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AN ANALOGUE OF THE DESCARTES-EULER FORMULA FOR INFINITE GRAPHS AND HIGUCHI'S CONJECTURE

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An analogue of the Descartes-Euler formula for infinite graphs and Higuchi's conjecture

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Abstract

Let $\mathcal{R}$ be a connected 2-manifold without boundary obtained from a (possibly infinite) collection of polygons by identifying them along edges of equal length. Let V be the set of vertices, and for every $v \in V$, let $\kappa(v)$ denote the (Gaussian) curvature of v: 2π minus the sum of incident polygon angles. Descartes showed that $\sum_{v\in V} \kappa(v) = 4\pi$ whenever $\mathcal{R}$ may be realized as the surface of a convex polytope in $\mathbb{R}^3$. More generally, if $\mathcal{R}$ is made of finitely many polygons, Euler's formula is equivalent to the equation $\sum_{v\in V} \kappa(v) = 2\pi\chi(\mathcal{R})$ where $\chi(\mathcal{R})$ is the Euler characteristic of $\mathcal{R}$. Our main theorem shows that whenever $\sum_{v\in V:\kappa(v)<0} \kappa(v)$ converges and there is a positive lower bound on the distance between any pair of vertices in $\mathcal{R}$, there exists a compact closed 2-manifold $\mathcal{S}$ and an integer t so that $\mathcal{R}$ is homeomorphic to $\mathcal{S}$ minus t points, and further $\sum_{v\in V} \kappa(v) \leq 2\pi\chi(\mathcal{S}) - 2\pi t$.

In the special case when every polygon is regular of side length one and $\kappa(v) > 0$ for every vertex v, we apply our main theorem to deduce that $\mathcal{R}$ is made of finitely many polygons and is homeomorphic to either the 2-sphere or to the projective plane. Further, we show that unless $\mathcal{R}$ is a prism, antiprism, or the projective planar analogue of one of these that $|V| \leq 3444$. This resolves a recent conjecture of Higuchi.

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1 Introduction

A *polygonal surface* $\mathcal{R}$ is a simply connected 2-manifold without boundary which is obtained from a (possibly infinite) collection of disjoint simple polygons in $\mathbb{R}^2$ by identifying them along edges of equal length. Based on this construction, $\mathcal{R}$ may be viewed as an embedded graph, and accordingly, we equip it with three distinguished sets: *vertices*, *edges*, and *faces*, respectively denoted $V(\mathcal{R})$, $E(\mathcal{R})$, and $F(\mathcal{R})$, and defined to be the set of all subsets of $\mathcal{R}$ which correspond (respectively) to a vertex of a polygon, edge of a polygon, or polygon itself. We view vertices, edges, and faces both combinatorially and as subsets of $\mathcal{R}$; two such objects are defined to be *incident* if one is a proper subset of the other. The space $\mathcal{R}$ is also equipped with a natural intrinsic metric: the distance between two points is the length of the shortest rectifiable curve joining them. This metric gives each point $x \in \mathcal{R}$ a Gaussian curvature which we denote by $\kappa(x)$. If x is not a vertex, then $\kappa(x) = 0$. If x is a vertex, then $\kappa(x)$ is equal to 2π minus the sum over all faces f incident with x of the angle of f at x (if a face is incident with v more than once, then all incident angles count). Descartes (see, e.g., [2] proved the following important theorem regarding the curvature of vertices in a 3-polytope.

Theorem 1.1 (Descartes) *If $\mathcal{R}$ is a polygonal surface which may be realized as the surface of a convex polytope in $\mathbb{R}^3$, then*

$$\sum_{v \in V(\mathcal{R})} \kappa(v) = 4\pi.$$

We define the polygonal surface $\mathcal{R}$ to be *finite* if $F(\mathcal{R})$ is finite. For finite polygonal surfaces, the Gauss-Bonnet formula gives us a curvature identity which is a generalization of Descartes theorem. In fact, this generalization is equivalent to Euler's formula and may be stated as follows (here we let $\chi(\mathcal{S})$ denote the Euler characteristic of a surface $\mathcal{S}$).

Theorem 1.2 (Euler-Gauss-Bonnet formula) *If $\mathcal{R}$ is a finite polygonal surface, then*

$$\sum_{v \in V(\mathcal{R})} \kappa(v) = 2\pi\chi(\mathcal{R}).$$

Our main theorem, stated below, gives a related formula which holds for infinite polygonal metric spaces. For that, a technical condition, which assures that the metric space $\mathcal{R}$ is complete, is needed. We define $\nu = \nu(\mathcal{R})$

to be the infimum of all distances between distinct vertices, and we say that $\mathcal{R}$ is *tame* if $\nu > 0$.

Theorem 1.3 *If $\mathcal{R}$ is a tame polygonal surface and $\sum_{v \in V(\mathcal{R}):\kappa(v)<0} \kappa(v)$ converges, then there exists a compact closed 2-manifold $\mathcal{S}$ and an integer t so that $\mathcal{R}$ is homeomorphic to $\mathcal{S}$ minus t points, and further*

$$\sum_{v \in V(\mathcal{R})} \kappa(v) \leq 2\pi\chi(\mathcal{S}) - 2\pi t.$$

For instance, if $\kappa(v)$ is everywhere non-negative, and takes a positive value on at least one vertex, then the above theorem implies that either $\mathcal{R}$ is homeomorphic to the plane, or $\mathcal{R}$ is finite and homeomorphic to either the sphere or the projective plane.

A relatively simple example shows that the above theorem does not hold without the assumption that $\mathcal{R}$ is tame. Let $\mathcal{S}$ be the surface obtained from the unit sphere S^2 in $\mathbb{R}^3$ by removing an infinite set C consisting of a sequence of points together with their (unique) accumulation point. (More generally, one can take as C any infinite set that is homeomorphic to a closed subset of the Cantor.) If $\mathcal{T}$ is a triangulation of $\mathcal{S}$ with all edges being geodesic segments in S^2, then $\mathcal{T}$ gives rise to a polygonal surface $\mathcal{T}'$ whose vertices are the vertices of $\mathcal{T}$ and edges are the corresponding straight line segments in $\mathbb{R}^3$. The Gaussian curvature at every vertex is positive. However, the inequality of Theorem 1.3 cannot hold since $t = |C|$ is infinite.

An interesting special case is when each face of $\mathcal{R}$ is a regular polygon of side length one. In this case, the space $\mathcal{R}$ may be described by an embedded graph, and several authors have used this connection to define a combinatorial notion of curvature for an embedded graph. We offer a definition below which is identical to one made by Stone [8, 9], Woess [11] and Higuchi [4], and is equivalent to one given by Gromov [3].

For any graph G, let $deg(v)$ denote the degree of a vertex $v \in V(G)$. If G is embedded in a surface, and f is a face, we let $deg(f)$ denote the size of f – which we shall also call its *degree*. We define the *combinatorial curvature* of the vertex $v \in V(G)$ to be

$$\phi(v) = 1 - \frac{deg(v)}{2} + \sum_{f \sim v} \frac{1}{deg(f)}$$

where the sum is taken over all faces f incident with v, counted according to multiplicity of incidences. As suggested above, the motivation for this

definition is that replacing each face of G by a regular polygon of side length one yields a polygonal surface for which $\kappa(v) = 2\pi\phi(v)$ for every vertex v. By way of this correspondence, our main theorem has a natural corollary for combinatorial curvature which we offer after introducing some added terminology.

A one-way infinite path in a graph G is called a *ray*. Two rays A and B are *equivalent* if for every finite set X of vertices of G, the (unique) component of $G - X$ containing an infinite number of vertices in A is also the (unique) component of $G-X$ containing an infinite number of vertices in B. It is straightforward to verify that this relation is an equivalence relation. Equivalence classes of rays in G are called *ends*, and we let $Ends(G)$ denote the set of ends of the graph G.

Corollary 1.4 *Let G be a graph that is 2-cell embedded in the surface $\mathcal{S}$ so that each face has finite size and let $t = |Ends(G)|$. If $\sum_{v\in V(G):\phi(v)<0} \phi(v)$ is bounded, then there exists a compact closed 2-manifold $\mathcal{T}$ so that $\mathcal{S}$ is homeomorphic to $\mathcal{T}$ minus t points, and further*

$$\sum_{v\in V(G)} \phi(v) \leq \chi(\mathcal{T}) - t.$$

If all faces of an embedded graph have the same size, the combinatorial curvature of a vertex may be expressed in terms of its degree. Such families provide a natural setting for the above result. For instance, applying this result to triangulations yields the following.

Corollary 1.5 *Let G be a graph which is a triangulation of the surface $\mathcal{S}$ and let $t = |Ends(G)|$. If G has only finitely many vertices of degree at least 7, then there exists a compact closed 2-manifold $\mathcal{T}$ so that $\mathcal{S}$ is homeomorphic to $\mathcal{T}$ minus t points, and further*

$$\sum_{v\in V(G)} (6 - deg(v)) \leq 6\chi(\mathcal{T}) - 6t.$$

It is fairly easy to find families of triangulations for which the above bound is tight. Begin with a triangulation G of a compact closed 2-manifold and a facial triangle f with vertices x_1, x_2, x_3 (and note that G is tight with respect to the above inequality). Next, draw an infinite sequence of nested triangles in f converging to a point z which we remove from the surface, and add edges between these nested triangles so that all newly created vertices are of degree six and so that the sum of the degrees of x_1, x_2, and x_3

increases by six. This newly formed graph is tight with respect to the above inequality, and by repeating this construction, we may form tight instances for any surface and any number of ends of the graph.

In a recent paper, Higuchi made a conjecture equivalent to the one given below concerning graphs with everywhere positive combinatorial curvature.

Conjecture 1.6 (Higuchi) *Let G be a graph which is 2-cell embedded in a surface $\mathcal{S}$ so that every vertex and face has degree ≥ 3. If $\mathcal{S}$ is homeomorphic to a subset of the 2-sphere and ϕ is everywhere positive, then G is finite.*

A special case of this conjecture when G is 3-regular was recently resolved by Sun and Yu [10]. Using Theorem 1.3 we prove the following result, which resolves Higuchi's conjecture in full.

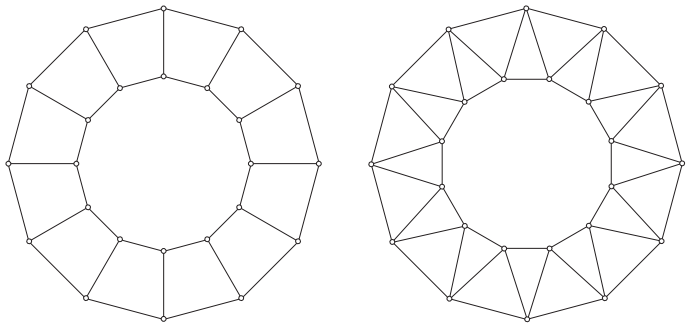

Figure 1: Examples of a prism and an antiprism

Theorem 1.7 *Let G be a graph which is 2-cell embedded in a surface $\mathcal{S}$ so that every vertex and face has degree ≥ 3. If ϕ is everywhere positive, then $\mathcal{S}$ is homeomorphic to either the 2-sphere or the projective plane and G is finite. Furthermore, if G is not a prism, antiprism, or the projective planar analogue of one of these, then $|V(G)| \leq 3444$.*

Examples of a prism and an antiprism (planar and projective planar) are shown in Figures 1 and 2. The projective planar prisms are also known as *Möbius ladder graphs*.

It would be interesting to find the smallest number C which could take the place of 3444 in the above theorem. The best lower bound we know of comes from the great rhombic icosidodecahedron. This polyhedron has 120 vertices, each of which is incident with one square, one hexagon, and one

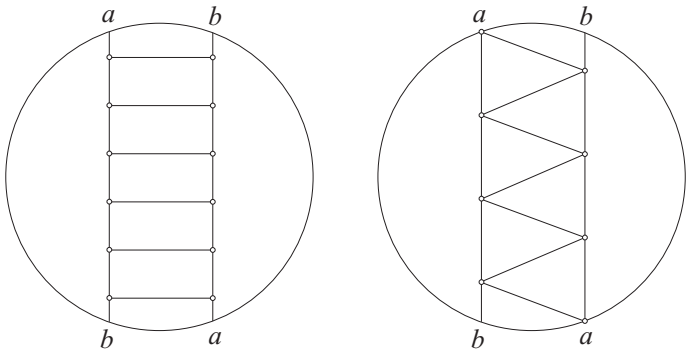

Figure 2: Projective planar prism and antiprism

decagon. Thus $120 \leq C \leq 3444$. Although little effort has been made here to optimize our upper bound, it appears that a considerably more detailed analysis would be required to get close to the correct value for C.

Although all of our results concern polygonal metric spaces, many of these results can be extended to more general spaces by way of approximation. As noted by Higuchi, Theorem 1.7 can be viewed as a discrete analogue of a result of Myers [7] who proved that every complete Riemannian manifold with Ricci curvature bounded below by a positive constant κ_0 is compact, has volume bounded in terms of κ_0 and has finite fundamental group. On the other hand, Theorem 1.3 shows other possible directions for improvements of such results. One such extension may be a conjecture of Milnor [5] that every complete Riemannian manifold with non-negative Ricci curvature has finitely generated fundamental group.

2 Preliminaries

In this section we shall introduce the main terms and objects we will use in the remainder of the paper. Many of these terms are standard in the context of polygonal surfaces. For a good introduction to this subject, we recommend [1].

In a polygonal metric space $\mathcal{R}$, the distance function gives rise to a length function ℓ which assigns length to any rectifiable curve in $\mathcal{R}$. A *geodesic* is an arc $\alpha \subseteq \mathcal{R}$ which is locally length minimizing. We think of a geodesic as having no direction, and if $x, y \in \mathcal{R}$ are the ends of α, then we say that α *joins* x and y. A geodesic is *nice* if both of its ends are vertices and no

interior point of α is a vertex.

The distance function also gives rise to a concept of angle. If x is a point in the boundary of the set $\mathcal{T} \subseteq \mathcal{R}$ and there exists a neighborhood N of x so that the intersection of the boundary of $\mathcal{T}$ and N is a rectifiable curve containing x, then we call $\mathcal{T}$ a *half-neighborhood* of x and we let $\angle(\mathcal{T}, x)$ denote the angle of $\mathcal{T}$ at x. The *inner boundary curvature* of $\mathcal{T}$ at x is defined to be $\iota(\mathcal{T}, x) = \pi - \angle(\mathcal{T}, x)$. If α is a curve containing x in its interior, and there is a neighborhood N of x so that α and the boundary of $\mathcal{T}$ coincide inside N, then we call $\mathcal{T}$ a *half-neighborhood* of x *along* α. Notice that if $\mathcal{T}_1$ and $\mathcal{T}_2$ are disjoint half-neighborhoods of x along α, then $\iota(\mathcal{T}_1, x) + \iota(\mathcal{T}_2, x) = \kappa(x)$. The following useful observation follows from definitions.

Observation 2.1 *Let α be a simple rectifiable curve in a polygonal surface $\mathcal{R}$. Then the following statements are equivalent:*

(i) *α is a geodesic.*

(ii) *For every x in the interior of α and every half-neighborhood $\mathcal{T}$ of x along α, we have $\angle(\mathcal{T}, x) \geq \pi$.*

(iii) *For every face f, every subarc of α contained in f is a line segment. For every point x in the interior of α which is not in the interior of a face and every half-neighborhood $\mathcal{T}$ of x along α, we have $\angle(\mathcal{T}, x) = \pi$ if x is in the interior of an edge and $\angle(\mathcal{T}, x) \geq \pi$ if x is a vertex.*

In particular, the above observation shows that every edge is a geodesic.

A *geodesic walk* W is a sequence $x_0, \alpha_1, x_1, \alpha_2, \ldots, x_{k-1}, \alpha_k, x_k$ where every x_i is a point in $\mathcal{R}$, every α_i is a geodesic with ends x_{i-1} and x_i, and the points x_{i-1} and x_i are distinct whenever $1 \leq i \leq k$. Every x_i is called a *corner* of W, and the sequence $x_0, x_1, \ldots, x_k$ is called the *sequence of corners*. We say that W is *closed* if $x_0 = x_k$, and we say that W is *simple* if every α_i is simple, any two geodesics α_i and α_j $(i \neq j)$ intersect only at their ends, and either W is closed and $x_0, \ldots, x_{k-1}$ are distinct, or W is not closed and $x_0, \ldots, x_k$ are distinct. We say that W is *nice* if every geodesic α_i is nice. The *length* of W is $\ell(W) = \sum_{i=1}^{k} \ell(\alpha_i)$ and the *combinatorial length* of W is $\lambda(W) = k$. Note that the combinatorial length is in general unrelated to the (ordinary) length. For convenience of notation, we shall associate the walk W with the subset $\cup_{i=1}^{k} \alpha_i \subseteq \mathcal{R}$.

For any subset $\mathcal{S} \subseteq \mathcal{R}$, we let $V^i(\mathcal{S})$ denote the set of vertices in the interior of $\mathcal{S}$ and $V^b(\mathcal{S})$ the set of vertices in the boundary of $\mathcal{S}$. The following basic observation is quite useful, since it permits us to reason about certain triangles using plane geometry.

Observation 2.2 *Let $x_0, \alpha_1, x_1, \alpha_2, x_2, \alpha_3, x_0$ be a nice closed geodesic walk which bounds a disc D with $V^i(D) = \emptyset$. Then there exist three points A, B, C in $\mathbb{R}^2$ so that the triangle ABC is congruent to D (i.e. $\overline{AB} = \ell(\alpha_1)$, $\angle CAB = \angle(D, x_1)$, etc.).*

Next we introduce an object which will be the main subject for much of the rest of this article. If (s, t, u) is a triple of nonnegative integers, then an (s, t, u)-*front* is a subset $\mathcal{Q} \subseteq \mathcal{R}$ with the following properties:

(i) There exists a surface $\mathcal{S}$ of Euler characteristic s so that $\mathcal{Q}$ is homeomorphic to the space obtained from $\mathcal{S}$ by removing t disjoint open discs. (In particular, $\mathcal{Q}$ is closed.)

(ii) The boundary of $\mathcal{Q}$ can be written as a union of nice simple closed geodesic walks $W_1, \ldots, W_t$, called *boundary walks*, so that the intersection of any two such walks is a subset of $V(\mathcal{R})$.

(iii) There are exactly u path-connected components of $\mathcal{R} \setminus \mathcal{Q}$.

(iv) The set $V^i(\mathcal{Q})$ is finite.

Extending some earlier definitions, we define the *length* of a front $\mathcal{Q}$ to be $\ell(\mathcal{Q}) = \sum_{i=1}^{t} \ell(W_i)$ and the *combinatorial length* of $\mathcal{Q}$ to be $\lambda(\mathcal{Q}) = \sum_{i=1}^{t} \lambda(W_i)$. For $x \in V^b(\mathcal{Q})$, let $J = \{j \mid 1 \leq j \leq k, x \in W_j\}$, and for every $j \in J$, let $\mathcal{T}_j$ be an open half-neighborhood of x along W_i which is disjoint from $\mathcal{Q}$. We define the *total angle of* $\mathcal{Q}$ *at* x to be $\angle(\mathcal{Q}, x) = 2\pi - \kappa(x) - \sum_{j \in J} \angle(\mathcal{T}_j, x)$, and we define the *inner boundary curvature* of $\mathcal{Q}$ at x to be $\iota(\mathcal{Q}, x) = \pi(2 - |J|) - \angle(\mathcal{Q}, x) = \kappa(x) - \sum_{j \in J} \iota(\mathcal{T}_j, x)$. The following useful identity follows from these definitions:

$$\sum_{x \in V^b(\mathcal{Q})} \iota(\mathcal{Q}, x) = 2\pi |V^b(\mathcal{Q})| - \pi\lambda(\mathcal{Q}) - \sum_{x \in V^b(\mathcal{Q})} \angle(\mathcal{Q}, x).$$

3 The Gauss-Bonnet formula

The Gauss-Bonnet formula relates the Gaussian curvature in the interior of a surface to the curvature of the boundary. We require a version of this result for the special case of polygonal metric spaces, which we will state in terms of fronts. In this special case, the Gauss-Bonnet formula may be deduced from Euler's formula, and for the sake of completeness, we include its derivation.

Theorem 3.1 (Gauss-Bonnet) *If $\mathcal{Q}$ is an (s,t,u)-front in $\mathcal{R}$, then*

$$\sum_{x\in V^i(\mathcal{Q})} \kappa(x) + \sum_{y\in V^b(\mathcal{Q})} \iota(\mathcal{Q},y) = 2\pi s - 2\pi t.$$

Proof. We shall modify our polygonal surface $\mathcal{R}$ to form a new polygonal surface $\mathcal{R}'$ by introducing new edges and vertices (and thus faces) without changing the structure of the metric space. First, at each point x in the boundary of $\mathcal{Q}$ through which an edge passes from $\mathcal{Q}$ to $\mathcal{R}\setminus\mathcal{Q}$, subdivide this edge by placing a new vertex at x. Then, whenever x and y are consecutive vertices on the boundary of $\mathcal{Q}$, we add an edge from x to y along this stretch of the boundary if such an edge does not already exist. Note that only finitely many new vertices are added in this process, since there are only finitely many vertices in $\mathcal{Q}$, each of finite degree. Now, $\mathcal{Q}$ is a front of $\mathcal{R}'$ with the special property that all points in the boundary of $\mathcal{Q}$ lie in $V(\mathcal{R}')\cup E(\mathcal{R}')$. Furthermore, every new vertex x introduced on the boundary of $\mathcal{Q}$ has $\iota(\mathcal{Q},x)=0$, so to prove the above result, it suffices to prove it for $\mathcal{Q}$ considered as a front in the space $\mathcal{R}'$. Let H be the embedded graph consisting of all vertices and edges of $\mathcal{R}'$ which lie in $\mathcal{Q}$. By construction, H is 2-cell embedded in $\mathcal{Q}$ and each boundary component of $\mathcal{Q}$ is bounded by a cycle of H. The following equation relies on the identity at the close of the previous section, the observation that the sum of the interior angles of a plane k-gon is $\pi(k-2)$, and Euler's formula applied to the graph H:

$$\begin{aligned}\sum_{x\in V^i(\mathcal{Q})} \kappa(x) + \sum_{y\in V^b(\mathcal{Q})} \iota(\mathcal{Q},y) &= 2\pi|V(H)| - \pi\lambda(\mathcal{Q}) - \pi\sum_{f\in F(H)}(deg(f)-2)\\ &= 2\pi(|V(H)|-|E(H)|+|F(H)|)\\ &= 2\pi s - 2\pi t.\end{aligned}$$

This completes the proof. □

The Descartes-Euler formula may be viewed as a corollary of this theorem applied to the front $\mathcal{Q}=\mathcal{R}$.

Let $\mathcal{Q}$ be an (s,t,u)-front with $t>0$. We define the value $\Theta(\mathcal{Q})$ by the following rule:

$$\Theta(\mathcal{Q}) = \frac{\nu}{\ell(\mathcal{Q})}\Big(\sum_{x\in V^i(\mathcal{Q})}\kappa(x) - 2\pi s + 2\pi u + \sum_{y\in V(\mathcal{R})\setminus V^i(Q):\kappa(y)<0}\kappa(y)\Big)$$

where ν is the infimum of all distances among distinct vertices of $\mathcal{R}$. The motivation for this definition is the following corollary, which is all we shall require from this section in the sequel.

Corollary 3.2 *If $\mathcal{Q}$ is a nonempty front in $\mathcal{R}$ with $\Theta(\mathcal{Q}) \geq 0$, then there exists a boundary walk W of $\mathcal{Q}$, a vertex $x \in W$, and an open half-neighborhood $\mathcal{T}$ of x along W which is disjoint from $\mathcal{Q}$ so that $\iota(\mathcal{T}, x) \geq \Theta(\mathcal{Q})$.*

Proof. We shall assume that $\mathcal{Q}$ is an (s,t,u)-front. Let $W_1, \ldots, W_t$ be the boundary walks of $\mathcal{Q}$ and for $1 \leq i \leq t$, let $\mathcal{T}_i$ be an open subset of $\mathcal{R} \setminus \mathcal{Q}$ which is a half-neighborhood of every vertex in W_i along W_i. For convenience, we will further assume that $V^b(\mathcal{T}_i)$ is precisely the vertex set of W_i. This brings us to the following inequality (in which the second line follows from the Gauss-Bonnet formula and the observation that $t \geq u$):

$$\begin{aligned}\sum_{i=1}^{t} \sum_{x \in V^b(\mathcal{T}_i)} \iota(\mathcal{T}_i, x) &= \sum_{x \in V^b(\mathcal{Q})} (\kappa(x) - \iota(\mathcal{Q}, x)) \\ &\geq \tfrac{\ell(\mathcal{Q})}{\nu} \Theta(\mathcal{Q}) \\ &\geq \lambda(\mathcal{Q})\Theta(\mathcal{Q}).\end{aligned}$$

Since there are exactly $\lambda(\mathcal{Q})$ pairs $(x, \mathcal{T}_i)$ with $x \in V^b(\mathcal{T}_i)$, it follows by averaging that we may choose a pair with $\iota(\mathcal{T}_i, x) \geq \Theta(\mathcal{Q})$, as desired. □

4 Some Plane Geometry

In this section we prove an easy but essential inequality from plane geometry which will be used later.

Lemma 4.1 *Let X, Y, Z, W be points in the plane, and assume that W lies in the triangle XYZ and that W is at distance at least d from Y. Let A, B, C, E, F be the respective lengths of the line segments XY, YZ, XZ, XW, WZ, and let θ be the angle XYZ. If $A, B \geq d$, then*

(a) $(A + B) - (E + F) \geq \frac{d}{2}(\cos\theta + 1)$.

(b) $A + B - C \geq \frac{d}{2}(\cos\theta + 1)$.

Proof. Let X' and Z' be the points at distance d from Y which are, respectively, on the line segments YX and YZ. Now, the line through X' and Z' separates Y and W, so we may choose points X'' on $X'X$ and Z'' on $Z'Z$ so that W lies on the line segment $X''Z''$. Let A'', B'', R_1, R_2 denote

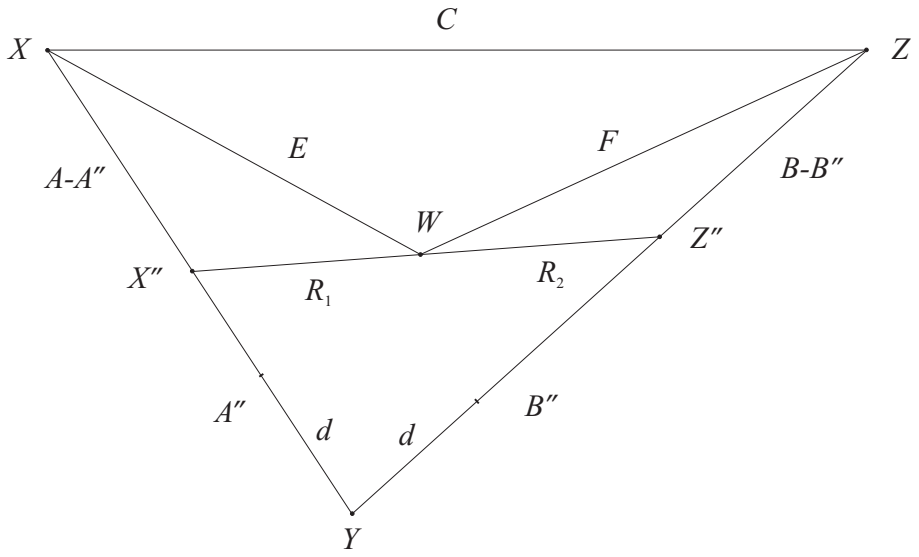

Figure 3: A plane geometry figure

the respective lengths of the line segments YX'', YZ'', $X''W$, and $Z''W$. We also let $R = R_1 + R_2$.

By the law of cosines in the triangle $X''YZ''$ we have $R^2 = A''^2 + B''^2 - 2A''B''\cos\theta$, so $(A''+B'')^2 - R^2 = 2A''B''(\cos\theta+1)$. Now, assuming (without loss of generality) that $A'' \geq B''$, we have

$$\begin{aligned} A'' + B'' - R &= \frac{2A''B''}{A'' + B'' + R}(\cos\theta + 1) \\ &\geq \frac{A''B''}{A'' + B''}(\cos\theta + 1) \\ &\geq \frac{B''}{2}(\cos\theta + 1) \\ &\geq \frac{d}{2}(\cos\theta + 1). \end{aligned}$$

Now, by the triangle inequality and the above, we have

$$\begin{aligned} A + B &= (A'' + B'' - R_1 - R_2) + (A - A'' + R_1) + (B - B'' + R_2) \\ &\geq \frac{d}{2}(\cos\theta + 1) + E + F \end{aligned}$$

which yields (a).

The inequality (b) is just a special case of (a) when $W = X$. This completes the proof. □

5 The Main Theorem

The main theorem requires a few preliminary lemmas. Our first observation and first lemma are technical arguments which show the existence of geodesic walks with useful properties. Throughout this section we shall assume that $\mathcal{R}$ is a tame polygonal surface. If W is a geodesic or a simple geodesic walk and x, y are points in W, then we let $W[x, y]$ denote the walk or geodesic obtained by restricting W to the interval between x and y.

Observation 5.1 *Let W be a geodesic walk in $\mathcal{R}$ given by $u, \alpha_1, v, \alpha_2, w$, let $\mathcal{T}$ be a half-neighborhood of v along W, and assume that $\angle(\mathcal{T}, v) < \pi$. Then there exists a point q on α_1 and a geodesic walk W' from w to q so that the concatenation of $q, \alpha_1[q, v], v, \alpha_2, w$ and W' bounds a disc $\mathcal{D}$ with the following properties:*

(1) *$\mathcal{D}$ has no vertices in its interior.*

(2) *There exists a half-neighborhood $\mathcal{T}'$ of v with $\mathcal{T}' \subseteq \mathcal{T} \cap \mathcal{D}$.*

(3) *$\iota(\mathcal{D}, z) = 0$ for every z in the interior of W'.*

(4) *Either $q = u$ or there exists a vertex in the interior of W'.*

Sketch of proof: Say that a point t in α_1 is *good* if there exists a geodesic walk W' from w to t so that the concatenation of $t, \alpha_1[t, v], v, \alpha_2, w$ and W' bounds a disc $\mathcal{D}$ which satisfies properties (1)–(3) above. It follows from the assumption that $\mathcal{R}$ is tame and Observation 2.1 that there exists a good point. Further, the set of all good points together with v form a geodesic of the form $\alpha_1[q, v]$ and the walk W' and disc $\mathcal{D}$ for the point q satisfy (1)–(4) above. □

Lemma 5.2 *Let W be a nice geodesic walk in $\mathcal{R}$ given by $u, \alpha_1, v, \alpha_2, w$, let $\mathcal{T}$ be a half-neighborhood of v along W, and assume that $\angle(\mathcal{T}, v) < \pi$. Then there exists a nice geodesic walk W' from w to u so that the concatenation of W and W' bounds a disc $\mathcal{D}$ which satisfies the following properties:*

(1) *$\iota(\mathcal{D}, x) \leq 0$ for every vertex x in the interior of W'.*

(2) *There exists a half-neighborhood $\mathcal{T}'$ of v with $\mathcal{T}' \subseteq \mathcal{T} \cap \mathcal{D}$.*

(3) $\ell(W') \leq \ell(W)$.

Proof. We shall form W' in stages by the following procedure. Initially, set $v_0 = v$, $w_0 = w$, $\beta_0 = \alpha_2$, and set W' to be the one vertex walk given by the sequence w. For step i, we apply Observation 5.1 to the walk $u, \alpha_1[u, v_{i-1}], v_{i-1}, \beta_{i-1}, w_{i-1}$ to choose a point v_i on $\alpha_1[u, v_{i-1}]$ and a walk W_i from w_{i-1} to v_i. If $v_i = u$, then update W' by concatenating it with W_i, and stop. If $v_i \neq u$, then let w_i be the vertex on W_i which is closest (along this walk) to v_i, set β_i to be the geodesic $W_i[w_i, v_i]$, update W' by concatenating it with $W_i[w_{i-1}, w_i]$, and pass to the next step. If our procedure does not terminate until after step i, then by the triangle inequality, we have

$$\begin{aligned} \ell(\alpha_1[u, v_i]) + \ell(\beta_i) + \nu &\leq \ell(\alpha_1[u, v_i]) + \ell(W_i) \\ &\leq \ell(\alpha_1[u, v_{i-1}]) + \ell(\beta_{i-1}) \end{aligned}$$

so it follows that our procedure always terminates. An easy induction shows that the resulting walk satisfies properties (1), (2), and (3) above. □

For any (positive) real number ζ and any walk W, we define the *ζ-weight* of W to be $w_\zeta(W) = \zeta^{-1}\ell(W) - \lambda(W)$. Similarly, define the *ζ-weight* of a front $\mathcal{Q}$ to be $w_\zeta(\mathcal{Q}) = \zeta^{-1}\ell(\mathcal{Q}) - \lambda(\mathcal{Q})$. We define the front $\mathcal{Q}$ to be *full* if every connected component of $\mathcal{R} \setminus \mathcal{Q}$ contains infinitely many vertices.

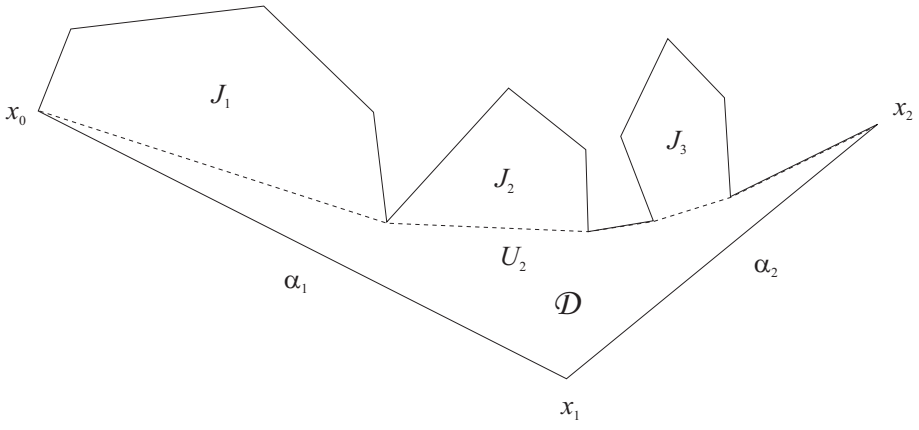

Figure 4: Change of the boundary walk W

Lemma 5.3 *If $\sum_{v \in V(\mathcal{R}):\kappa(v)<0} \kappa(v)$ converges and $V(\mathcal{R})$ is infinite, then $\Theta(\mathcal{Q}) \leq 0$ for every full front $\mathcal{Q}$.*

Proof. We suppose (for a contradiction) that $\mathcal{Q}_0$ is a full front and $\Theta(\mathcal{Q}_0) = \theta_0 > 0$. Set $\zeta = \frac{\nu}{4}(1 - \cos\theta_0)$. It follows from Corollary 3.2 that $\theta_0 < \pi$, so we have $0 < \zeta < \nu$. It follows from this that the ζ-weight of any front is nonnegative. Now, choose a full (s,t,u)-front $\mathcal{Q}$ with the following properties:

(1) $\Theta(\mathcal{Q}) \geq \theta_0$

(2) $\lceil w_\zeta(\mathcal{Q}) \rceil$ is minimum subject to (1).

It follows from Corollary 3.2 that we may choose a boundary walk W of $\mathcal{Q}$ with sequence $x_0, \alpha_1, x_1, \ldots, x_k$, and an open half-neighborhood $\mathcal{T}$ of x_1 along W so that $\mathcal{T} \cap \mathcal{Q} = \emptyset$ and $\iota(\mathcal{T}, x_1) \geq \Theta(\mathcal{Q}) \geq \theta_0$. Apply Lemma 5.2 to the walk $U_1 = x_0, \alpha_1, x_1, \alpha_2, x_2$ and the half-neighborhood $\mathcal{T}$ to choose a walk U_2 from x_2 to x_0, let $\mathcal{D}$ be the corresponding closed disc, and set $\mathcal{Q}' = \mathcal{Q} \cup \mathcal{D}$. (Figure 4 shows the described change of W. The geodesic walks J_1, J_2, J_3 are the boundary walks of Q' that will replace W.) It follows from (1) of Lemma 5.2 and the assumption that $\iota(\mathcal{T}, x_1) > 0$ that no point in the interior of $\mathcal{D}$ may appear in a boundary walk of $\mathcal{Q}$. It follows easily from this that $\mathcal{Q}'$ is also a front. Let $\mathcal{Q}''$ be the front obtained by taking the union of $\mathcal{Q}'$ with every connected component of $\mathcal{R} \setminus \mathcal{Q}'$ which has only finitely many vertices. It follows from (3) of Lemma 5.2 that $\ell(\mathcal{Q}'') \leq \ell(\mathcal{Q}') \leq \ell(\mathcal{Q})$. Further, if $\mathcal{Q}''$ is a (s'', t'', u'')-front, then $s'' \leq s$, and it follows from the assumption that $\mathcal{Q}$ is full that $u'' \geq u$. All of these conclusions imply that $\Theta(\mathcal{Q}'') \geq \Theta(\mathcal{Q}) \geq \theta_0$. The following equation also follows from this construction:

$$w_\zeta(\mathcal{Q}) - w_\zeta(\mathcal{Q}'') \geq w_\zeta(\mathcal{Q}) - w_\zeta(\mathcal{Q}') \geq w_\zeta(U_1) - w_\zeta(U_2).$$

If $\lambda(U_2) > \lambda(U_1)$ then by (3) of Lemma 5.2 we have $w_\zeta(U_1) - w_\zeta(U_2) \geq 1$ but this gives a contradiction to the choice of $\mathcal{Q}$. If $\lambda(U_2) \leq \lambda(U_1)$, then either $\lambda(U_2) = 2$ or $\lambda(U_2) = 1$. It follows from Observation 2.2 and part (a) or (b) of Lemma 4.1 that $w_\zeta(U_1) - w_\zeta(U_2) \geq \frac{\nu}{2\zeta}(\cos(\pi - \theta_0) + 1) - 1 = 1$ again giving us a contradiction to the choice of $\mathcal{Q}$. □

If G is a graph, $X \subseteq V(G)$ and A_1, A_2 are distinct ends of G, then we say that X *separates* A_1 and A_2 if there exist distinct components H_1, H_2 of $G \setminus X$ so that every ray in A_i has all but finitely many vertices in H_i for $i = 1, 2$. If $\mathcal{R}$ is a polygonal metric space, G is the graph $(V(\mathcal{R}), E(\mathcal{R}))$, $\mathcal{T} \subseteq \mathcal{R}$, and A is an end of G, then we say that $\mathcal{T}$ *captures* A if every ray in A has all but finitely many vertices in $\mathcal{T}$. Our next lemma shows the existence of fronts with certain useful properties.

Lemma 5.4 *Let $\mathcal{R}$ be a polygonal surface, let G be the graph $(V(\mathcal{R}), E(\mathcal{R}))$, let Y be a finite set of vertices, let $\mathcal{E}$ be a finite subset of $Ends(G)$, and let Ξ be a finite collection of closed curves in $\mathcal{R}$. Then there exists a full front $\mathcal{Q}$ with the following properties:*

(i) $Y \subseteq V^i(\mathcal{Q})$.

(ii) *Every closed curve in Ξ is contained in $\mathcal{Q}$.*

(iii) *Every end in $\mathcal{E}$ is captured by a component of $\mathcal{R} \setminus \mathcal{Q}$, and no two distinct ends from $\mathcal{E}$ are captured by the same component.*

Proof. It follows from the definition of ends that we may choose a finite set $Z \subseteq V(G)$ so that Z separates every pair of ends in $\mathcal{E}$. Let J denote the set of all faces which contain a point in $Y \cup Z$ or contain a point in some curve in Ξ. It follows from the fact that each closed curve is compact that J is finite. Since $\mathcal{R}$ is connected, we may choose a finite set of faces $J' \supseteq J$ so that $\mathcal{Q}' = \cup J'$ is connected. Now, by construction, $\mathcal{Q}'$ is a front which satisfies (i)–(iii). Finally, let $\mathcal{Q}$ be the union of $\mathcal{Q}'$ and all components of $\mathcal{R} \setminus \mathcal{Q}'$ that contain only finitely many vertices. Since $\mathcal{Q}'$ is the union of finitely many faces, $V^i(\mathcal{Q})$ is finite and $\mathcal{Q}$ is a full front satisfying (i)–(iii). □

Proof of Theorem 1.3. Let $\kappa^- = \sum_{v \in V(\mathcal{R}):\kappa(v)<0} \kappa(v)$ and let G be the graph $(V(\mathcal{R}), E(\mathcal{R}))$. If G has infinitely many ends, then choose a finite $\mathcal{E} \subseteq Ends(G)$ so that $\kappa^- - 4\pi + 2\pi|\mathcal{E}| > 0$. Now, by applying Lemma 5.4 to $\mathcal{E}$, $Y = \emptyset$ and $\Xi = \emptyset$ we get a front which contradicts Lemma 5.3. Thus, $Ends(G)$ is finite. Similarly, if the fundamental group of $\mathcal{R}$, hereafter denoted $\pi_1(\mathcal{R})$, is not finitely generated, then by choosing a sufficiently large subset Ξ of a minimal generating set of curves for $\pi_1(\mathcal{R})$ and applying Lemma 5.4 and then Lemma 5.3 we obtain a contradiction. Thus, we may choose a finite set Ξ of closed curves in $\mathcal{R}$ which generate $\pi_1(\mathcal{R})$. By a similar argument, we find that $\sum_{v \in V(\mathcal{R})} \kappa(v)$ converges.

Now, choose a finite set $Y \subseteq \{v \in V(\mathcal{R}) : \kappa(v) > 0\}$ so that

$$\left\lceil \frac{1}{2\pi} \sum_{v \in V(\mathcal{R})} \kappa(v) \right\rceil = \left\lceil \frac{1}{2\pi}\kappa^- + \frac{1}{2\pi} \sum_{y \in Y} \kappa(y) \right\rceil$$

and apply Lemma 5.4 to $\mathcal{E} = Ends(G)$, Ξ, and Y to obtain a full (s,t,u)-front $\mathcal{Q}$. Let $\mathcal{T}$ be a connected component of $\mathcal{R} \setminus \mathcal{Q}$. If there exist two boundary walks W_1, W_2 of $\mathcal{Q}$ which are both in the boundary of $\mathcal{T}$, we have

a contradiction to the assumption that Ξ generates $\pi_1(\mathcal{R})$. (In particular, it follows that $t = u$). Thus, there is a unique boundary walk W of $\mathcal{Q}$ which is in the boundary of $\mathcal{T}$. Modify $\mathcal{T}$ to form $\mathcal{T}'$ by identifying it with a closed disc along the boundary walk W. If $\mathcal{T}'$ is not homeomorphic to the plane, then we have again a contradiction to the assumption that Ξ generates $\pi_1(\mathcal{R})$. Thus, $\mathcal{T}$ is homeomorphic to $S^1 \times (0, 1)$ and it follows that $\mathcal{R}$ is homeomorphic to a space obtained from a compact closed 2-maninfold of Euler characteristic s by removing a set of t points. Furthermore, by Lemma 5.3 we have that $\sum_{y \in Y} \kappa(y) - 2\pi s + 2\pi t + \kappa^- \leq 0$, so it follows from our choice of Y that $\sum_{v \in V(\mathcal{R})} \kappa(v) - 2\pi s + 2\pi t \leq 0$ as desired. □

6 Higuchi's Conjecture

In this section we shall apply the key lemma from the previous section to prove Theorem 1.7 (which implies Higuchi's conjecture). We begin with some additional terminology. Let G be a plane graph with $\phi(v) > 0$ for every $v \in V(G)$. If $a_1, \ldots, a_k$ is a list of positive integers with $a_1 \leq \cdots \leq a_k$, then we say that a vertex v has *pattern* $(a_1, \ldots, a_k)$ if the faces incident with v may be ordered $f_1, \ldots, f_k$ so that f_i has size a_i for $1 \leq i \leq k$. The following observation is straightforward to verify.

Observation 6.1 *If v is a vertex of a 2-cell embedded graph G with degree at least 3 and $\phi(v) > 0$, then v has one of the patterns in Table 1.*

Motivated by this observation, we define a face to be *big* if it has size ≥ 42. The following lemma shows that distinct big faces cannot be too close, unless the graph is either a prism or an antiprism. Examples of graphs of a prism and an antiprism and their projective planar analogues are indicated in Figures 1 and 2.

Lemma 6.2 *Let G be a 2-cell embedded graph with minimum degree at least 3, all faces of finite size, and everywhere positive combinatorial curvature ϕ. If uv is an edge of G that is not incident with a big face but u and v are both incident with a big face, then G is either a prism, an antiprism, a projective prism or a projective antiprism.*

Proof. If u or v has pattern $(3, i, j)$ where $j \geq 42$, then the vertex of the facial triangle other than u, v lies on two big faces, which is impossible. Thus, the pattern $(3, i, j)$ for $j \geq 42$ is not possible at u and v. Consequently, according to Table 1, u, v either both have pattern $(4, 4, j)$ or both have

Pattern of v	Parameters	$\phi(v)$
$(3,3,3,3,j)$	$3 \leq j \leq 5$	$\frac{1}{j} - \frac{1}{6}$
$(3,3,3,j)$	$3 \leq j$	$\frac{1}{j}$
$(3,3,4,j)$	$4 \leq j \leq 11$	$\frac{1}{j} - \frac{1}{12}$
$(3,3,5,j)$	$5 \leq j \leq 7$	$\frac{1}{j} - \frac{2}{15}$
$(3,4,4,j)$	$4 \leq j \leq 5$	$\frac{1}{j} - \frac{1}{6}$
$(3,i,j)$	$3 \leq i \leq 6$ and $i \leq j$	$\frac{1}{i} + \frac{1}{j} - \frac{1}{6}$
$(3,7,j)$	$7 \leq j \leq 41$	$\frac{1}{j} - \frac{1}{42}$
$(3,8,j)$	$8 \leq j \leq 23$	$\frac{1}{j} - \frac{1}{24}$
$(3,9,j)$	$9 \leq j \leq 17$	$\frac{1}{j} - \frac{1}{18}$
$(3,10,j)$	$10 \leq j \leq 14$	$\frac{1}{j} - \frac{1}{15}$
$(3,11,j)$	$11 \leq j \leq 13$	$\frac{1}{j} - \frac{5}{66}$
$(4,4,j)$	$4 \leq j$	$\frac{1}{j}$
$(4,5,j)$	$5 \leq j \leq 19$	$\frac{1}{j} - \frac{1}{20}$
$(4,6,j)$	$6 \leq j \leq 11$	$\frac{1}{j} - \frac{1}{12}$
$(4,7,j)$	$7 \leq j \leq 9$	$\frac{1}{j} - \frac{3}{28}$
$(5,5,j)$	$5 \leq j \leq 9$	$\frac{1}{j} - \frac{1}{10}$
$(5,6,j)$	$6 \leq j \leq 7$	$\frac{1}{j} - \frac{2}{15}$

Table 1: Possible patterns in positively curved graphs

pattern $(3,3,3,j)$ for some $j \geq 42$. In the first case, choose u', v' so that $uu'v'vu$ is the facial walk of a 4-face. Then both u' and v' have pattern $(4,4,j)$ and we may choose u'', v'' distinct from u, v so that $u'u''v''v'u'$ is a facial walk. Since this process may be repeated ad infinitum, the graph G must be either a prism or a projective prism. Similarly, if u, v both have pattern $(3,3,3,j)$, then we may choose a vertex u' so that v, u, u' lie on a triangular face and choose v' so that v, v', u' lie on a triangular face. Next, we may choose u'' distinct from v so that u', v', u'' lie on a triangular face and choose v'' distinct from u' so that v', u'', v'' lie on a triangular face. As before, this process may be repeated ad infinitum, and it follows that G must be an antiprism or a projective antiprism. □

For any subset $X \subseteq V(G)$, we let $N(X)$ denote the set of vertices adjacent to a vertex in X. The following lemma gives a lower bound on the sum of $\phi(x)$ over a set of the form $X \cup N(X)$.

Lemma 6.3 *If G is neither ordinary nor projective prism or antiprism, and ϕ is everywhere positive, then $\sum_{x\in X\cup N(X)} \phi(x) \geq \frac{|X|}{1722}$ for every $X \subseteq V(G)$.*

Proof. Call a vertex *bad* if it is incident with a big face, *good* otherwise. Starting with the weight function ϕ, we create a new function ϕ' by the following discharging rule. If v is a good vertex of degree d, send $\frac{\phi(v)}{d+1}$ to each bad neighbor of v. Suppose that v is a good vertex which is adjacent to a bad vertex. Then v must either be incident with two squares, or with one triangle and another face of size ≤ 6. An easy check of Table 1 shows that this implies $\phi(v) \geq \frac{1}{132}$. Therefore, $\phi'(v) \geq \frac{1}{132(d+1)} \geq \frac{1}{792}$. The same inequality holds for the modified curvature of every bad vertex in X since it receives such a value from some good neighbor by Lemma 6.2. It follows from this that $\phi'(w) \geq \frac{1}{1722}$ for every vertex w. Thus, $\sum_{x\in X\cup N(X)} \phi(x) \geq \sum_{x\in X} \phi'(x) \geq \frac{|X|}{1722}$ as desired. □

With this, we are ready to prove the theorem.

Proof of Theorem 1.7. Let G be a 2-cell embedded graph with minimum degree ≥ 3 and every face of finite size. Suppose that G is neither a prism nor an antiprism (planar or projective planar) and that ϕ is everywhere positive. We form a polygonal metric space $\mathcal{R}$ by associating each face in G of size k with a regular polygon in the plane having k-sides of length one. Note that by this construction, $\nu = 1$, and $\kappa(v) = 2\pi\phi(v)$ for every vertex v. Now, it follows from Lemma 6.3 and Theorem 1.3 that G is finite. By the Descartes-Euler formula (Theorem 1.1) we have that $\sum_{v\in V(G)} \phi(v) = 2$, so by applying Lemma 6.3 to $V(G)$, we deduce that $|V(G)| \leq 3444$ as claimed. □

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References

[1] A. D. Aleksandrov, V. A. Zalgaller, Intrinsic Geometry of Surfaces, Amer. Math. Soc., 1967.

[2] `http://www.ams.org/new-in-math/cover/descartes1.html`

[3] M. Gromov, Hyperbolic groups, in "Essays in Group Theory," S. M. Gersten (Editor), M.S.R.I. Publ. 8, Springer, 1987, pp. 75–263.

[4] Y. Higuchi, Combinatorial curvature for planar graphs, J. Graph Theory 38 (2001) 220–229.

[5] J. Milnor, A note on curvature and fundamental group, J. Differential Geometry 2 (1968) 1–7.

[6] B. Mohar, Embeddings of infinite graphs, J. Combin. Theory, Ser. B 44 (1988) 29–43.

[7] S. B. Myers, Riemannian manifolds with positive mean curvature, Duke Math. J. 8 (1941) 401–404.

[8] D. A. Stone, A combinatorial analogue of a theorem of Myers, Illinois J. Math. 20 (1976) 12–21.

[9] D. A. Stone, Correction to my paper: "A combinatorial analogue of a theorem of Myers", Illinois J. Math. 20 (1976) 551–554.

[10] L. Sun, X. Yu, Positively curved cubic plane graphs are finite, preprint, 2003.

[11] W. Woess, A note on tilings and strong isoperimetric inequality, Math. Proc. Camb. Phil. Soc. 124 (1998) 385–393.