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LOCALLY PLANAR GRAPHS ARE 5-CHOOSABLE

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It is proved that every graph embedded in a fixed surface with sufficiently large edge-width is 5-choosable.

Key Words: List coloring, choosability, surface, edge-width, locally planar graph

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1. INTRODUCTION

Let G be a graph. A *list-assignment* is a function L which assigns to every vertex $v \in V(G)$ a set $L(v)$ of natural numbers, which are called *admissible colors* for that vertex. An *L-coloring* of the graph G is an assignment of admissible colors to all vertices of G, i.e., a function $c : V(G) \to \mathbb{N}$ such that $c(v) \in L(v)$ for every $v \in V(G)$, and for every edge uv we have $c(u) \neq c(v)$. If k is an integer and $|L(v)| \geq k$ for every $v \in V(G)$, then L is a *k-list-assignment*. The graph is *k-choosable* if it admits an L-coloring for every k-list-assignment L. If $L(v) = \{1, 2, \ldots, k\}$ for every v, then every L-coloring is referred to as a *k-coloring* of G. If G admits an L-coloring (k-coloring), then we say that G is *L-colorable* (*k-colorable*).

It is not difficult to prove that every planar graph is 5-colorable. In fact, the Four Color Theorem [2, 11] shows that every planar graphs is 4-colorable. Concerning list colorings, Voigt [16] proved that not every planar graph is 4-choosable, and Thomassen proved in his renowned paper [13] that all planar graphs are 5-choosable. In fact, Thomassen proved a strengthening of this result which will be applied in our paper:

THEOREM 1.1 (Thomassen [13]). *Let G be a plane graph, and let L be a list-assignment for G such that every vertex has at least three admissible colors and every vertex that is not on the boundary of the outer face has at least five admissible colors. Then G can be L-colored.*

Leaving the plane to consider graphs on surfaces of higher genus, the chromatic and list chromatic number can increase. However, for graphs which obey certain local planarity conditions, one can deduce similar properties as for planar ones. We say that a graph G embedded in a surface S is *locally planar* if it does not contain short noncontractible cycles. Quantitatively, we introduce the *edge-width* of G as the length of a shortest cycle which is noncontractible in S. Thomassen proved in [12] that graphs in S with sufficiently large edge-width are 5-colorable. Many people asked if sufficiently large edge-width also implies 5-choosability. In our paper we answer this question affirmatively.

THEOREM 1.2. *For every surface S there exists a constant w such that every graph that can be embedded in S with edge-width at least w is* 5-*choosable.*

The proof of Theorem 1.2 is given in Section 4.

If G is a graph of girth at least w, then its edge-width in every surface is at least w. For arbitrarily large values of w, there exist graphs of girth w with arbitrarily large chromatic number [4]. Therefore, the constant w in Theorem 1.2 necessarily depends on the surface.

Our proof of Theorem 1.2 uses a result of Robertson and Seymour (cf. Theorem 2.1), whose proof in [9] does not yield an explicit bound on the value of $w = w(S)$ needed in the proof. However, there are more specific results which show that one can take $w = 2^{O(g)}$, where g is the Euler genus of S. See [8, Chapter 5] for more details.

Böhme et al. [3] found the best possible value of w for the projective plane by proving:

THEOREM 1.3 (Böhme, Mohar, and Stiebitz [3]). *A graph G embedded in the projective plane is 5-choosable if and only if it does not contain K_6 as a subgraph.*

Since K_6 has edge-width 3, every graph in the projective plane having edge-width 4 or more is 5-choosable. The value $w = 4$ for the projective plane is the only known minimum value of the width forcing 5-choosability of graphs in some surface.

Our proof also gives a polynomial time algorithm to 5-list-color graphs with an embedding of sufficiently large edge-width.

THEOREM 1.4. *There exists an algorithm, whose input is a graph G embedded in a surface of Euler genus g with edge-width at least $2^{\Theta(g)}$ and a 5-list-assignment L for G, which finds an L-coloring of G in polynomial time.*

Some comments about the algorithm of Theorem 1.4 can be found in the concluding section.

Let us point out that deciding about the choice number of planar graphs is actually hard. In fact, the following was proved by Gutner [6] (see also survey [17]).

THEOREM 1.5. *The problems of deciding whether a given planar graph is 4-choosable and of deciding whether a given triangle-free planar graph is 3-choosable are both Π_2^p-complete.*

So, Theorem 1.4 is best possible in a sense that one cannot replace 5-list-colorings with 4-list-colorings (if P $\neq$ NP). On the other hand, we believe that the edge-width condition is perhaps not necessary. We propose the following:

CONJECTURE 1.6. *For any fixed surface S, there is a polynomial time algorithm to decide whether a given graph embedded on S is 5-choosable.*

In fact, Thomassen posed the following conjecture, which would immediately imply ours.

Conjecture 1.7 (Thomassen [14]). *For every fixed surface S, there are only finitely many 5-list-critical graphs that can be embedded in S.*

The definition of k-list-critical graphs is given after Corollary 3.1.

Albertson [1] conjectured that for every surface S there exists an integer $q = q(S)$ such that any graph G embedded in S contains a set U of at most q vertices such that $G - U$ is 4-colorable. Such a result does not hold for list colorings since there exist planar graphs that are not 4-choosable [16]. However, Theorem 1.2 implies such a result for 5-list-colorings.

Corollary 1.1. *For every surface S there is a constant k such that for every graph G embedded in S, there exists a vertex set U of at most k vertices such that $G - U$ is 5-choosable.*

Proof. The proof is by induction on the Euler genus g of S. The base case when $g = 0$ follows by Theorem 1.1 with $k = 0$. Let k' be the maximum value of k taken over all surfaces whose Euler genus is less than g, and let w be the value from Theorem 1.2. If the edge-width of G is at least w, then G is 5-choosable by Theorem 1.2. Otherwise, let C be a noncontractible cycle of length at most $w - 1$. The Euler genus of the induced embedding of $G - C$ is less than g, so there is a set of at most k' vertices whose removal yields a 5-choosable graph. This shows that we may take $k = k' + w - 1$. ∎

2. PLANARIZATION

Local planarity in this paper is defined by requesting large edge-width. However, sometimes a stronger condition is appropriate. We say that a graph G embedded in a surface S has *face-width* at least k, $\mathtt{fw}(G) \geq k$, if every noncontractible closed curve in the surface intersects the graph in at least k points.

It is known that under the assumption of an embedding with large edge-width, there exist pairwise disjoint simple closed curves in the surface such that (a) any intersections of distinct curves with the graph are far apart in the graph, and (b) after cutting the surface along these curves, a surface of genus 0 is obtained. If the face-width is large, then these curves can be chosen to correspond to cycles in the graph. This result was established by Robertson and Seymour [9]; for a simple proof see, e.g., [8, Theorem 5.11.1]. After cutting the surface, known properties of planar graphs can be applied to the graph obtained after cutting, and this approach gave rise to many results about graphs on general surfaces embedded with large edge- or face-width, cf. [8, Chapter 5].

We will also use the approach described above. However, we will need a more elaborate way of cutting the surface. In particular, we want the surface obtained after the cutting to be a disk. Before introducing additional conditions, we need some definitions.

Let H be a graph. We will assume that H is connected, without vertices of degree 1 and that it has a vertex of degree more than 2. Every vertex of H of degree more than 2 is called a *branch vertex*. A path in H from a branch vertex x to a branch vertex y whose intermediate vertices all have degree 2 is said to be a *branch* of H and x and y are the *endvertices* of this branch. Let H° be the graph obtained from H by replacing every branch with a single edge. Then we say that H is a *subdivision* of H°.

Suppose that H is a subgraph of a graph W, and that the maximum degree of H is equal to 3. Suppose that for every branch ε of H, there is an induced subgraph of W that contains ε and is isomorphic to a subdivision of the graph shown in Figure 1, where ε is represented by the bold edges. Only the vertices on the left and the right in the figure are incident with edges of W that are not shown. The shown structure extends along ε all the way towards its endvertices. Suppose also that around the branch vertices of H, W is isomorphic to a subdivision of the graph shown in Figure 2. Then we say that W is a *wall* of *height* 3 around H. In the same way we define walls around H of height bigger than 3 by increasing the number of layers on each side of the branches. The minimum number of 6-cycles along a branch of H is called the *width* of the wall.

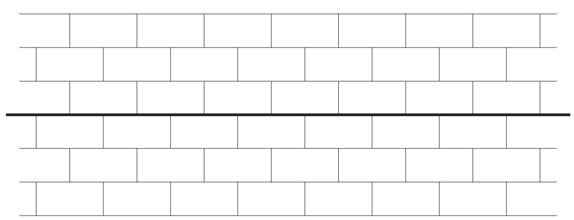


FIG. 1. A part of the wall of height 3 around a branch

Suppose now that G is a graph that is embedded in some surface. Suppose that G contains a subgraph W which is a wall of height h around its subgraph H, and that H has all the properties stated above. Suppose moreover that every cycle C in W, which corresponds to one of the cycles of length 5 or 6 shown in Figures 1 and 2, is contractible in S and the closed disk D bounded by C in S has its interior disjoint from W. Let B be the subgraph of G contained in D. Then B is a planar graph with outer

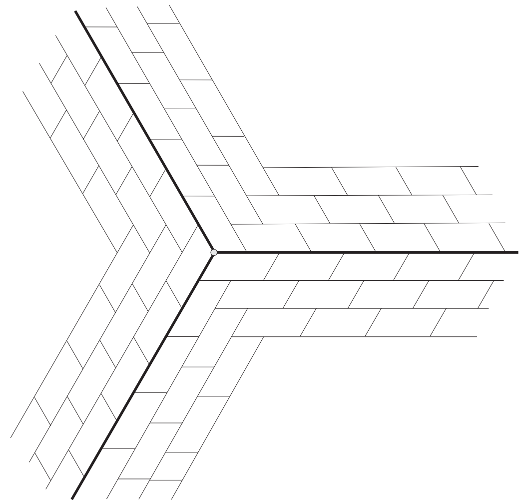


FIG. 2. A part of the wall of height 3 around a branch vertex

cycle C, and we say that B is a *brick* of the wall W and that C is a *brick cycle*.

Suppose that the wall W around H has height h. If b is a vertex of H and $d \leq h$ is a non-negative integer, then we define the brick neighborhood $B_d(b)$ around b as follows. If $d = 0$, then $B_d(b) = \{b\}$. For $1 \leq d \leq h$, let $B_d(b)$ be the union of $B_{d-1}(b)$ and all bricks that have some vertex in common with $B_{d-1}(b)$. The subgraph $B_d(b)$ is planar and is called the *d-brick neighborhood* of b.

Later we will assume that G triangulates the surface S. In that case, every brick will be a near-triangulation – a planar graph whose all faces except the outer face are triangles. Our goal will be to cut the surface along the branches of H, and we will ask the resulting surface to be a disk. This is true if and only if the induced embedding of H in S is a 2-cell embedding with precisely one face. In fact, we shall also require a wall W of width 100 and height 100 around H. Such a graph W can be drawn in S, but it is not true that every triangulation G will contain it as a subgraph. However, if G has large face-width, then G contains an induced subgraph which is a wall of width and height 100 around a subgraph H whose properties are as requested above. This was proved by Robertson and Seymour in [9].

Theorem 2.1 ([9]). *Let S be a surface, and let W be a cubic graph that is embedded in S. Then there is a constant w such that every graph embedded in S with face-width at least w contains a subgraph W' which is isomorphic to a subdivision of W and whose induced embedding is combinatorially the same as the embedding of W.*

Theorem 2.1 holds also when W is not cubic, but in that case we may only conclude that W' can be contracted to W.

We shall need a large face-width condition in order to apply Theorem 2.1. However, this does not follow from our assumption that the edge-width is large. This is resolved by applying the following construction. Let F be a facial walk of length $k \geq 4$ in the embedded graph G. Next we define a graph G_1 embedded in the same surface as G such that G_1 contains G as an embedded subgraph and such that the only face of G which is not a face of G_1 is F. We start by drawing k nested circuits $C_1, C_2, \ldots, C_k$, each of length k, inside the face F. For $1 \leq i \leq k$, we add an "antiprism" between the vertices of C_{i-1} and C_i, where instead of C_0 we consider F. Finally, we add one additional vertex and join it to all vertices of C_k. The construction is illustrated in Figure 3.

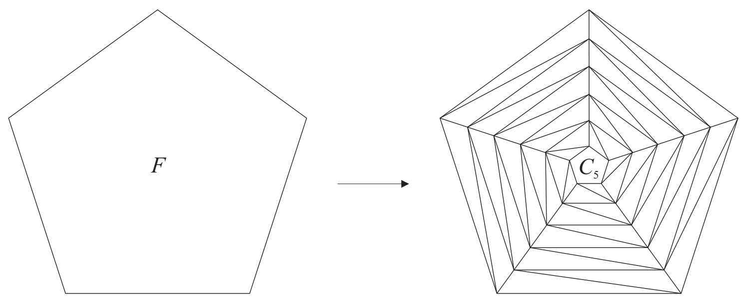


FIG. 3. Adding a chimney in a face of length 5

We say that this new embedded graph is obtained from G by adding a *chimney* to F. Let us observe that F may not be a cycle. The following proposition will enable us to restrict our attention to triangulations.

PROPOSITION 2.1. *Let G' be obtained from an embedded graph G by adding a chimney to one or more faces of length at least* 4. *Then the edge-width of G' is equal to the edge-width of G, and G' is 5-choosable if and only if G is 5-choosable.*

Proof. If C' is a noncontractible cycle in G', then C' can be changed by a homotopy into a closed walk C which does not use any edges inside the chimneys. It is easy to argue that the length of C is not larger than the length of C'. So C is also a noncontractible closed walk in G. Since the edge set of C contains a noncontractible cycle in G, we conclude that the edge-width of G is not larger than the edge-width of G'. The converse inequality is obviously true since G is a subgraph of G', so we conclude that the first statement of the proposition holds.

Clearly, if G' is 5-choosable, then so is its induced subgraph G. To prove the converse, suppose that G is 5-choosable, and let L be a 5-list assignment for G'. Since G is 5-choosable, there exists an L-coloring of G. Every vertex of $G' - V(G)$ has at most two neighbors in G. Therefore, $G' - V(G)$ consists of disjoint planar graphs to which Theorem 1.1 can be applied. This shows that the L-coloring of G can be extended to an L-coloring of the whole G'. This completes the proof. ∎

Chimneys will be used to triangulate faces of size at least 7. Shorter faces will be triangulated by adding edges only in order to keep control of short contractible cycles.

Proposition 2.2. *Let G' be obtained from an embedded graph G by adding a chimney to every face of length at least* 7*, and by triangulating the faces of lengths* 4, 5, 6 *by adding edges in those faces. Then G' is a triangulation whose edge-width is at least one third of the edge-width of G. If G' is 5-choosable, then so is G.*

Proof. The proof of the first part is similar to the proof of Proposition 2.1, except that one has to note that the cycle C may become shorter. However, it can shrink at most by the factor of 3 (which can be caused by adding diagonals across 6-faces of G). The second part is obvious since G is a subgraph of G'. ∎

The graph G' described in Proposition 2.2 will be called the *triangular completion* of G.

3. LIST-CRITICAL GRAPHS AND SHORT CONTRACTIBLE CYCLES

In [13], Thomassen proved a result which is slightly stronger than stated in Theorem 1.1.

Theorem 3.1 (Thomassen [13]). *Let G be a plane graph with outer facial walk C, and let x, y be adjacent vertices on C. Let L be a list-assignment for G such that $L(x) = \{\alpha\}$, $L(y) = \{\beta\}$, where $\beta \neq \alpha$, every vertex on $C \backslash \{x, y\}$ has at least three admissible colors, and every vertex that is not on C has at least five admissible colors. Then G can be L-colored.*

Theorem 3.1 has the following useful corollary.

Corollary 3.1. *Let G be a plane graph whose outer face C is a triangle. Let L be a* 5*-list-assignment for G. Then every L-coloring of C can be extended to an L-coloring of the whole graph G.*

Proof. Let $C = xyz$. Let $c(x) = \alpha$, $c(y) = \beta$, and $c(z) = \gamma$ be a precoloring of C. Let L' be the list-assignment for G which agrees with L on $V(G) \setminus V(C)$, and $L'(x) = \{\alpha\}$, $L'(y) = \{\beta\}$, $L'(z) = \{\alpha, \beta, \gamma\}$. Now the proof is complete by applying Theorem 3.1 to G and L'. ▮

Let G_0 be a graph and let L be a list assignment for G. We say that G_0 is *L-critical* if G_0 is not L-colorable but every proper subgraph of G_0 is. The graph G_0 is *5-list-critical* if there is a 5-list-assignment L such that G_0 is L-critical.

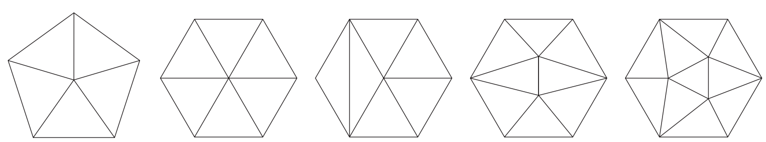
Suppose that G_0 is embedded in some surface, and let G be a triangular completion of G_0, cf. Proposition 2.2. Let L be a 5-list-assignment for which G_0 is L-critical. Let us recall that G is said to be *internally 6-connected* if it is 5-connected and whenever $G = G_1 \cup G_2$ with $|V(G_1 \cap G_2)| = 5$, either $G_1 \setminus G_2$ or $G_2 \setminus G_1$ contains at most one vertex.

Lemma 3.1. *Let G_0 and G be as stated above. Then the following holds:*

(a) *If $\mathtt{ew}(G_0) \geq 10$, then G is 4-connected.*

(b) *Suppose that C is a cycle of G of length $k \leq 6$. If C is contractible in the surface and the disk D bounded by C contains at least one vertex in its interior, then $G \cap D$ is as shown in Figure 4.*

(c) *If $\mathtt{ew}(G_0) \geq 19$, then G is internally 6-connected.*



(a) (b) (c) (d) (e)

FIG. 4. Short contractible cycles

Proof. Throughout the proof, let L be the 5-list assignment as introduced above. We also let R be the subgraph of G_0 obtained by deleting all vertices and edges of G_0 that lie in the interior D° of D. Finally, let $Q = G \cap D$ be the planar near-triangulation with outer cycle C.

(a) By Proposition 2.2, $\mathtt{ew}(G) \geq 4$. Since G is a triangulation, it is 3-connected (cf. [8, Proposition 5.3.1]). If vertices x, y, z separate G, they form a minimal separator and hence they induce a triangle in G. Since the edge-width of G is at least 4, the triangle $C = xyz$ is contractible. Clearly, C cannot be inside a chimney, so the disk D bounded by C (and containing one of the components of $G - \{x, y, z\}$ in its interior) contains at least one edge of G_0. Since G_0 is L-critical, R has an L-coloring c. By Corollary 3.1, the coloring c can be extended to Q, and hence to the whole of G_0. This contradicts the fact that G_0 is not L-colorable, and proves (a).

(b) By means of contradiction, we assume that we have an example for which the claims in (b) do not hold and that this counterexample is chosen in such a way that k is minimum possible and, subject to that, the number of vertices in D° is minimum. Let us observe that C is not the face of G_0 since the triangular completion of G_0 in Proposition 2.2 adds only edges into the disk bounded by a face of size at most 6.

First of all, it is easy to see that D contains at least one edge of G_0 in its interior. Hence, there is an L-coloring c of R. By the minimality of k, D contains no chords of C, i.e. R is an induced subgraph of G_0. Let us first assume thet Q contains a vertex v that is adjacent to three or more vertices on C. The minimality of k and $|V(Q)|$ assures that the following is true. If $k = 4$, then v is the only vertex in D°. So v is of degree 4, and c can be extended to the whole G_0. If $k = 5$, then we have in D° two adjacent vertices, one of which is of degree 4 in G. This again yields a contradiction.

So, we have $k = 6$. Moreover, v is adjacent to precisely three vertices on $C = x_1x_2 \ldots x_6$, and they are consecutive on C. Suppose that they are x_1, x_2, x_3. Consider the cycle $C' = x_1vx_3x_4x_5x_6$. By the minimality of our counterexample, C' looks like shown in Figure 4. Since v must be of degree bigger than 4, this gives one of the cases in Figure 4 also for the cycle C, except for the case when inside the disk of C' we have three mutually adjacent vertices (see Figure 4(e)), two of which are adjacent to v. In this case it is easy to see that c can be extended to Q, a contradiction.

Now we may assume that no vertex in D has three or more neighbors on C. Therefore the list coloring c can be extended to Q by Corollary 3.1, and hence to the whole of G_0. This contradicts the fact that G_0 is not L-colorable, and proves (b).

(c) Let k be the order of a minimal separation and suppose that $k \in \{4, 5\}$. Since G is a triangulation, every minimal separation induces a connected subgraph, denote it by C in our case. By Proposition 2.2 we know that $\mathtt{ew}(G) \geq 7$, so C contains no noncontractible cycles. Of course, C should contain a cycle C' in order to separate. By (a), this cycle is of length 4 or 5, and now we easily complete the proof by using (b). ∎

4. PROOF OF THE MAIN THEOREM

This section is devoted to the proof of Theorem 1.2. Having a non-5-choosable graph in a fixed surface S with large edge-width, we first take its 5-list-critical subgraph, which is henceforth extended to a triangulation of large face-width. Next, we apply Theorem 2.1 to obtain a 2-cell embedded subgraph H surrounded by a wall of width 70 and height 30. The graph H will be used to cut the surface in order to obtain a disk.

The graph H, along which we will cut the surface, will be colored in such a way that Thomassen's theorem 1.1 can be applied to list-color the rest of the graph. This step is rather complicated, and H has to be changed and extended accordingly in order to achieve this task. The details of the whole procedure are given in the sequel.

Initial reductions

Let G_1 be a graph embedded in a surface of Euler genus g with edge-width at least w. If G_1 is not 5-choosable, then there is a 5-list assignment L' such that G_1 is not L'-colorable. Let G_0 be an L'-critical subgraph of G_1. The induced embedding of G_0 (cf. [8] for details) is an embedding in a (possibly different) surface S of Euler genus at most g and with $\mathtt{ew}(G_0) \geq \mathtt{ew}(G_1) \geq w$. Let G be a triangular completion of G_1 in S. By Proposition 2.2,

$$\mathtt{fw}(G) = \mathtt{ew}(G) \geq \frac{1}{3}\mathtt{ew}(G_0) \geq \frac{1}{3}\mathtt{ew}(G_1).$$

Extend L' (arbitrarily) to a 5-list-assignment L for G. We will show that G is L-colorable, contradicting the assumption that its subgraph G_0 is not L'-colorable. This contradiction will complete the proof.

We are assuming that $w \geq 19$, so $\mathtt{fw}(G) \geq 7$, and hence Lemma 3.1 can be applied to conclude that G is internally 6-connected.

The cutting subgraph

Let H' be a cubic graph which is 2-cell embedded in S with a single face, and let W' be a wall of height 30 and width 70 around H'. We are assuming that w is large enough so that G contains a subdivision W_0 of W' as a subgraph; cf. Theorem 2.1. (Let us observe that this is the only condition imposed on the edge-width of G_1. See the discussion after the statement of Theorem 1.2.) We also let H_0 be the subgraph of W_0 corresponding to H'.

We shall modify H_0 to obtain a subgraph H of G, by changing H_0 only locally, with the goal to achieve the following properties:

(1) Every branch vertex s of H_0 is replaced by another branch vertex s' of degree 3 which is contained within the 25-brick neighborhood around s in W_0.

(2) Every branch ε of H_0 joining branch vertices s and t (say) is changed outside the 27-brick neighborhoods of s and t to a path which is contained within the 4-brick neighborhood of ε.

(3) H is an induced subgraph of G homeomorphic to H_0.

(4) H satisfies the *consecutive neighbors property* (abbreviated CNP), which means that for every vertex $v \in V(G) \setminus V(H)$, if its neighbors in H are contained in a single branch ε of H, then they induce a path on ε.

(5) Every branch of H contains a break segment (see the next subsection for the definition).

Break segments on the branches

In the next two subsections we shall use the following notation. If P is a path and $a, b \in V(P)$, then $P[a, b]$ denotes the subpath of P from a to b. By replacing brackets with parentheses, i.e., writing $P[a, b)$, $P(a, b]$, or $P(a, b)$, we denote the same segment except that one, the other, or both ends are excluded. By $d_P(a, b)$ we denote the distance between a and b on P, while $d_G(a, b)$ denotes their distance in G. The distance notation in G is also extended to the distance between a vertex and a subgraph, and the distance between two subgraphs in the standard way.

Let ε be a branch joining branch vertices a and b. Suppose that $\varepsilon = L \cup S \cup R$ where L and S have only a vertex p in common, S and R have precisely a vertex w in common, and $S = \varepsilon[p, w] = pqrstuvw$ is a segment of ε of length 7. We say that S is a *break segment* on ε if the following conditions are satisfied:

(BS1) For every $x, y \in V(S)$, $d_\varepsilon(x, y) = d_G(x, y)$.
(BS2) For every $x \in V(L) \setminus \{p\}$ and every $y \in V(S)$,

$$d_G(x, y) \geq d_\varepsilon(p, y) + 1.$$

(BS3) For every $x \in V(R) \setminus \{w\}$ and every $y \in V(S)$,

$$d_G(x, y) \geq d_\varepsilon(w, y) + 1.$$

(BS4) For every $x \in V(L)$ and every $y \in V(R)$, $d_G(x, y) \geq 7$, with equality holding if and only if $x = p$ and $y = w$.

Let ε be a branch in H_0. We shall show how to change ε to another branch ε' joining the same branch vertices, a and b, which will contain a break segment in the middle.

First, let us fix some notation. Let $B_1, \ldots, B_{70}$ be the consecutive bricks along ε on the "upper" side of ε. Let z_0 be the vertex on ε in which the bricks B_{35} and B_{36} intersect. Let us consider all pairs of vertices $p \in \varepsilon(a, z_0)$ and $w \in \varepsilon(z_0, b)$ such that $d_G(p, w) = 7$, and choose a pair (p, w) whose ε-segment $\varepsilon[p, w]$ has maximum number of vertices. Let $S = pqrstuvw$ be a path of length 7 joining p and w, and let $\varepsilon' = \varepsilon[a, p] \cup S \cup \varepsilon[w, b]$.

It is easy to see that ε' is a path in G and that S is a break segment on ε': (BS1) is clear by construction, while (BS2)–(BS4) hold because the pair (p, w) was selected so that $\varepsilon[p, w]$ has maximum number of vertices. Let us observe that S is contained within the 7-brick neighborhood of z_0, so it does not intersect the 27-brick neighborhood of any branch vertex. It is also within the 4-brick neighborhood of ε.

We repeat this procedure for all branches ε of H_0 and replace ε with ε'. Let us denote by H_1 the resulting subgraph of G.

Changing branch vertices

Let s be a branch vertex of H_0. Let us consider the 25-brick neighborhood N of s. There are three branches meeting at s, and we denote them by α, β, γ. We will replace the branches inside N with three new paths α', β', γ' (respectively), meeting at a new branch vertex s'. Let us observe that N contains only contractible cycles, so we may consider an embedding of N in the plane. Under this embedding, we will assume that formerly $\alpha \cup \beta$, and afterwards $\alpha' \cup \beta'$, are drawn as a "horizontal" line, and that γ, and later γ', arrives to this line from below, as shown in Figures 5 and 6. Accordingly, we shall speak of "*up*" and "*down*", of the "*left*" and "*right*" side of γ, etc.

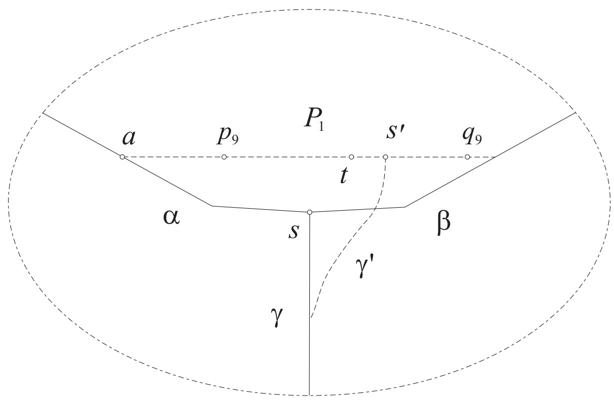


FIG. 5. Changes in a neighborhood of a branch vertex

For a path $P = p_1 p_2 \dots p_k$ in G we say that it is *geodesic* if for any vertices p_i, p_j on P, $d_G(p_i, p_j) = d_P(p_i, p_j) = |j - i|$. Suppose that P is a geodesic path in a graph embedded in a surface and that there is another path P' that contains P in its interior. Then we can speak of the left and the right hand-side at every vertex in P. The edges which are incident with vertices on P and do not lie on P' are divided to those on the left and those on the right according to their local rotation with respect to the two edges of P', and the choice of the left and right is made consistent when we follow the edges of P from p_1 to p_k. We say that P is *strongly geodesic* on the right side (or on the left side) if every path Q in G which joins two vertices p_i, p_j of P, which is internally disjoint from P, and leaves p_i and enters p_j at an edge on the right (respectively, left) side of P, has length strictly greater than $|j-i|$.

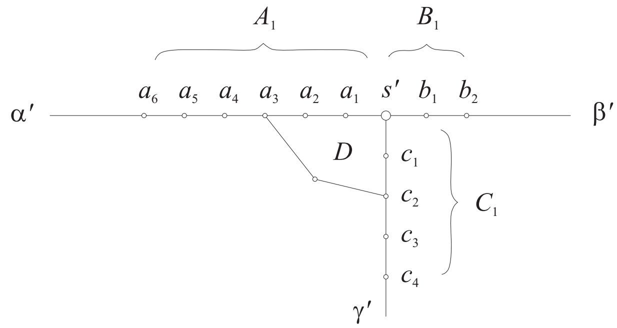


FIG. 6. Strongly geodesic neighborhood of a new branch vertex

Our goal, when making the indicated changes, is to achieve strong geodesic property on the upper side of a part of $\alpha' \cup \beta'$ and on the right side of γ'. See Figure 6. More precisely, if A_1, B_1, C_1 are are the initial segments of α', β', and γ' of length 6, 2, and 4 (respectively), we will require that

(B1) $A_1 \cup B_1$ is geodesic and is strongly geodesic on its upper side in N.

(B2) C_1 is geodesic and is strongly geodesic on the right side in N.

(B3) The ith vertex c_i of C_1 $(1 \le i \le 4)$ has distance i from $\alpha' \cup \beta'$ in G.

(B4) $d_G(\alpha' - A_1, \beta' - B_1) \ge 3$, $d_G(\alpha' - A_1, \gamma' - C_1) \ge 3$, and $d_G(\beta' - B_1, \gamma' - C_1) \ge 3$.

(B5) For every vertex in $(\alpha' - A_1) \cup (\beta' - B_1) \cup (\gamma' - C_1)$, its distance from vertices $s', a_1, a_2, a_3, a_4, c_1$, and c_2 is at least 3.

Let us now traverse α from the boundary of N towards s, and let a be the first vertex reached in this way whose distance in G from $\beta \cup \gamma$ is equal to 25. We may assume that $d_G(a, \beta \cup \gamma) = d_G(a, \beta) = 25$. Let P_1 be the upper-most path of length 25 joining a and β. Observe that P_1 is contained in N. Let t be the vertex on P_1 whose distance from a is 13.

For $1 \le i \le 9$, let p_i, q_i be the vertices on P_1 that are at distance i from t. We assume that p_i is to the left, and q_i is to the right of t. Since G is a triangulation and the triangles in N are contained in a topological disk, there is a unique path Q_i in N joining p_i and q_i such that the following holds. The path Q_i is below P_1, every vertex on Q_i is at distance i from t, and every vertex of G that lies in N below P_1 and is at distance at most i from t is contained in the disk bounded by $P_1[p_i, q_i] \cup Q_i$. This is obvious for Q_1, and can be proved easily by induction for $i = 2, \ldots, 9$.

Let us now consider Q_9. Since P_1 is geodesic in G, $d_G(p_9, q_9) = 18$. Therefore, Q_9 contains a vertex r_9 whose distance from p_9 and from q_9 is at least 9. Let $R = tr_1r_2 \dots r_9$ be a path of length 9 joining t with r_9. Let us now traverse R from t towards r_9, and let u be the first vertex whose distance from P_1 is equal to 4. We claim that u exists and that it is one of the vertices $r_4, \dots, r_8$. To prove this, it suffices to see that $d_G(r_8, P_1) \geq 4$. If this distance would be less than 4, then we would have $d_G(r_9, P_1) \leq 4$. However, this would imply that $d_G(r_9, \{p_9, t, q_9\}) \leq 8$, a contradiction.

Now let us consider paths of length 4 from P_1 to u. Every such path is contained inside the disk bounded by $P_1[p_9, q_9] \cup Q_9$. Among all such paths we select the right-most one, and we denote it by $C_1 = s'c_1c_2c_3c_4$, where $s' \in V(P_1)$ and $c_4 = u$. The right-most choice assures that (B2) is satisfied. The vertex s' will be our new branch vertex, and we just observe that $s' \in V(P_1[p_8, q_8])$. (If not, a pair of vertices in $\{p_9, t, q_9, r_9\}$ would be at distance less than 9.) We also define $A_1 = s'a_1a_2 \dots a_6$ and $B_1 = s'b_1b_2$ to be the segments of P_1 to the left and right of s', which are of lengths 6 and 2, respectively. Observe that the choice of t and our earlier observation that $s' \in V(P_1[p_8, q_8])$ guarantee that A_1 and B_1 are contained in P_1. Hence, (B1) holds. It is also clear that (B3) holds.

Now, we define α' as the branch which is obtained from α by replacing $\alpha[a, s]$ with $P_1[a, s']$. Similarly, we replace the last segment of β with the appropriate segment of P_1 to get β'. Finally, we let γ' be a path starting at s' with C_1, continuing on R from $u = c_4$ to r_9, and finally taking a path towards γ such that we never approach $\alpha' \cup \beta'$ to the distance 2 or less.

Existence of such a path γ' needs an argument which shall be provided in this paragraph. Let us first observe that Q_9 is at distance at least 3 from $\beta \cup \gamma$ since every vertex $x \in V(Q_9)$ satisfies:

$$d_G(a, x) \leq d_G(a, t) + d_G(t, x) = 13 + 9 = 22.$$

Similarly as we have argued about paths $Q_1, \dots, Q_9$, one can show that there are paths R_1, R_2, R_3 that lie below $\alpha' \cup \beta'$ such that each vertex on R_i is at distance i from $\alpha' \cup \beta'$ $(i = 1, 2, 3)$. Let $D \subset N$ be the disk below R_3. Clearly, $\beta \cup \gamma$ contains a path in D joining the boundary of N with a vertex y on R_3. There is a path in Q_9 joining r_9 with R_3 (on the right side). Continuing this path along R_3 to y, we obtain the required path γ'. This is sketched in Figure 7.

The final conclusion is that α', β', γ' satisfy (B4) and (B5). As mentioned above, they also satisfy (B1)–(B3), hence our goal is complete.

After performing such a change around every branch vertex of H_1, a new subgraph of G is obtained, and we denote it by H_2.

Let us recall that W_0 is a wall around H_0 of width 70 and height 30. Therefore, the changes made around the branch vertices and the changes made to obtain break segments on the branches do not interfere with each other. Actually, much smaller width and height would suffice for our ar-

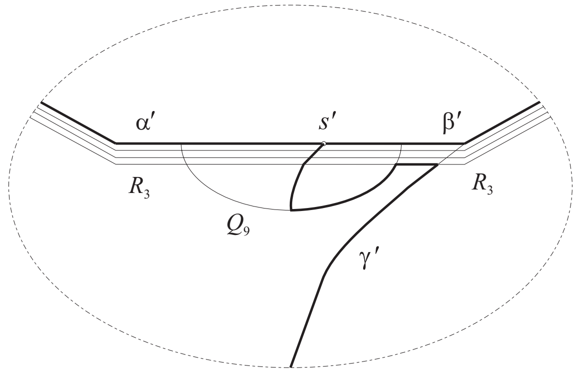


FIG. 7. How to construct γ'

guments to work, but we care about simplicity of the proof, and not that much about best possible estimates on the edge-width.

Consecutive neighbors property

Let ε be a branch of H_2 and let s' be one of its endvertices. Recall that, when changing H_1 around the branch vertex s which was eventually replaced by s', the new branch vertex became the end of three branches α', β', γ'. These branches start with geodesic walks A_1, B_1, C_1 (respectively), where $A_1 = s'a_1 \ldots a_6$ is of length 6, $B_1 = s'b_1b_2$ is of length 2, and $C_1 = s'c_1c_2c_3c_4$ is of length 4. If $\varepsilon = \beta'$, then we set $x = s'$. If $\varepsilon = \gamma'$, then we set $x = c_2$. In this case we also set $x' = c_1$. If $\varepsilon = \alpha'$, then we consider the largest index $i \in \{0, \ldots, 4\}$ such that a_i is at distance 2 from c_2 (where we conveniently set $a_0 = s'$). Then we take $x = a_i$ and set $x' = a_{i-1}$ (if $i > 0$). Let us observe for further reference that the vertices on $\varepsilon[s', x]$ will be colored before we will start coloring other vertices on ε. See the next two subsections.

Let $S = pqrstuvw$ be the break segment in the "middle" of ε defined previously. If p is closer to x than w, then we take $y = p$; otherwise we take $y = w$. We also denote the neighbor of y outside $\varepsilon[x, y]$ by y'. Observe that $y' \in \{q, v\}$.

We shall change the segment $\varepsilon[x, y]$ such that CNP will be satisfied for all vertices with a neighbor in $\varepsilon(x, y)$ and such that all vertices on the new branch will be at distance at most 2 from ε.

First of all, we want ε to be an induced path in G. If ε contains adjacent vertices u, v that are not consecutive on ε, then we can shorten ε by replacing $\varepsilon[u, v]$ by the edge uv. Let us observe that such changes do not affect break segments or the initial segments A_1, B_1, C_1 around branch vertices in H_2. In particular, they preserve all properties established earlier.

Suppose now that v is a vertex in $V(G) \setminus V(H_2)$ that has at least three neighbors in ε, at least one of which is in $\varepsilon(x, y)$. Let $u \in \varepsilon(x, y)$ be a neighbor of v, and let u' be a neighbor of v in $\varepsilon - \varepsilon[x, y]$ such that $uu' \notin E(G)$. Because of properties (BS1)–(BS4) and (B1)–(B4), it follows that $u' \in \{x', y'\}$, where x' and y' are the neighbors of x or y (respectively) on $\varepsilon - \varepsilon[x, y]$ as introduced above.

Suppose that the neighbors of v do not induce a path on ε. Let v_1, v_2 be distinct nonadjacent neighbors of v on ε such that $\varepsilon(v_1, v_2)$ contains no neighbors of v. It is clear by 4-connectivity of G that $d_\varepsilon(v_1, v_2) > 2$. By using property (B5), it can be verified easily that $v_1, v_2 \neq x'$. Since G is a 4-connected triangulation, the neighbors of v inside the disk bounded by the path $v_1 v v_2$ and the segment $\varepsilon[v_1, v_2]$ induce a path $P_{v_1 v_2} \subseteq G$ from v_1 to v_2. Now, we replace $\varepsilon[v_1, v_2]$ by $P_{v_1 v_2}$. See Figure 8.

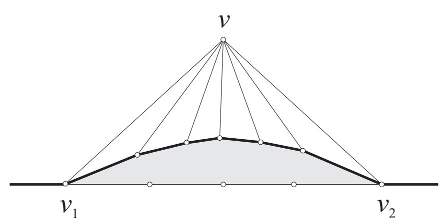


FIG. 8. Elimination of a bad triple

It may happen that v_1 (or v_2) is equal to y'. In such a case, the corresponding vertex p or w of the break segment on ε is replaced by another vertex on $P_{v_1 v_2}$. It is easy to see that such a change preserves the essential propoerties (BS1)–(BS4) of break segments.

Let us call the triple (v, v_1, v_2) *bad* if it corresponds to a situation as encountered above. It is clear that the change described above removes the bad triple (v, v_1, v_2) and we claim that it does not introduce any new bad triples (u, u_1, u_2). If it would, then u would be contained in the (closed) disk bounded by $\varepsilon[v_1, v_2] \cup P_{v_1 v_2}$, and u_1, u_2 would be vertices on $P_{v_1 v_2}$. However, in such a case, the vertices u, u_1, u_2, and v would separate the graph G, contrary to Lemma 3.1. This contradiction proves that after a finite number of steps, we can get rid of all bad triples. It is also clear that every bad triple elimination preserves properties (BS1)–(BS4) of break segments and (B1)–(B4) of neighborhoods of branch vertices.

Once there are no bad triples, the resulting graph, which we will denote by H, satisfies the CNP.

Obtaining a precoloring of H

As mentioned before, we are going to color H and some other vertices of G which are either close to the branch vertices or close to the break segments of H. The ultimate goal is to be able to apply Theorem 1.1 to the uncolored part of the graph G.

For every branch vertex b of H, we color b and some vertices close to b. Next, we color every branch starting with the colored vertices around branch vertices and continuing towards the break segments. Finally, the break segments and some other vertices close to them are colored. The requirement for this precoloring is that all vertices of H are colored. We also color some additional vertices forming a connected subgraph together with H. This ensures that all vertices of the remaining graph, which have a precolored neighbor, are on the outer face of the corresponding planar embedding. Lastly, and this is where the difficulty lies, we require that every uncolored vertex still has at least three available colors in its list after the colors used on its neighbors are removed. All of these properties ensure that Theorem 3.1 can be applied.

During the proof given below, we shall have some vertices of G already colored (precolored). If we color a vertex v by color a, then this color can no longer be used to color the neighbors of v. We therefore consider the reduced list $L'(u) = L(u) \setminus \{a\}$ for every neighbor u of v. To simplify notation, we shall assume that the current list of admissible colors is always updated according to the colors used on the precolored neighbors of any vertex. When we color the subgraph H, our main goal will be to make sure that every vertex not in H is left with at least three admissible colors. This will be automatically satisfied for all vertices which have at most two neighbors in H. For others we adopt the following terminology. We say that v is a *peak* if it is a vertex in $V(G) \setminus V(H)$ and has at least 3 neighbors in H. Having partially colored the vertices of H, we say that a peak v is *endangered* if it has only three admissible colors left. Our only concern will be not to lose any more colors at endangered vertices.

Coloring around a branch vertex

Let us recall the notation for $P_1, \alpha', \beta', \gamma'$, etc. around a branch vertex s' introduced before and conveniently shown in Figure 6.

Consider all paths of length 2 from c_2 to P_1. They all reach P_1 to the left of s'. Choose the one, call it D_1 which is the lower-most one (possibly $D_1 = c_2c_1s'$), and let D be the disk (possibly just the path c_2c_1s') bounded by D_1, C_1, A_1. Observe that $D_1 \cap A_1$ is the vertex x in $\{s', a_1, a_2, a_3, a_4\}$, which we have introduced when speaking how to get the CNP on α'.

The vertices in D can be L-colored by Theorem 3.1. We claim that, after doing this, every vertex adjacent to D still has at least three admissible colors. If there was a vertex v with at most two, then v would have at least three neighbors in D. Now, v cannot be above P_1 because of (B1), and it cannot be to the right of C_1 because of (B2). If it were below P_1 (and hence to the left of C_1), it would be adjacent to the three vertices in

D_1. However, this would contradict our choice of D_1 as the lowest path of length 2 from c_2 to P_1.

There are two details that should be made clear at this point. First, the above coloring procedure is performed simultaneously around all branch vertices s' of H. Secondly, if ε is a branch of H incident with s', let x be the vertex on ε as defined in the subsection on Consecutive neighbors property. Then it is precisely the segment $\varepsilon[s', x]$ that has been precolored at this step.

Extending the coloring along a branch

Let ε be a branch of H and let $v_0, v_1, \dots, v_N$ be the consecutive vertices on ε, where v_0 is a branch vertex of H and v_N belongs to the break segment S on ε such that $S \cap \varepsilon[v_0, v_N] = \{v_N\}$. In the previous step we have already colored one or up to 5 initial vertices $v_0, \dots, v_j$ on ε, where $0 \le j \le 4$, and in the general step we assume that $v_0, \dots, v_{i-1}$ $(i > j)$ have been precolored, and that we want to color v_i next.

Let us recall that v_{j+1} and v_{j+2} are part of a geodesic and on one side strongly geodesic segment A_1, B_1, or C_1, see the subsection on changing the branch vertices and confirm conditions (B1)–(B3). This ensures that, when we come to color v_{j+1} and afterwards v_{j+2}, at most one endangered vertex is adjacent to that vertex.

We claim that v_i is adjacent to at most one endangered vertex also for $i \ge j+3$. If there were two, say a and b, consider their precolored neighbors in H. Because of the CNP, their neigbors are on ε, immediately preceding v_i. In particular, they are both adjacent to v_{i-2}. Consequently, $v_i a v_{i-2} b v_i$ is a (contractible) 4-cycle whose interior contains v_{i-1}. This contradicts Lemma 3.1(b).

Since ε is an induced subgraph of G, the only precolored neighbor of v_i is v_{i-1}. Therefore, v_i has at least 4 admissible colors when we come to color it. If v_i is adjacent to an endagered vertex u, let c be an admissible color for v_i that is not among admissible colors of u. Otherwise, let c be any admissible color of v_i. Now we color v_i by c. We repeat this for $i = j+1, \dots, N$.

Note that the above procedure has not much flexibility. It may happen that all vertices have endangered neighbors and that the coloring along ε is uniquely determined. Therefore, a special care is needed when the precolorings of ε from the left and the right side come together. Here we use special properties of the break segments. This is dealt with in the next subsection. Let us also observe that coloring different branches or different segments of the same branch do not interfere with each other by (BS4).

Coloring at break segments

Let ε be a branch of H with endvertices a and b and break segment $S = pqrstuvw$. Let us denote $L = \varepsilon[a,p]$ and $R = \varepsilon[w,b]$, so that $\varepsilon = L \cup S \cup R$ and $L \cap S = \{p\}$ and $S \cap R = \{w\}$.

For $x, y \in V(S)$, define $D(x,y)$ as the subgraph consisting of all geodesic paths from x to y in G. In particular, $S[x,y] \subseteq D(x,y)$. We choose $x \in \{p,q\}$ and $y \in \{v,w\}$ as follows. First we consider $D(q,v)$. Suppose that q is of degree 2 in $D(q,v)$ and that α, β are its neighbors. Let γ, δ be vertices distinct from p, q such that $pq\gamma$ and $pq\delta$ are the triangles containing pq. If $\alpha\gamma$ and $\beta\delta$ (or $\alpha\delta$ and $\beta\gamma$) are edges of G, then we set $x = p$ (see Figure 9). In all other cases we set $x = q$. Similarly, we set $y = v$, unless v has precisely two neighbors α, β in $D(q,v)$ and there exist vertices γ, δ with γ adjacent to v, w, α, and δ adjacent to v, w, β; in the latter case we set $y = w$.

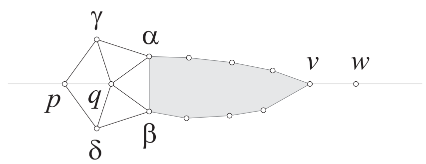


FIG. 9. The case when we select $x = p$

Let us now consider $D(x,y)$. Since $D(x,y)$ is contained within the wall around ε, there is a (closed) disk (possibly with some "degeneracy") bounded by the "upper-most" and the "lower-most" (x,y)-path in $D(x,y)$. We let D be the set of vertices of G contained in this disk. Let x', x'' (where possibly $x' = x''$ if the disk is degenerate at that point) be the neighbor(s) of x on the boundary of $D(x,y)$. Define similarly the neighbor(s) y', y'' of y.

To color the whole branch ε, we first color it along L until reaching p, as explained in the preceding subsection. Next we color x if $x = q$. Same arguments as before show that this is possible. After that, we also color x' and x'', taking care of possible endangered vertices. This may not be possible only when x' and x'' are adjacent and they each have an endangered neighbor. If this were the case, then x', x'' and their endangered neighbors would play the role of $\alpha, \beta, \gamma, \delta$ above, and hence x would not be equal to q. However, knowing that the situation in Figure 9 occurs around q, it is easy to see that the same cannot happen at $x = p$. So, we conclude that x' and x'' can be precolored.

If $x' \neq x''$ and there is a vertex x''' adjacent to x, x', and x'', then we also color x'''. Note that such a vertex x''' belongs to D.

On the "right" part of ε, we color R in the similar way, starting at b. Finally, we color y', y'', and y''' (if it exists).

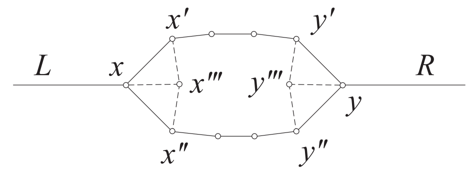


FIG. 10. Coloring the disk around a break segment

We claim that every vertex in D has at most two colored neighbors. Clearly, $z \in V(D)$ cannot be adjacent to one of x, x', x'', x''' and to one of y, y', y'', y''' at the same time, since $d_G(x, y) \geq d_G(q, v) = 5$. So, if a vertex $z \in D$ has three colored neighbors, they are x', x'', x''' (or y', y'', y'''). However, this contradicts Lemma 3.1. For convenience, we exhibit the situation in Figure 10.

Therefore, we can apply Theorem 3.1 and extend the coloring to all vertices in D. We claim that all uncolored vertices still have at least 3 admissible colors. To see this, recall that we took care of endangered vertices while coloring $L \cup \{x', x''\}$ and $R \cup \{y', y''\}$. May it be that by extending the coloring to D, we overlooked some endangered vertex z? Then z has at least 3 neighbors in $D \cup L \cup R$. Since the outer paths of D are geodesic (x, y)-paths in G, z cannot have three neighbors in D. If so, it would be included in D. So, we may assume that one of its neighbors, say α (possibly $\alpha = p$), is in $L - \{x\}$. Another neighbor is in the "upper" path in D and is different from x, x', x''. Then $d_G(\alpha, y) \leq d_G(x, y)$, which is a contradiction to the defining property of a break segment, (BS1) if $\alpha = p$, or (BS2) if $\alpha \neq p$. This shows that z does not exist.

Extending into the remaining disk

After coloring H and some additional vertices contained in the disks D close to branch vertices or close to break segments, the remaining graph K is planar. Moreover, all vertices of K which have a precolored neighbor are on the outer face of a plane embedding of K. (K may be disconnected, though.) Therefore, we can apply Thomassen's Theorem 3.1 and color K using only admissible colors. This completes the proof of our main theorem.

5. CONCLUDING REMARKS

In concluding, we should first remark that the proof of Theorem 1.2 also gives a polynomial-time algorithm for 5-list-coloring graphs embedded in a fixed surface S with sufficiently large edge-width. The value of w imposed on the edge-width in Theorem 1.2 depends on Robertson and Seymour Theorem 2.1, whose proof in [9] does not yield an explicit bound on w. However, the description of the algorithm below suggests how to overcome this trouble by a direct approach, which yields an explicit bound $w = 2^{O(g)}$, where g is the Euler genus of S.

All steps in the proof of Theorem 1.2 can be performed in polynomial time. The only step, which is not obvious, is how to find the initial subgraph H_0 surrounded by a wall of height 30 and width 70. Actually, this can be done by an algorithm of Robertson amd Seymour in their series of Graph Minors papers [10]. But there is also a simpler direct way: First we find a planarizing collection of cycles which are far apart. This can be done rather easily if the edge-width is at least $2^{\Theta(g)}$, see, e.g., [8, Section 5.11] for more details. Next, repeat the following procedure starting with the graph formed by the planarizing cycles, until the resulting graph becomes connected. Let P be a shortest path in G joining distinct components of the current graph. Then add this path to the current graph, thus reducing the number of components by one. The final resulting subgraph L of G has the property that there is a large wall surrounding it, and inside this wall we can find a cubic subgraph H_0 surrounded by a wall W_0 whose width and height satisfy requirements imposed in the proof of Theorem 1.2.

Similarly to the 5-choosability of arbitrary planar graphs, it can be shown easily that planar graphs of girth at least 4 are 4-choosable, and those of girth at least 6 are 3-choosable. Thomassen [15] strengthened the latter fact by showing that all planar graphs of girth 5 are 3-choosable. These results can be generalized to the setting of locally planar graphs.

Proposition 5.1. *For every surface S there exists an integer w such that every triangle-free graph G embedded in S with edge-width at least w is 4-choosable. Similarly, every graph of girth at least 6 embedded in S with edge-width at least w is 3-choosable.*

Let us observe that the value of w in Proposition 5.1 is of order $O(\log g)$, where g is the genus of S. Compare this with the constant from Theorem 1.2 which is much bigger.

We shall only sketch the proof of Proposition 5.1 since it only needs Euler's formula and an application of a theorem of Gallai. This was actually proved in [5] for usual colorings. Fisk and Mohar proved in [5] that a graph of girth 4 and minimum degree 4 whose edge-width is bigger than $c \log g$ (where c is a constant), contains a vertex of degree 4 contained in four faces of size 4, all of whose vertices have degree 4. Similarly, for girth 6 and minimum degree 3, there is a vertex of degree 3 surrounded by hexagons, all of whose vertices have degree 3. Assuming that the graph G is critical for 4-list or 3-list-colorings (respectively), the list coloring version of Gallai theorem (see [7]) tells us that every block of the subgraph of G induced on vertices of degree 4 or 3 (respectively) is either a clique or an odd cycle. This yields a contradiction to the result of [5] stated above.

Thomassen [15] proved that for each surface S, there are only finitely many 3-critical graphs of girth at least 5 that can be embedded in S. This implies that graphs of large edge-width on S having girth at least 5 are 3-colorable and raises the following question: Is it true that graphs of girth 5 and with sufficiently large edge-width on a fixed surface are 3-choosable?

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