List-Color-Critical Graphs on a Fixed Surface

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Abstract

A *k-list-assignment* for a graph G assigns to each vertex v of G a *list* $L(v)$ of *admissible colors*, where $|L(v)| \geq k$. A graph is *k-list-colorable* (or *k-choosable*) if it can be properly colored from the lists for every k-list-assignment.

We prove the following conjecture posed by Thomassen in 1994: "There are only finitely many list-color-critical graphs with all lists of cardinality at least 5 on any fixed surface." This generalizes the well-known result of Thomassen on the usual graph coloring case. We use this theorem and specific parts of its proof to resolve the complexity status of the following problem about k-list-coloring graphs on a fixed surface S, where k is a fixed positive integer.

Input: A graph G embedded in the surface S.
Question: Is G k-choosable? If not, provide a certificate (a list-color-critical subgraph and the corresponding k-list-assignment).

The cases $k = 3, 4$ are known to be NP-hard (actually even Π_2^p-complete), and the cases $k = 1, 2$ are easy. Our main results imply that the problem is tractable for every $k \geq 5$. In fact, together with our recent algorithmic result, we are able to solve it in linear time when $k \geq 5$. Our proof yields even more: if the input graph is k-list-colorable, then for any k-list-assignment L, we can construct an L-coloring of G in linear time. This generalizes the well-known linear-time algorithms for planar graphs by Nishizeki and Chiba (for 5-coloring), and Thomassen (for 5-list-coloring).

We also give a polynomial-time algorithm to resolve the following question:

Input: A graph G in the surface S, and a k-list-assignment L, where $k \geq 5$.
Question: Does G admit an L-coloring? If not, provide a certificate for this. If yes, then return an L-coloring.

If the graph G is k-list-colorable, then our first result gives a linear time solution. However, the second problem is more general, since it provides a coloring (or a small obstruction) for an arbitrary graph in S.

We also use our main theorem to prove another conjecture that was proposed recently by Thomassen: "For every fixed surface S, there exists a positive constant c such that every 5-list-colorable graph with n vertices embedded on S, has at least $c \cdot 2^n$ distinct 5-list-colorings for every 5-list-assignment for G." Thomassen himself proved that this conjecture holds for usual 5-colorings.

In addition to all these results, we also made partial progress towards a conjecture of Albertson concerning coloring extensions and a progress on similar questions for triangle-free graphs and graphs of larger girth.

Keywords: List-Coloring, Critical Graphs, Surface

1 List-coloring

Graph coloring is arguably the most popular subject in graph theory. Also, it is one of the central problems in combinatorial optimization, since it is one of the hardest problems to approximate. An interesting variant of the classical problem of properly coloring the vertices of a graph with the minimum possible number of colors arises when one imposes some restrictions on the colors or the number of colors available to particular vertices. This variant received a considerable amount of attention by many researchers, and that led to several beautiful conjectures and results. This notion, known as *list-coloring*, was first introduced in the 1970s, in two papers by Vizing [34] and independently by Erdős, Rubin and Taylor [10].

Let G be a graph. A *list-assignment* is a function L which assigns to every vertex $v \in V(G)$ a set $L(v)$ of natural numbers, which are called *admissible colors* for that vertex. An *L-coloring* of G is an assignment of admissible colors to all vertices of G, i.e., a function $c : V(G) \to \mathbb{N}$ such that $c(v) \in L(v)$ for every $v \in V(G)$, such that for every edge uv we have $c(u) \neq c(v)$. If k is an integer and $|L(v)| \geq k$ for every $v \in V(G)$, then L is a *k-list-assignment*. The graph is *k-list-colorable* (or *k-choosable*) if it admits an L-coloring for every k-list-assignment L. If $L(v) = \{1, 2, \ldots, k\}$ for every v, then every L-coloring is referred to as a *k-coloring* of G. If

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G admits an L-coloring (k-coloring), then we say that G is *L-colorable* (*k-colorable*).

The smallest integer k such that G is k-choosable is the *list-chromatic number* $\chi_l(G)$. Clearly, $\chi(G) \leq \chi_l(G)$, and there are many graphs for which $\chi(G) < \chi_l(G)$. A simple example is the complete bipartite graph $K_{2,4}$, which is not 2-choosable. Another well-known example is the complete bipartite graph $K_{3,3}$. In fact, it is easy to show that for every k, there exist bipartite graphs whose list-chromatic number is bigger than k. Let G be a graph and let L be a list assignment for G. We say that G is *L-critical* if G is not L-colorable but every proper subgraph of G is. The graph G is *k-list-critical* if there is a k-list-assignment L such that G is L-critical.

The problem of computing or determining the list-chromatic number of a given graph is notoriously difficult, even for small graphs with a simple structure. One example is that the complete bipartite graph $K_{5,8}$ is 3-choosable, but a proof given in [18] is lengthly and nontrivial. It is shown in [13] that k-list-colorability is Π_2^p-complete for every $k \geq 3$; see also Theorem 1.1. Hence if the complexity classes NP and $coNP$ are different, as is commonly believed, the problem is strictly harder than the NP-complete problems.

Although there are many negative results as stated above, there are some positive results, which are mainly related to the Four Color Theorem. The most celebrated example is Thomassen's result that planar graphs are 5-choosable [27]. Its beautiful short proof gives rise to a linear-time algorithm to 5-list-color planar graphs. In contrast with the Four Color Theorem, there are planar graphs that are not 4-choosable [32].

Let us point out that deciding about the choice number of planar graphs is actually hard. In fact, the following was proved by Gutner [13] (see also survey [33]).

THEOREM 1.1. *The problems of deciding whether a given planar graph is 4-choosable and of deciding whether a given triangle-free planar graph is 3-choosable are both Π_2^p-complete.*

Leaving the plane to consider graphs on surfaces of higher genus, the chromatic and list chromatic number can increase. However, for graphs which obey certain local planarity conditions, one can deduce similar properties as for planar ones. We say that a graph G embedded in a surface S is *locally planar* if it does not contain short non-contractible cycles. Quantitatively, we introduce the *edge-width* of G as the length of a shortest cycle which is non-contractible in S. We also define the *face-width* of G as the minimum number of points of G that some non-contractible closed curve in S intersects. Thomassen proved in [28] that graphs embedded in S with sufficiently large edge-width are 5-colorable. More than 10 years ago, he asked if they are also 5-choosable. This was answered recently in the affirmative in [6].

THEOREM 1.2. (DEVOS, KAWARABAYASHI, MOHAR) *For every surface S there exists a constant w such that every graph that can be embedded in S with edge-width at least w is 5-choosable.*

If G is a graph of girth at least w, then its edge-width in every surface is at least w. For arbitrarily large values of w, there exist graphs of girth w with arbitrarily large chromatic number [9]. Therefore, the constant w in Theorem 1.2 necessarily depends on the surface.

The proof of Theorem 1.2 in [6] uses a result of Robertson and Seymour, whose proof in [21] does not yield an explicit bound on the value of $w = w(S)$ needed in the proof. However, there are more specific results which show that one can take $w = 2^{O(g)}$, where g is the Euler genus of S. See [19, Chapter 5] for more details. Böhme et al. [4] proved that the best possible value of w for the projective plane is 4. Apart from the planar case, this is the only surface, for which the minimum width forcing 5-choosability is known.

THEOREM 1.3. (BÖHME, MOHAR, AND STIEBITZ [4]) *A graph G embedded in the projective plane is 5-choosable if and only if it does not contain K_6 as a subgraph.*

From an algorithmic point of view, the following is perhaps the most interesting question in this area.

CONJECTURE 1.1. *For any fixed surface S, there is a polynomial-time algorithm to decide whether a given graph embedded on S is 5-choosable.*

In general it is hard to provide a certificate for k-choosability since there are exponentionally many distinct k-list-assignments for which list-colorability has to be established. The following conjecture of Thomassen is therefore a key property neeeded for the resolution of Conjecture 1.1.

CONJECTURE 1.2. (THOMASSEN [28]) *For every fixed surface S, there are only finitely many 5-list-critical graphs that can be embedded in S.*

2 Our Main Results

The main purpose of this paper is to prove both Conjectures 1.1 and 1.2, and moreover, provide a linear-time algorithm for deciding 5-choosability of graphs on a fixed surface.

THEOREM 2.1. *For every fixed surface S, there are only finitely many 5-list-critical graphs that can be embedded in S.*

This generalizes a well-known result on the usual graph coloring case proved by Thomassen [28]. We use Theorem 2.1 and the details from its proof to address the complexity status of the following question on list-coloring graphs on a fixed surface S.

Input: A graph G in the surface S.

Question: Is G k-list-colorable? If so, give a k-list-coloring for any prescribed k-list-assignment.

When k is 3 or 4, Theorem 1.1 shows that the problem is NP-hard. Actually, it is Π_2^p-complete, as proved by Gutner [13]. Since the problem is easy when $k \leq 2$, the remaining cases are when $k \geq 5$.

Theorem 2.1 yields a positive answer to Conjecture 1.2. However, our next result yields much better time complexity.

THEOREM 2.2. *For each fixed surface S and each $k \geq 5$, there is a linear-time algorithm to decide about k-list-colorability of any graph G embeddable in S. In fact, if G is not k-list-colorable, then the algorithm returns a certificate (a list-color-critical subgraph) of constant size. Moreover, if the answer is yes, then for any given k-list-assignment L, the algorithm returns an L-coloring of G in linear time.*

This generalizes the well-known linear-time algorithms for planar graphs by Nishizeki and Chiba [20] (for 5-coloring), and by Thomassen [27] (for 5-list-coloring).

We also use Theorem 2.1 to prove the following, more general algorithmic result.

THEOREM 2.3. *Given a graph G in a fixed surface S, and given a k-list-assignment L (with $k \geq 5$), there is a polynomial-time algorithm to decide if G has an L-coloring or not. If the answer is no, then the algorithm gives a certificate of bounded size for this, and if the answer is yes, the algorithm returns a desired L-coloring.*

If the graph G is k-list-colorable, then our first result gives a linear time solution. However, the algorithm in Theorem 2.2 is more general, since it provides a coloring (or a small obstruction) for an arbitrary graph embedded in S.

We also answer a question posed by Thomassen [30, 31]. Namely:

THEOREM 2.4. *For every fixed surface S, there exists a positive constant c such that every 5-list-colorable graph with n vertices on S has at least $c \cdot 2^{n/12}$ distinct 5-list-colorings from any given 5-list-assignment.*

We give an analogous result for k-list-colorings for every integer $k > 5$.

Albertson [1] conjectured that for every surface S there exists an integer $q = q(S)$ such that any graph G embedded in S contains a set U of at most q vertices such that $G - U$ is 4-colorable. Such a result does not hold for list colorings since there exist planar graphs that are not 4-choosable [32]. However, Theorem 2.1 implies such a result for 5-list-colorings. In fact, in [16], it was shown that $q = q(S) \leq 1000g$, where g is the Euler genus of S. Our results yield the same result (with worse upper bound on $q(S)$).

The following question was asked by Albertson [2] (see also Thomassen [28]):

Let G be a planar graph and $W \subseteq V(G)$ such that any two vertices in W have distance at least 100 from each other. If L is a 5-list-assignment for G, can any precoloring of W be extended to an L-coloring of G?

Albertson himself [2] answered this question for the usual graph coloring case. In fact, his theorem says that the distance 4 is enough instead of 100. However, his proof uses the Four Color Theorem, and his proof method breaks down for the list-coloring case.

We partially answer this question. Namely, when $|W| \leq k$, and the distance between any two vertices in W is at least $f(k)$ for some integer value $f(k)$ depending only on k, then the above question has a positive answer. We also generalize this result to graphs on a fixed surface with large edge-width. See Theorem 6.1.

Related Work. In early 1990's, Thomassen formulated the following program concerning study of graph colorings on a fixed surface.

Question 1. Suppose G is embedded into the surface S with large edge-width. What is the chromatic number of G? Is it 5-colorable? Is it even 4-colorable?

Question 2. Is there a polynomial-time algorithm to decide, for a fixed integer $k \geq 4$ and a fixed surface S, whether a given graph G embedded in S is k-colorable?

Question 3. Given the surface S, is the number of $(k+1)$-color-critical graphs embeddable in S finite?

Let us first observe that positive answer to Question 3 for a fixed k would imply both Questions 1 and 2 in the affirmative for the same k. To see this, simply test if the input graph G has a subgraph isomorphic to one of the finitely many $(k+1)$-color-critical graphs. In fact, this algorithm can be implemented to run in linear time using the result that was proved later by Eppstein [11]. Furthermore, when the number of $(k+1)$-color-critical graphs is finite, we also have an explicit bound on the maximum order of such graphs, which only depends on the Euler genus of S, and hence we not only know that the algorithm exists, but we can actually construct it. Note also that the cases $k \leq 2$ for Question 2 are easy, while the case $k = 3$ has negative answer for all questions, if $P \neq NP$, since it is NP-complete to test 3-colorability of planar graphs. Therefore, Questions 1–3 make sense only when $k \geq 4$.

Question 1 was resolved by Thomassen [26] for $k \geq 5$, while the case $k = 4$ is negative, see [19]. Thomassen [28] also answered Question 3 in the affirmative when $k \geq 5$, while the case $k = 4$ is negative, see [19]. Thus Question 2 is also answered in the affirmative when $k \geq 5$. The only remaining case is $k = 4$ of Question 2. When S is the sphere, the result follows from the Four Color Theorem, but the prospects for a general solution are not at all bright, since we would need a far generalization of the Four Color Theorem.

In the late 1990's, Thomassen also posed versions of Questions 1–3 for triangle-free graphs and for graphs of girth at least 5. In [29], he solved all questions for graphs of girth at least 5. Recently, Dvorák, Král and Thomas [8] have settled the second question for all k for triangle-free graphs.

In the middle 1990's, Thomassen has also proposed to study Questions 1–3 for list-colorings. In [27] he proved that every planar graph is 5-list-colorable, and Voigt [32] has constructed a planar graph that is not 4-list-colorable. So, Questions 1 and 3 make sense only when $k \geq 5$ or $k \leq 2$. Also by Theorem 1.1, the cases $k = 3, 4$ for Question 2 do not make sense, while the cases $k = 1, 2$ are easy by the result in [9]. Question 1 was answered in the affirmative recently, see Theorem 1.2. There is another direction for these problems. Kawarabayashi and Thomassen [16] proved that every graph on the surface S of Euler genus g has a vertex set X of order at most $1000g$ such that $G - X$ is 5-list-colorable.

In this paper, together with the previous results, we completely solve Questions 2 and 3 for list-coloring. In the last section, we shall discuss Questions 1–3 for list-colorings of triangle-free graphs and graphs of girth at least 5.

An overview of our approach. All algorithmic applications will use Theorem 2.1. Its lengthy proof also shows how the algorithms work. So we need to give an overview for the proof of this theorem. In the next section, we shall discuss the algorithmic applications towards Theorems 2.2 and 2.3.

Let G be a graph embedded on a fixed surface S, whose Euler genus is g. If G has edge-width at least $w(g)$, as in Theorem 1.2, then G is 5-list-colorable by Theorem 1.2. Thus the edge-width of G is "small" when G is not 5-list-colorable. To handle graphs of small edge-width, we will cut the graph G and the surface S along shortest non-contractible cycles. That simplifies the surface, but duplicates some vertices, and we have to make sure that the two copies of the same vertex receive the same color. In other words, we have to extend Theorem 1.2 to graphs that are partially precolored. To formalize this approach, we introduce the following definition. For $i = 1, \dots, l$, let $C_i \subseteq V(G)$ be a set such that all vertices in C_i lie on the boundary of some face F_i, where $F_1, \dots, F_l$ are pairwise distinct faces. We call $C_1, \dots, C_l$ *cuffs*, since one can make them to lie on distinct boundary components, after cutting holes in $F_1, \dots, F_l$. We say that G is a *cuffed graph* with cuffs $C_1, \dots, C_l$ if G is embedded into S and the cuffs are disjoint. We define the *breadth* of the cuffed graph G as the sum $|C_1| + \dots + |C_l|$.

We will show that if G is a cuffed graph of breadth $f(l)$ with the cuffs $C_1, \dots, C_l$, the edge-width is sufficiently large, and in addition, the cuffs $C_1, \dots, C_l$ are pairwise "far apart", then any precoloring of $C_1 \cup \dots \cup C_l$ extends to an L-coloring of G (if the lists have size at least 5), unless there is a local obstruction for such a coloring extension. To clarify what "far apart" means, we need the metric developed by Robertson and Seymour [23, 24] in the Graph Minors Project. Roughly speaking, if S is not sphere and the face-width is large, then the metric is as follows: for two points a, b on the surface, the distance between a and b is small if there is a contractible closed curve J that contains both a and b in its (closed) interior and intersects the graph only in a small number of points. If S is the sphere, then this metric is close to the usual "face-distance", except that we also consider contractible curves J with a, b in the different component of $S - J$ into account.

If some pair of the cuffs are not "far apart", then we just take a shortest curve J (certifying for them to be close), cut along J, and either merge the two cuffs or simplify the problem by either reducing the genus or the number of cuffs. If the face-width is small, then we can simplify the surface. In both cases, we can simplify the cuffed graph. Since Euler genus is fixed, therefore, if we simplify the cuffed graph as much as possible, then the resulting graph G' is a cuffed graph of breadth $f(l)$ with the cuffs $C_1, \dots, C_{l'}$, where l' is bounded in terms of l and g, and the edge-width is sufficiently large (or the resulting graph is planar). In addition, the cuffs $C_1, \dots, C_{l'}$ are pairwise far apart.

In summary, the proof proceeds as follows:

1. First we simplify the cuffed graph as much as possible, keeping the breadth and the number of cuffs bounded.

2. Then we prove the resulting cuffed graph has an L-coloring that extends the given precoloring of the cuffs $C_1, \dots, C_{l'}$, unless there is a local obstruction from the very beginning.

In the next section, we provide detailed statement of both results. In both cases, the proofs are algorithmic.

Using a recent result by the authors [14], we can actually implement all the steps in linear time. This will be discussed in Section 4. In Section 5, we prove Theorem 2.4 using Theorem 2.1 and its proof. In Section 6, we relate our proof to Albertson's conjecture.

3 Statements of the Main Results

As discussed in the previous subsection, we shall now provide two results that correspond to each step in the previous subsection. For notation not defined here, we refer the reader to the Appendix.

THEOREM 3.1. *For any non-negative integer g and positive integers q, c, there exist natural numbers $f(g, c, q)$, $l(g, c)$ and $r(g, c)$ satisfying the following. Suppose that G is a graph embedded on a surface S of Euler genus g, and H is a subgraph of G with at most q vertices and having at most c connected components. Then there is a subgraph H' of G with at most $f(g, c, q)$ vertices* *such that $H \subseteq H'$ and such that for every precoloring $c_0 : V(H) \to \mathbb{N}$ and every 5-list-assignment L for $G - V(H)$, one of the following holds:*

(1) *c_0 cannot be extended to an L-coloring of H'.*

(2) *c_0 extends to an L-coloring of H', and in addition, every vertex v in $G - H'$ is either joined to at most two colors in its list or v is a vertex of degree 5 such that two of its neighbors receive the same color under the L-coloring of H'. Furthermore, the following properties are satisfied:*

   (a) In $G - H'$, there are disjoint cuffs $C_1, \ldots, C_l$ (with $l \leq l(g, c)$) such that all vertices in $G - H'$ that are joined to a vertex of H' are in one of the cuffs $C_1, \ldots, C_l$.

   (b) Any two cuffs among $C_1, \ldots, C_l$ have distance at least $r(g, c)$.

   (c) If the induced embedding of $G - H'$ is into a surface of positive Euler genus, then the edge-width of $G - H'$ is at least $r(g, c)$.

   (d) If the induced embedding of $G - H'$ is into a surface of positive Euler genus, then there is no contractible cycle C' of order at most $r(g, c)$ with $int(C')$ containing at least one of the cuffs $C_1, \ldots, C_l$ (except when two cuffs C_i and C_j induce a cylinder; in this case, we allow such a cycle C' in the cylinder).

   (e) If the induced embedding of $G - H'$ is into the sphere S, then there is no contractible cycle C' of order at most $r(g, c)$ such that one component of $S - C'$ contains exactly one of the cuffs $C_1, \ldots, C_l$. There is one exception to this condition, when one of the components is a cylynder with two two cuffs C_i and C_j; in this case, we allow such a cycle C' that separates the two cuffs.

   (f) If the induced embedding of $G - H'$ is into a surface of positive Euler genus, then there is no path P of length at most $r(g, c)$ joining two vertices of one of the cuffs, say C_i, such that the path P together with the cuff C_i contains a non-contractible cycle.

Note that the quantifiers in Theorem 3.1 give the strongest possible version – the same graph H' serves for all precolorings of H and for all 5-list-assignments of $G - V(H)$.

The second result is the following. Again, for notation not defined here, we refer the reader to the appendix. We say that G is *of type* (g, l, w, t) with $w \geq t$ if G satisfies the following:

1. G is embedded in a surface of Euler genus g.
2. G is a cuffed graph with the cuffs $C_1, \ldots, C_l$.
3. The face-width of this embedding of G is at least w (if $g > 0$).
4. $d(C_i, C_j) \geq t$ for any i, j with $i \neq j$, where d is the Robertson-Seymour metric (as defined in the appendix).

Proposition 3.1. *For any non-negative integers g and l, there are integers $w(g, l)$ and $t(g, l)$, where $w(g, l) \geq t(g, l)$, satisfying the following. Suppose G is of type $(g, l, w(g, l), t(g, l))$. Suppose furthermore that each vertex in any of the cuffs C_i $(1 \leq i \leq l)$ has a list with at least three available colors, and every vertex not on the cuffs has a list with at least 5 available colors. Then G has an L-coloring.*

We now clarify how Theorem 3.1 and Proposition 3.1 imply Theorem 2.1.

Suppose that G is a 5-list-critical graph on the surface S, and has at least $f(g - 1, 2, 2w(g))$ vertices, where the function f comes from Theorem 3.1, and $w(g)$ comes from Theorem 1.2. If the edge-width of G is at least $w(g)$, then by Theorem 1.2, G is 5-list-colorable, a contradiction. Therefore, the edge-width of G is small. We take a shortest non-contractible cycle C, and cut the graph G and the surface S along C (cf. [19]). In the resulting graph G', the cycle C corresponds to a cycle C' with $2|C|$ vertices (if C is one-sided) or to two cycles, which we call C' and C'' (if C is two-sided). Then the resulting graph G' has smaller Euler genus, and there are one or two cuffs C', C'' obtained from C, containing together $2|C|$ vertices. So this procedure simplifies the surface on the expense of adding cuffs.

For each L-coloring of C, we apply Theorem 3.1 to G' and $H = C' \cup C''$ (if C is two-sided) or $H = C'$ (if C is one-sided). Since G is 5-list-critical and has at least $f(g - 1, 2, 2w(g))$ vertices, Theorem 3.1 implies that there is a subgraph H' of G' such that any coloring c_0 of H extends to an L-coloring of H', and H' satisfies the second conclusion of Theorem 3.1. In particular, in $G - H'$, each vertex in any of the cuffs $C_1, \ldots, C_l$ has a list with at least three available colors. We now prove that $G - H'$ satisfies the assumptions in Proposition 3.1. We fist observe that $G - H'$ may consist of more than two components. In this case, we shall apply Proposition 3.1 to each component. So, hereafter, we assume that $G - H'$ is connected. Note that the conditions (a)–(f) imply that $d(C_i, C_j) \geq t(g, l)$ for any two distinct cuffs. The only thing we need to verify is the face-width condition.

If $G - H'$ is planar, then clearly $G - H'$ gives rise to the assumption in Proposition 3.1.

Suppose $G - H'$ is embedded into a surface of positive Euler genus g. We only need to verify the face-width condition in Proposition 3.1. Right now, the edge-width of $G - H'$ is at least $r(g, c)$. By applying a method developed in [6], we can ensure that this is also possible.

Since Proposition 3.1 implies that there is an L-coloring in $G - H'$ (and hence G is 5-list-colorable, a contradiction), Theorem 3.1 and Proposition 3.1 imply that G has at most $f(g-1, 2, w(g))$ vertices. Thus, Theorem 2.1 follows.

4 Algorithmic results: Linear time algorithms and another polynomial-time algorithm

We first prove the following result, using Theorem 2.1.

THEOREM 4.1. *For $k \geq 5$, there is a linear-time algorithm for the following problem.*
Input: *A graph G in the surface S.*
Question: *Is G k-list-colorable? If not, provide a certificate for this. If yes, then given a k-list-assignment L, return an L-coloring of G.*

Proof. For simplicity, we only consider the case $k = 5$, since this case is the hardest. Some more details for the other cases, when $k \geq 6$, will be presented towards the end of the paper.

We first apply Theorem 2.1 to give a linear-time algorithm to answer the decision problem. By Theorem 2.1, there are at most $f(g)$ list-critical graphs with all lists of cardinality 5 on the surface S of Euler genus g. For each such list-color-critical graph H, we test if G contains a subgraph isomorphic to H. We can implement this task to run in linear time using a result of Eppstein [11]. If G contains one of them, then we output this subgraph, and the answer is clearly that G is not 5-list-colorable.

Suppose now that G contains none of them. Then G is 5-list-colorable. Given a 5-list-assignment L, we want to list-color the graph G in linear time in order to fulfill our second task. By Theorem 3.1 (with $q = c = 0$), there is a subgraph H' of G that satisfies the second conclusion, i.e, there is an L-coloring c' of H' that satisfies (a)–(f) of Theorem 3.1. Our algorithm proceeds by finding this subgraph H' and the coloring c' of H'.

In order to get H', we need the following subroutine that is provided in [14].

THEOREM 4.2. *Suppose G is embedded into a fixed surface S. For any fixed k, there is a linear-time algorithm to decide if the face-width is at least k. If the face-width is at most k, then the algorithm finds a non-contractible curve intersecting G in at most k points.*

Our algorithm repeatedly applies Theorem 4.2, and an algorithm to find a shortest path between two points in G. In our proof of Theorem 3.1, we only need to find a path of length at most $r(g, c)$ between two points, and a non-contractible cycle of length at most $r(g, c)$ in the surface. So these processes can be done in linear time by repeatedly applying Theorem 4.2.

After applying the two operations introduced above at most $4g$ times, we can clearly get the subgraph H' that has at most $f(g)$ vertices, where $f(g)$ is appropriate constant. Since H' has at most $f(g)$ vertices, we can provide all possible L-colorings for H' in constant time (by brute force enumeration). It follows from Theorem 3.1 that there is at least one L-coloring c' that satisfies (a)–(f) of Theorem 3.1.

As discussed in the previous section, $G - H'$ satisfies the assumptions in Proposition 3.1. We now apply Proposition 3.1 to get an L-coloring of the whole graph G that extends some precoloring c_0 of H'. To do so, we need the following subroutine, Theorem 4.3, to get a minor W'' (as described in Theorem 4.3) rooted at the vertices obtained from the cuffs by contracting each of them into a single vertex. This result is also provided in [14].

THEOREM 4.3. *For any non-negative integers g, k, there are constants $f(g, k)$, $t(g, k)$ and $w(g, k)$ satisfying the following. Suppose G is embedded into a surface S of Euler genus g with face-width at least $w(g, k)$. Let $\{v_1, \ldots, v_k\}$ be vertices in G such that $d(v_i, v_j) \geq t(g, k)$ if $i \neq j$, where d is the Robertson-Seymour metric. Then for any fixed cuffed graph W' embedded in S and having at most $f(k, g)$ vertices, there is a linear-time algorithm to find a rooted minor W'' in G with roots $\{v_1, \ldots, v_k\}$ and isomorphic to W'.*

The whole argument in the proof of Proposition 3.1 can be translated into a linear-time algorithm once we get the rooted minor, since we can reduce to problems on planar graphs, and this case was done by Thomassen [27, 30]. These problems can be solved in linear time. This completes the proof of Theorem 4.1. $\square$

We also prove the following theorem, using Theorem 2.1. As pointed out in the introduction, the following theorem solves a more general problem than Theorem 4.1, on the expense of losing linear time complexity.

THEOREM 4.4. *Let S be a surface. For every $k \geq 5$, there is a polynomial-time algorithm for the following problem.*
Input: *A graph G embedded in S and a k-list-assignment L.*
Task: *Is G L-colorable? If not, provide a certificate for this (an L-critical subgraph of constant size). If yes, then return an L-coloring of G.*

Proof. For simplicity, we only consider the case $k = 5$, since other cases are easier. (We shall discuss the cases $k \geq 6$ towards the end of the paper.)

We first apply Theorem 2.1 to obtain a polynomial-time algorithm to answer the decision problem. By Theorem 2.1, there are at most $f(g)$ list-critical graphs with all lists of cardinality 5 on the surface S.

The following lemma, whose proof easily follows from Theorem 2.1, is useful for the proof.

LEMMA 4.1. *For every surface S of Euler genus g and any $k \geq 5$, every list-color-critical graph G with all lists of cardinality k on S has at most $t(g, k)$ (non-isomorphic) descriptions of k-list-assignments for which G is not k-list-colorable, where $t(g, k)$ is appropriate integer value depending on g and k only.*

For each list-color-critical graph H, we first figure out which 5-assignments have no valid coloring for the subgraph H. By Lemma 4.1, there are at most $t(g,k)$ descriptions of 5-list-assignments for which H is not 5-list-colorable. Therefore, we can provide all 5-assignments that have no valid coloring for it.

For an input graph G and an input 5-list-assignment L, we test if G contains a list-color-critical graph H with a 5-list-assignment that has no L-coloring. If it does, then this is certainly a good certificate for non-colorability of G, and we output this subgraph with its 5-list-assignment.

Suppose now that G is L-colorable, and we have to find an L-coloring. It follows from Theorem 3.1 (with $q = c = 0$) that there is a subgraph H' of G that satisfies the second conclusion, i.e., there is an L-coloring c_0 of H' that satisfies (a)–(f) of Theorem 3.1. Our algorithm follows the proof of Theorem 3.1 to find this subgraph H' and its coloring c'. Then we use Proposition 3.1 to extend the coloring of c' to the whole graph G.

The rest of the proof is exactly the same as that of Theorem 4.1, so we omit the details. □

5 Exponentially many 5-list-colorings

We now describe a proof of Theorem 2.4. Our proof uses similar ideas as that of Thomassen [31]. Suppose G is 5-list-colorable and embedded into a surface of Euler genus g. It follows from Theorem 3.1 (with $q = c = 0$) that there is a subgraph H' of G satisfying (a)–(f) of Theorem 3.1. Moreover, since H' has at most $f(g,0,0)$ vertices, it follows that the number of vertices in one of the cuffs $C_1,\dots,C_l$ that have only three available colors is at most $f(g,0,0)$.

When we prove Proposition 3.1, we find a subgraph Q of G which contains all the cuffs $C_1,\dots,C_l$, such that $G-Q$ can be embedded into a disk. In addition, each vertex of $G-Q$ that has a neighbor in Q is in the outer face boundary of $G-Q$. Moreover, we prove that there is a valid coloring of Q such that every vertex in the outer face boundary of $G-Q$ has a list with at least three available colors. By taking $r(g,0)$ in Theorem 3.1 and $w(g,l)$ in Proposition 3.1 large enough, we can prove that $|Q| \le |G|/101$. All arguments needed to get such a conclusion are given in [5]. It follows that there are at most $2|G|/101$ vertices in the outer face boundary of $G-Q$ that have only three available colors.

In [31], Thomassen proved the following theorem.

THEOREM 5.1. *Let G be a planar near-triangulation with the outer face boundary C. Let n be the number of vertices of G. Suppose that every vertex in $G-C$ has a list with five available colors, and that every vertex in C has a list with at least three available colors. Let r be the number of vertices of C with precisely three available colors. Then G has at least $2^{n/9-r/3}$ distinct L-colorings.*

Since we fix the coloring of Q, which has at most $|G|/101$ vertices, and there are at most $2|G|/101$ vertices in the outer face boundary of $G-Q$ that have only three available colors, it follows from Theorem 5.1 that G has at least $2^{9|G|/101-2|G|/303}$ distinct L-colorings. This proves Theorem 2.4.

6 List-coloring extension

We now apply Proposition 3.1 to give a partial answer to Albertson's problem. In fact, we can prove the following stronger theorem.

THEOREM 6.1. *For any non-negative integers g,l, there are integers $w'(g,l)$ and $t'(g,l)$ satisfying the following. Suppose G is embedded into a surface of Euler genus g with face-width at least $w'(g,l)$, and let L be a 5-list-assignment for G. Let $e_1,\dots,e_l$ be edges of G such that for any two edges e_i,e_j (with $i \neq j$), $d(e_i,e_j) \ge t'(g,l)$, where d is the Robertson-Seymour metric. Then for any precoloring c_0 of the endvertices of $e_1,\dots,e_l$, there is an L-coloring of G that extends the precoloring c_0.*

Proof. Set $w(g,l) = w'(g,l) + 2l$ and $t(g,l) = t'(g,l) + 2$ in Proposition 3.1. The graph $G' = G - \{e_1,\dots,e_l\}$ clearly satisfies the assumptions in Proposition 3.1. Moreover, each vertex in one of the cuffs $C_1,\dots,C_l$ in G' has a list with three available colors. Thus Theorem 6.1 follows from Proposition 3.1. □

7 Concluding remarks

It is not hard to see that Euler's formula and an application of a theorem of Gallai implies that there are only finitely many list-color-critical graphs with all lists of cardinality 6 on a fixed surface, see [15].

Similarly to the 5-choosability of arbitrary planar graphs, it can be shown easily that planar graphs of girth at least 4 are 4-choosable, and those of girth at least 6 are 3-choosable. Thomassen [29] strengthened the latter fact by showing that all planar graphs of girth 5 are 3-choosable. These results can be generalized to the setting of locally planar graphs. In fact, we can prove the following stronger results.

THEOREM 7.1.
1. *There are only finitely many k-list-critical triangle-free graphs on a fixed surface for $k \ge 4$. Consequently, there is a polynomial-time algorithm to decide, for any triangle-free graph on a fixed surface, whether or not it is k-list-colorable for $k \ge 4$.*
2. *There are only finitely many k-list-critical graphs of girth at least 6 on a fixed surface S for $k \ge 3$. Also, there is a polynomial-time algorithm to decide, for any graph of girth at least 6 on S, whether or not it is k-list-colorable for $k \ge 3$.*

We shall only sketch the proof of Theorem 7.1 since it only needs Euler's formula and an application of a theorem of Gallai.

Assuming that the graph G is critical for 4-list or 3-list-colorings (respectively), the list coloring version of

Gallai's theorem (see [17]) tells us that every block of the subgraph of G induced by vertices of degree 4 or 3 (respectively) is either a clique or an odd cycle. This already yields a contradiction if G is large since girth is at least 4 or 6, respectively, and the Euler formula tells us that there are $n - O_g(1)$ vertices of degree 4 or 3, respectively, where n is the number of vertices of G.

This theorem implies the following.

Corollary 7.1. *1. There is a polynomial-time algorithm to approximate the list-chromatic number of any triangle-free graph on a fixed surface, with the error of the approximation never exceeding one.*

2. There is a polynomial-time algorithm to determine the list-chromatic number $\chi_l(G)$ of an arbitrary graph G of girth at least 6 on any fixed surface.

The first corollary is essentially best possible in a sense since it is Π_2^p-complete to decide whether or not a given triangle-free planar graph is 3-list-colorable, see [33].

Thomassen [29] proved that for each surface S, there are only finitely many 4-critical graphs of girth at least 5 that can be embedded in S. This implies that graphs of large edge-width on S having girth at least 5 are 3-colorable and raises the following question: *"Is it true that graphs of girth 5 and with sufficiently large edge-width on a fixed surface are 3-choosable?"*

Appendix

Definitions and Preliminaries

Basic notation. For notation not defined here, we refer to the book [19]. But for the sake of completeness, let us repeat some important definitions.

A *surface* is a compact connected 2-manifold without boundary. We assume familiarity with basic notions of surface topology, like genus and Euler's formula. We define the *Euler genus* of a surface S as $2-\chi(S)$, where $\chi(S)$ is the Euler characteristic of S. An *arc* in S is subset of S homeomorphic to $[0, 1]$. An *O-arc* is a subset of S homeomorphic to a circle.

A graph G is *embedded* in a topological space X if the vertices of G are distinct elements of X and every edge of G is simple arc connecting the two vertices in X which it joins in G, such that its interior is disjoint from other edges and vertices. *Embedding* of a graph G in the topological space X is an isomorphism of G with a graph G' embedded in X. In this case, G' is said to be a representation of G in X. If there is an embedding of G into X, we say that G *can be embedded* into X.

Let G be a graph that is embedded in a surface S. To simplify notation we do not distinguish between a vertex of G and the point of S used in the embedding to represent the vertex, and we do not distinguish between an edge and the arc on the surface representing it. We also consider G as the union of the points corresponding to its vertices and edges.

A *region* or *face* of G in S is a connected component of $S \setminus (E(G) \cup V(G))$. Every region is an open set. We use the notation $F(G)$ for the set of regions of G. The embedding is said to be a *2-cell embedding* if every region is homeomorphic to a disc. In that case, the boundary of every region r can be represented by a closed walk in the graph, called a *facial walk* of r.

If C is a contractible cycle in a graph on a surface, then $int(C)$ denotes the set of vertices and edges inside the disk bounded by C (but not on C). If S is the sphere, the disk bounded by C is not uniquely defined. In this case, we fix a point in $S \setminus G$ and ask that the disk does not contain that point.

If G is a graph and A is a set of vertices of G, then $G(A)$ is the subgraph of G induced by A, that is, its vertex set is A and its edge set consists of all edges in G joining two vertices of A. The *edge-width* of an embedded graph G is the length of a shortest non-contractible cycle, and the *face-width* is the smallest possible cardinality of the intersection of G with a non-contractible curve on the surface. A shortest path in a graph is also called a *geodesic*.

If G is a 2-connected plane graph such that every facial cycles, except for possibly one, is a triangle, then G is a *near-triangulation*.

L-critical graphs. Let G be a graph and let L be a list-assignment for G. We say that G is *L-critical* if G is not L-colorable but every proper subgraph of G is. The graph G is *k-list-critical* if there is a k-list-assignment L such that G is L-critical.

If a vertex v in a colored (or partially colored) graph G is joined to vertices $v_1, \ldots, v_r$ of colors $c(v_1), \ldots, c(v_r)$ respectively, then we shall also say that v is *joined to the colors* $\{c(v_1), \ldots, c(v_r)\}$.

Robertson-Seymour metric and cuffed graphs on a surface. In our proofs we use the notion of a radial graph. Informally, the *radial graph* of a graph G that is 2-cell embedded in a surface is the bipartite graph R_G obtained by selecting a point in each region $r \in F(G)$ and connecting it to every vertex of G encountered on the facial walk of r. Note that we get multiple edges between r and a vertex v if v appears more than once on the facial walk of r.

Let $A(R_G)$ be the set of vertices, edges, and regions (collectively, *atoms*) in the radial graph R_G. According to Section 9 of [23] (see also [24]), the existence of a respectful tangle of order θ makes it possible to define a metric d on $A(R_G)$ as follows:

1. If $a = b$, then $d(a, b) = 0$.
2. If $a \neq b$, and a and b are interior to a contractible closed walk of the radial graph R_G of length $< 2\theta$, then $d(a, b)$ is half the minimum length of such a walk (here by *interior* we mean the direction in which the walk can be contracted).
3. Otherwise, $d(a, b) = \theta$.

We use these metrics very often in our proofs. We call it the *Robertson-Seymour metric*.

If H_1, H_2 are two disjoint subgraphs in a graph G, then the *Robertson-Seymour distance* $d(H_1, H_2)$ between them is the minimum value of $d(a, b)$, where a is a vertex of H_1 and $b \in V(H_2)$.

Let H be a graph which is 2-cell embedded in a surface S of Euler genus g. Suppose we specify l vertices $v_1, \ldots, v_l$. We say that H' *can be obtained from* H *with rooted vertices* $v_1, \ldots, v_l$ if the graph H' can be obtained from H by subdividing l edges once (let $v'_1, \ldots, v'_l$ be the vertices of degree 2 obtained from this subdividing) and add edges $e_1, \ldots, e_l$ such that the endpoints of e_i are v'_i and v_i, can be 2-cell embedded into the same surface S. So every vertex of H' has degree 3, except for $v_1, \ldots, v_l$, each of which has degree exactly 1. Let W' be a wall of height 100 and width 100 around H'. Note that a wall of height 100 and width 100 is around each rooted vertex.

We are now ready to state the rooted version of Robertson and Seymour's theorem that for any fixed graph W embedded in S, every graph that is embedded in S with large enough face-width contains W as a surface minor.

THEOREM 7.2. ([21]) *Let S be a surface of Euler genus $g \geq 0$, and let W be a cubic graph that is embedded in S. Then for any nonnegative integers l, there are functions $w(g, l)$ and $t(l)$ (with $w(g, l) \geq t(l)$) satisfying the following. Suppose G is of type $(g, l, w(g, l), t(l))$. Let G' be the graph obtained from G by contracting each cuffs C_i into a single point v_i. Then G' contains a subgraph W_0 which is isomorphic to a subdivision of W' obtained from W with rooted vertices $v_1, \ldots, v_l$, and whose induced embedding is combinatorially the same as the embedding of W'.*

Metric on planar graphs. If the surface S is the sphere, there are some complications which require slightly different approach and different notion of the metric d.

The main problem with the sphere case is the lack of unique "interior". This does not allow us to define a unique respectful tangle, and hence we cannot define the Robertson-Seymour metric. We now look at the case when a given graph is planar. In this case, we only look at a cuffed graph G, with cuffs $C_1, \ldots, C_l$.

Suppose the surface S is sphere. Then $d(C_i, C_j)$ $(i \neq j)$ would be the minimum of the following two values.

1. The length of the shortest I-arc between C_i and C_j.
2. The length of the shortest O-arc J such that C_i and C_j are in different components of $S - J$.

This metric allows us to give the sphere case of Theorem 7.2, see [21].

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