) \neq \emptyset$, then we say that the pair (A, B) is a *separation* of G. The *order* of this separation is equal to $|V(A \cap B)|$. Let $Z \subseteq V(G)$ be a vertex set. We say that the separation (A, B) of G is *Z-essential* if $(A - Z, B - Z)$ is a separation of $G - Z$. If l is a positive integer, we say that G is *l-connected relative to Z* if it has no Z-essential separations of order less than l.

We will need the following corollary of Theorem 1.5:

Theorem 1.6 *For any integers k and z, there exists a constant $N_1(k,z)$ such that for every graph G and every vertex set $Z \subseteq V(G)$ of cardinality at most z, if G is $(3k+2)$-connected relative to Z, the degree of every vertex in $V(G) \setminus Z$ is at least $15.5k$, and G has at least $N_1(k,z)$ vertices, then G contains the complete graph K_k as a minor.*

Proof. Let G and Z be as assumed in the statement of the theorem. Let Z' be the set of all vertices in Z whose degree is at most $3k + 1 + z$. Let D be a set of vertices in $G - Z$ of cardinality $3k + 2$ such that no vertex in Z' is adjacent to D. If $|V(G)| \geq (3k + 2 + z)z + 3k + 2$ (which we may assume), then D exists. Let G' be the graph obtained from G by adding all edges between Z' and D.

In G', every vertex in Z has at least $3k + 2$ neighbors that are not in Z. Since G is $(3k + 2)$-connected relative to Z and is a spanning subgraph of G', this implies that G' is $(3k + 2)$-connected. Suppose that $|V(G)| \geq N_0(k, s, t)$. By Theorem 1.5, G' either contains a subdivision of $K_{k,st}$ or a minor isomorphic to s disjoint copies of $K_{k,t}$. Let us take $s = z + 1$ and $t = 3k + 2 + z$. If G' has s copies of $K_{k,t}$ as a minor, then G' contains a $K_{k,k}$-minor (and hence also a K_k-minor) that is disjoint from Z. As for the other alternative, when G' contains a subgraph K which is a subdivision of $K_{k,st}$, none of the vertices of degree st in K belong to Z' since the vertices in Z' have degree less than $(3k+2+z)z+3k+2 < (3k+2+z)(z+1) = st$. Therefore, $G' - Z'$ contains a subgraph which is a subdivision of $K_{k,st-z}$. Since $st - z \geq k$, G has $K_{k,k}$ and hence also K_k as a minor. So, the theorem holds for the value $N_1(k, z) = N_0(k, z + 1, 3k + 2 + z)$. □

2 Proof of Theorem 1.4

In this section we fix a positive integer k and a number $\tau = \tau(k) > 6k+1$ for which there exists a constant $N = N(k,\tau)$ such that for every graph G and a vertex set $Z \subseteq V(G)$ of cardinality at most $6k+1$, if G is $2k$-connected relative to Z, every vertex in $V(G) \setminus Z$ has degree at least τ, and $|V(G)| \geq N$, then G contains K_k as a minor. According to Theorem 1.6, we can take $\tau = 15.5k$ and take as $N(k,\tau)$ the value $N_1(k, 6k+1)$ from Theorem 1.6.

The proof of Theorem 1.4 is by induction on $|V(G)|$. For the induction purpose, we shall prove the following stronger statement:

Theorem 2.1 *Let k, $\tau = \tau(k)$ and $N(k,\tau)$ be as above. Let $f(k)$ be the maximum of $N(k,\tau(k))$ and $\tau(k)$. Suppose that G is a graph without K_k as a minor, L is a τ-list-assignment, $Z \subseteq V(G)$ is a vertex set with $|Z| \leq 6k+1$, and $c : Z \to \mathbb{N}$ is a mapping such that $c(z) \in L(z)$ for every $z \in Z$. Then c can be extended to a mapping $c_0 : V(G) \to \mathbb{N}$ with the following properties:*

(a) *For every $v \in V(G)$, $c_0(v) \in L(v)$.*

(b) *For every $i \in \mathbb{N}$, the subgraph G_i of G induced on $V_i = c_0^{-1}(i)$ has only components whose order is smaller than $f(k)$.*

(c) *If a vertex $v \in V(G) - Z$ is adjacent to a vertex $z \in Z$, then $c_0(v) \neq c(z)$.*

Proof. Throughout the proof, the mapping c_0 will be called a *coloring*, the mapping c a *precoloring*, and the set Z will be referred to as the *precolored set*. We also consider the corresponding *color classes* V_i. All these terms refer to G or to its minors on which the induction hypothesis will be applied.

We prove this statement by induction on $|V(G)|$. If $V(G) = Z$, there is nothing to prove. We claim that for any vertex $v \in V(G-Z)$, degree of v is at least τ. Suppose $G-Z$ has a vertex v of degree at most $\tau - 1$. Then, by the induction hypothesis, $G - v$ has a desired coloring, and since v has degree at most $\tau - 1$, we can set $c_0(v) = i$, where $i \in L(v)$ is such that v has no neighbors in V_i. So, we may assume that every vertex $v \in V(G-Z)$ has degree at least τ. In particular, $|V(G)| > \tau$.

Next, we claim that G is $(3k+2)$-connected relative to Z. Suppose that there is a Z-essential separation (A,B) of order at most $3k+1$. We assume that (A,B) is a minimal Z-essential separation, and we let $S = A \cap B$. Note that the minimum degree of G is at least τ. Since $|S| \leq 3k+1$ and $|Z| \leq 6k+1$, it follows that either $|S \cup (A \cap Z)| \leq 6k+1$ or $|S \cup (B \cap Z)| \leq$

$6k+1$, say $|S \cup (A \cap Z)| \le 6k+1$. Then we first apply induction to the subgraph of G induced on $B \cup Z$ with Z precolored. Since (A, B) is a Z-essential separation, the subgraph on $B \cup Z$ is smaller than G, and hence the induction hypothesis can be applied.

Let $S' = S - Z$. Then, after coloring B, each vertex $s \in S'$ has an assignment $c(s)$. Now, we apply induction to A with $Z' = S \cup (A \cap Z)$ precolored. Recall that $|Z'| \le 6k+1$. Finally, the combination of the obtained colorings of B and A yields a coloring c_0 of G. Every vertex in S satisfies requirement (c) under the coloring of A. Therefore, every component of some V_i is either contained in $B \cup Z$ (and is also a component of the coloring of $B \cup Z$), or is contained in $A - (B \cup Z)$. This shows that the coloring c_0 of G satisfies (b). Conditions (a) and (c) hold for inductively obtained colorings, so they also hold for c_0.

To conclude, we may now assume that G is $(3k+2)$-connected relative to Z. Since G has minimum degree at least τ and since G has at least $f(k) \ge N(k, \tau)$ vertices, G contains K_k as a minor. This contradiction completes the proof. $\square$

To conclude, let us observe that there is some room for improvement. Certainly, the function N_0 from [3] in Theorem 1.5 which is used to define the constant $f(k)$ can be considerably improved (but not to anything small). Also the $15.5k$ bound can be improved slightly by improving parts of the proof in [3]. However, new methods would be needed to go below $10k$.

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