Approximation Algorithms via Contraction Decomposition

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Abstract

We prove that the edges of every graph of bounded (Euler) genus can be partitioned into any prescribed number k of pieces such that contracting any piece results in a graph of bounded treewidth (where the bound depends on k). This decomposition result parallels an analogous, simpler result for edge deletions instead of contractions, obtained in [Bak94, Epp00, DDO+04, DHK05], and it generalizes a similar result for "compression" (a variant of contraction) in planar graphs [Kle05]. Our decomposition result is a powerful tool for obtaining PTASs for contraction-closed problems (whose optimal solution only improves under contraction), a much more general class than minor-closed problems. We prove that any contraction-closed problem satisfying just a few simple conditions has a PTAS in bounded-genus graphs. In particular, our framework yields PTASs for the weighted Traveling Salesman Problem and for minimum-weight c-edge-connected submultigraph on bounded-genus graphs, improving and generalizing previous algorithms of [GKP95, AGK+98, Kle05, Gri00, CGSZ04, BCGZ05]. We also highlight the only main difficulty in extending our results to general H-minor-free graphs.

1 Introduction

A fundamental way to design graph algorithms is decomposition or partitioning of graphs into smaller pieces. Lipton and Tarjan's divide-and-conquer separator decomposition for planar graphs [LT80] (generalized to arbitrary graphs via sparsest cut [ARV04, LR99]) is one of the most famous such decompositions. The main technique in these decompositions is to find relatively small cuts in the graph that minimize the interaction between the pieces. To make the pieces relatively small, the decompositions cut the graph into many pieces. An alternative approach of recent study is to partition the graph into a small number of computationally simpler (but not necessarily small) pieces, allowing large interaction between the pieces. For instance, we can solve many optimization problems efficiently on graphs of bounded treewidth. If a graph can be partitioned into a small number s of bounded-treewidth pieces, then in many cases, each piece gives a lower/upper bound on the optimal solution for the entire graph, so solving the problem exactly in each piece gives an s-approximation to the problem. Many NP-hard optimization problems are now solved in practice using dynamic programming on low-treewidth graphs—see, e.g., [Bod05, Ami01, Tho98]—so such a partition into bounded-treewidth graphs may also be practical. Recently, this decomposition approach has been successfully used to obtain constant-factor approximations for many graph problems, including a 2-approximation for graph coloring in any H-minor-free graph family [DHK05] (a problem which on general graphs is inapproximable within $n^{1-\varepsilon}$ for any $\varepsilon > 0$ unless ZPP = NP [FK98]).

A generalization of this decomposition approach leads to PTASs for many minimization and maximization problems, such as vertex cover, minimum color sum, and hereditary problems such as independent set and max-clique [Bak94, Epp00, DHK05]. The idea is to partition the vertices or edges of the graph into a small number k of pieces such that deleting any one of the pieces results in a bounded-treewidth graph (where the bound depends on k). Such a decomposition is known for planar graphs [Bak94], bounded-genus graphs [Epp00] (conjectured by Thomas [Tho95]), apex-minor-free graphs [Epp00], and H-minor-free graphs [DDO+04, DHK05].

This decomposition approach is effectively limited to problems whose optimal solution only improves when deleting edges or vertices from the graph. The bidimensionality theory introduced by Demaine, Fomin, Hajiaghayi, and Thilikos (see, e.g., [DFHT05a, DFHT05b, DFHT04]) highlights *contracted-closed* problems, whose optimal solution only improves when contracting edges, including classic problems such as dominating set (and its variations), minimum chordal completion, and the Traveling Salesman Problem (TSP). Indeed, these results are motivated by deletions and contractions being the basic operations of graph minors and thus algorithmic graph minor theory.

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Motivated by the applications to approximation algorithms for contraction-closed problems, as well as basic questions in structural graph minor theory, we find a new kind of decomposition problem: can the edges of a graph be partitioned into a small number k of pieces such that contracting any one of the pieces results in a bounded-treewidth graph (where the bound depends on k)? Recently, Klein [Kle05, Kle06] proved such a result for planar graphs with a variation of contraction called compression (deletion in the dual graph). However, no such decomposition result is known for more general graphs.

In this paper, we prove such a contraction decomposition result for bounded-genus graphs, paralleling the edge-deletion decompositions of [Epp00, DDO$^+$04, DHK05]. Our construction is much more difficult than what was required for the edge-deletion decomposition, using advanced techniques from topological graph theory for graphs on bounded-genus surfaces [BMR96, MT01, Moh01]. In particular, the type of "surgery" that we apply to the surface is quite different from the simpler surgery performed in previous algorithmic papers on this topic; see, e.g., [DHT06, DFHT05b, FT04]. Indeed, even the planar case requires significant new insights, and is completely different from the deletion case.

Our result gives a general approach for developing approximation algorithms on bounded-genus graphs for many graph problems that are closed under contractions. For example, we obtain a PTAS for weighted TSP in bounded-genus graphs, improving on the quasi-polynomial-time approximation scheme (QPTAS) for this problem (and solving an open problem) by Grigni [Gri00]. Indeed, TSP is a classic problem that has served as a testbed for almost every new algorithmic idea over the past 50 years, and it has been considered extensively in planar graphs and its generalizations, starting with a PTAS for unweighted planar graphs [GKP95] and a PTAS for weighted planar graphs [AGK$^+$98] (recently improved to linear time [Kle05]). Our result can also be viewed as a generalization of these results. Furthermore, we obtain a PTAS for minimum-weight c-edge-connected submultigraph[1] in bounded-genus graphs, for any constant $c \geq 2$, which generalizes and improves previous algorithms for $c = 2$ on planar graphs [BCGZ05, CGSZ04]. We also extend our results in Section 4 toward general H-minor-free graphs, where significant additional difficulties arise, and we show how to solve all but one.

[1]This problem allows using multiple copies of an edge in the input graph—hence sub*multi*graph—but the solution must pay for every copy.

Bounded-genus graphs have been studied extensively in the algorithms community; see, e.g., [CM05, DFHT05b, DHT06, DFT06, FT04, GHT84, Kel06, Moh99]. One attraction of this graph class is that it includes every graph, using a sufficiently large bound on the genus.

1.1 Our Results. First we state our decomposition result, whose proof is deferred to Section 3. See Section 2 for relevant definitions.

Theorem 1.1. *For a fixed genus g, and any integer $k \geq 2$ and for every graph G of Euler genus at most g, the edges of G can be partitioned into k sets such that contracting any one of the sets results in a graph of treewidth at most $O(g^2 k)$. Furthermore, such a partition can be found in $O(g^{5/2} n^{3/2} \log n)$ time.*

The following theorem describes a general family of PTASs for minimization problems on edge-weighted graphs. We include the proof to illustrate the power of Theorem 1.1. Define the *weight* $w(G)$ of a graph G with given edge weights to be the total weight of the edges of G. A minimization problem is *closed under contractions* if the optimal solution value after any edge contraction in G is at most the optimum solution value for G.

Theorem 1.2. *Consider a minimization problem P on weighted graphs that is closed under contractions, solvable in polynomial time on graphs of bounded treewidth, and satisfying the following properties:*

1. *There is a polynomial-time algorithm that, given a bounded-genus graph G and constant $\delta > 0$, computes a bounded-genus graph G' such that $\mathrm{OPT}(G') \geq \alpha \cdot w(G')$, for some constant $\alpha > 0$ (possibly depending on δ), and any c-approximate solution to G' can be converted into a $(1+\delta)c$-approximate solution to G in polynomial time. (G' is called a (δ, α)-*spanner *of G.)*
2. *There is a polynomial-time algorithm that, given a subset S of edges of a graph G, and given an optimal solution for G/S, constructs a solution for G of value at most $\mathrm{OPT}(G/S) + \beta\, w(S)$ for some constant $\beta > 0$.*

Then, for any fixed genus g and any $0 < \varepsilon \leq 1$, there is a polynomial-time $(1+\varepsilon)$-approximation algorithm for problem P in graphs of genus at most g.

Proof. We apply Property 1 to obtain a (δ, β)-spanner G' of G, which also has bounded genus. Then we apply Theorem 1.1, with a value of k to be determined later, to obtain a partition of the edges of G' into sets $S_1, S_2, \ldots, S_k$. For some i, S_i has weight $w(S_i) \le \frac{1}{k}\, w(G')$. The contracted graph G'/S_i has bounded treewidth and thus we can compute an optimal solution $\mathrm{OPT}(G'/S_i)$ in polynomial time. We apply Property 2 with $S = S_i$ to obtain a solution for G' whose value is at most $\mathrm{OPT}(G'/S_i) + \beta\, w(S_i)$. Because P is closed under contractions, $\mathrm{OPT}(G'/S_i) \le \mathrm{OPT}(G')$. Also, $w(S_i) \le \frac{1}{k}\, w(G') \le \frac{1}{\alpha k}\, \mathrm{OPT}(G')$. Hence, our solution for G' has value at most $\left(1 + \frac{\beta}{\alpha k}\right) \mathrm{OPT}(G')$. By the spanner construction, we can convert this solution into a solution for G with value at most $(1+\delta)\left(1 + \frac{\beta}{\alpha k}\right) \mathrm{OPT}(G)$. For any $\delta > 0$, we can choose k so that $1 + \frac{\beta}{\alpha k} \le 1 + \delta$, giving us a solution for G with value at most $(1+\delta)^2\, \mathrm{OPT}(G)$. Setting $\delta = \sqrt{1+\varepsilon} - 1$ gives us a $(1+\varepsilon)$-approximation. □

As a consequence of Theorem 1.2, we obtain the following particular approximation results:

COROLLARY 1.1. *For any fixed genus g, any constant $c \ge 2$, and any $0 < \varepsilon \le 1$, there is a polynomial-time $(1+\varepsilon)$-approximation algorithm for weighted TSP, and for minimum-weight c-edge-connected submultigraph, in graphs of genus g.*

Proof. TSP can be solved in graphs of bounded treewidth via dynamic programming; see [DFT06] for a particularly fast running time on graphs of bounded genus. Spanners for TSP in bounded-genus graphs are developed in [Gri00]. Finally, given an optimal solution to G/S, we can construct a solution of value at most $\mathrm{OPT}(G/S) + 3\, w(S)$ as follows. When we expand each vertex of G/S to the corresponding subgraph of G, we add an Eulerian tour of that subgraph (with some doubled edges), for a total cost of at most $2\, w(S)$. The resulting structure spans G and is connected, but some of the vertices of an Eulerian tour may have odd degree because of TSP edges from the solution for G/S attaching to the corresponding vertex of G/S. We also add a perfect matching among all odd-degree vertices on each Eulerian tour, routing the perfect matching along the Eulerian tour, choosing the perfect matching that has minimum weight for a cost of at most $w(S)$. Now the connected spanning structure of G has all vertices of even degree, so we can take one global Eulerian tour to obtain the desired TSP tour.

Minimum-weight c-edge-connected submultigraph can also be solved in graphs of bounded treewidth via dynamic programming, for any constant c; to simplify matters, it is helpful to first duplicate each edge in the input graph c times. The same spanner result of [Gri00] applies for any c because every edge of that spanner G' of weight w has a corresponding path of length $(1+\varepsilon)w$ in G, so we can convert a minimum-weight c-edge-connected submultigraph of G' into a c-edge-connected submultigraph of G (by duplicating edges in G according to their use in paths) at a multiplicative factor of $1+\varepsilon$. Finally, given an optimal solution to G/S, we can construct a solution of value at most $\mathrm{OPT}(G/S) + c\, w(S)$ by augmenting the tour with c copies of a spanning tree of the subgraph contracting to each vertex of G/S. Then c edge-disjoint paths in G/S can be expanded to c edge-disjoint paths in G by letting each path follow a different copy of each visited spanning tree. □

2 Definitions

First we define the basic notion of a graph minor. Given an edge $e = vw$ in a graph G, the *contraction* of e in G is the result of identifying vertices v and w in G and removing all loops and duplicate edges. A graph H obtained by a sequence of such edge contractions starting from G is said to be a *contraction* of G. A graph H is a *minor* of G if H is a subgraph of some contraction of G. A graph class $\mathcal{C}$ is *minor-closed* if any minor of any graph in $\mathcal{C}$ is also a member of $\mathcal{C}$. A minor-closed graph class $\mathcal{C}$ is *H-minor-free* if $H \notin \mathcal{C}$. More generally, we use the term "H-minor-free" to refer to any minor-closed graph class that excludes some fixed graph H.

Second we define the basic notion of treewidth, as introduced by Robertson and Seymour [RS86]. To define this notion, we consider representing a graph by a tree structure, called a tree decomposition. More precisely, a *tree decomposition* of a graph $G = (V, E)$ is a pair (T, χ) in which $T = (I, F)$ is a tree and $\chi = \{\chi_i \mid i \in I\}$ is a family of subsets of $V(G)$ such that

1. $\bigcup_{i \in I} \chi_i = V$;
2. for each edge $e = uv \in E$, there exists an $i \in I$ such that both u and v belong to χ_i; and
3. for every $v \in V$, the set of nodes $\{i \in I \mid v \in \chi_i\}$ forms a connected subtree of T.

To distinguish between vertices of the original graph G and vertices of T in the tree decomposition, we call vertices of T *nodes* and their corresponding χ_i's *bags*. The *width* of the tree decomposition is the maximum size of a bag in χ minus 1. The *treewidth* of a graph G, denoted $\mathrm{tw}(G)$, is the minimum width over all possible tree decompositions of G. A tree decomposition is called a *path decomposition* if $T = (I, F)$ is a path. The *pathwidth* of a graph G, denoted $\mathrm{pw}(G)$, is the minimum width over all possible path decompositions of G.

Third, we need a basic notion of embedding; see, e.g., [RS94, CM05]. In this paper, an *embedding* refers to a *2-cell embedding*, i.e., a drawing of the vertices and edges of the graph as points and arcs in a surface such that every face (connected component obtained after removing edges and vertices of the embedded graph) is homeomorphic to an open disk. We use basic terminology and notions about embeddings as introduced in [MT01]. We only consider compact surfaces without boundary. Occasionally we refer to embeddings in the plane, when we actually mean embeddings in the 2-sphere. If S is a surface, then for a graph G that is (2-cell) embedded in S with f facial walks, the number $g = 2 - |V(G)| + |E(G)| - f$ is independent of G and is called the *Euler genus* of S. The Euler genus coincides with the crosscap number if S is nonorientable, and equals twice the usual genus if the surface S is orientable.

3 Decomposition

In this section, we prove our main result, Theorem 1.1, that bounded-genus graphs have a partition of their edges into any number $k \geq 2$ of pieces such that contracting any piece results in bounded treewidth.

3.1 Preliminaries. We say that a graph G satisfies *property* $\mathcal{C}_k^w$, and write $G \in \mathcal{C}_k^w$, if $E(G)$ can be partitioned into k subsets $E_1, \ldots, E_k$ such that $\mathrm{tw}(G/E_i) \leq w$ for every $i = 1, \ldots, k$.

Lemma 3.1. *Let $F \subseteq E(G)$ be a set of edges and $H = G/F$. If $G \in \mathcal{C}_k^w$, then $H \in \mathcal{C}_k^w$.*

Proof. Let $E_1, \ldots, E_k$ be a partition of $E(G)$ showing that $G \in \mathcal{C}_k^w$. For $i = 1, \ldots, k$, let $E'_i = E_i \setminus F$. Clearly, $H_i = H/E'_i = (G/E_i)/(F \setminus E_i)$ is a minor of $G_i = G/E_i$. Therefore, $\mathrm{tw}(H_i) \leq \mathrm{tw}(G_i)$, and so the partition $E'_1, \ldots, E'_k$ of $E(H)$ shows that $H \in \mathcal{C}_k^w$. □

Lemma 3.2. *Let G be a graph, and let H be an induced subgraph of G that is obtained by deleting at most r vertices from G. If $H \in \mathcal{C}_k^w$, then $G \in \mathcal{C}_k^{w+r}$.*

Proof. Let $E'_1, \ldots, E'_k$ be a partition of $E(H)$ showing that $H \in \mathcal{C}_k^w$. Let $E_1 = E'_1 \cup (E(G) \setminus E(H))$, and let $E_i = E'_i$ for $2 \leq i \leq k$. Then H/E'_i is obtained from G/E'_i by deleting at most r vertices, so $\mathrm{tw}(G/E'_i) \leq \mathrm{tw}(H/E'_i) + r$. Moreover, because G/E_i is a minor of G/E'_i, we have $\mathrm{tw}(G/E_i) \leq \mathrm{tw}(G/E'_i)$. Both inequalities together show that $G \in \mathcal{C}_k^{w+r}$. □

3.2 Face-Distance on a Surface. The following result of Eppstein [Epp00] (see also [DH04a]) relates the diameter and the treewidth of a graph on a fixed surface.

Theorem 3.1. [Epp00] *Let G be a graph embedded in a fixed surface of genus g, and let $x_0 \in V(G)$. If every vertex of G is at distance at most d from x_0, then* $\mathrm{tw}(G) \leq 3d + 3$ *if $g = 0$, and* $\mathrm{tw}(G) = O(gd)$ *if $g \geq 1$.*

If G is embedded in some surface, one can define a distance function on $T(G) = V(G) \cup E(G)$ based on the smallest number of faces joining two elements in $T(G)$. More precisely, we define the *face-distance* $\varphi(x, y)$ on $T(G)$ inductively as follows. For every $x \in T(G)$ we have $\varphi(x, x) = 0$. If x and y are distinct and $d \geq 1$ is an integer, then $\varphi(x, y) \leq d$ if there exists $z \in T(G)$ such that $\varphi(x, z) \leq d - 1$ and z and y lie on the same facial walk. Finally, $\varphi(x, y) = d$ if $\varphi(x, y) \leq d$ and $\varphi(x, y) \not\leq d - 1$. It is clear that $\varphi(x, y) \leq d$ if and only if there exist facial walks $F_1, \ldots, F_d$ such that $x \in F_1$, $y \in F_d$ and $F_i \cap F_{i+1} \neq \emptyset$ for $i = 1, \ldots, d - 1$. In this case we say that $F_1, \ldots, F_d$ is a *face-chain* of *length* d *connecting* x and y. This shows that φ is a metric on $T(G)$.

Lemma 3.3. *Let G be a graph embedded in the plane and $x_0 \in V(G)$. Let a and b, $b \geq a$, be integers, and let $S(a, b)$ be the subgraph of G induced on all vertices and edges whose face-distance from x_0 is at least a and at most b. Then* $\mathrm{tw}(S(a, b)) \leq 3(b - a + 1)$.

Proof. For each face F of G, let c be the face-distance of $V(F)$ from x_0, and let x_F be a vertex of F at face-distance c from x_0. Then add to G all edges $x_F y$, where $y \in V(F)$ is at face-distance $c + 1$ from x_0. Clearly, the resulting graph $\tilde{G}$ has an embedding in the plane which extends the embedding of G and preserves the face-distance from x_0. Consider the corresponding graph $\tilde{S}(a, b) \supseteq S(a, b)$.

For every vertex at face-distance d from x_0 ($a \leq d \leq b$), the newly inserted edges of $\tilde{G}$ give rise to paths of length d to x_0. The initial $a - 1$ edges of such paths do not belong to $\tilde{S}(a, b)$, but they show that all vertices in $S(a)$ lie on the same face of $\tilde{S}(a, b)$. This implies that $\tilde{S}(a, b)$ is a subgraph of a plane graph Z in which each vertex is at distance at most $b - a + 1$ from some vertex y_0. By Theorem 3.1, we conclude that $\mathrm{tw}(S(a, b)) \leq \mathrm{tw}(Z) \leq 3(b - a + 2)$. □

Lemma 3.4. *Let G be a graph and suppose that for a vertex-set $A \subseteq V(G)$, the graph $H = G - A$ has an embedding in the plane such that every 2-connected component B of H has a vertex x_0 such that every vertex in B is at face-distance at most d from x_0. Then* $\mathrm{tw}(G) \leq |A| + 3(d + 1)$.

Proof. Clearly, $\mathrm{tw}(H) = \max \mathrm{tw}(B)$, where the maximum runs over all 2-connected components B of H. By assumption on the face-distance in 2-connected components of H, every B is of the form $S(0,d)$ for some choice of a base vertex $x_0 \in V(B)$. By Lemma 3.3 we have $\mathrm{tw}(B) \le 3(d+1)$. Consequently, $\mathrm{tw}(G) \le |A| + \mathrm{tw}(H) \le |A| + 3(d+1)$. □

3.3 Main Proof. If F is a subset of $E(G)$, we say that G has *property* $\mathcal{C}_k^w$ *with respect to* F if $E(G)$ can be partitioned into k subsets $E_1,\dots,E_k$ such that $\mathrm{tw}(G/E_i) \le w$ for every $i = 1,\dots,k$ and such that $F \subseteq E_1$.

For nonnegative integers k,q,w, we define *property* $\mathcal{C}_{k,q}^w$ as the class of all graphs G embedded in some surface such that for every collection $F_1,\dots,F_q$ of q faces, G has property $\mathcal{C}_k^w$ with respect to $F = E(F_1) \cup \cdots \cup E(F_q)$.

Our first result concerns planar graphs.

THEOREM 3.2. *Let $k \ge 1$ and $q \ge 0$ be integers. If $w \ge 6k(q+1)$, the class $\mathcal{C}_{k,q}^w$ contains all plane graphs.*

Proof. Let G be a plane graph and let $F_1,\dots,F_q$ be the prescribed distinguished faces, and let $F = E(F_1) \cup \cdots \cup E(F_q)$.

Let us fix a vertex x_0 of G. We first partition vertices and edges of G into *level sets* L_j ($j \ge 0$), so that L_j contains all vertices and edges whose face-distance from x_0 is equal to j. Observe that for every face R of G, we have $V(R) \cup E(R) \subseteq L_j \cup L_{j+1}$ for some $j \ge 0$. Next we define sets K_i for $i \ge 1$. Each of them is the union of one or more consecutive sets L_j. We define $K_0 = L_0 = \{x_0\}$. In general, having defined K_i, let L_j be the last level set that was included into K_i. If $i \not\equiv 0 \pmod{k}$, then we add L_{j+1} and L_{j+2} into K_{i+1} and consequently repeat the procedure with $i+1$ (unless L_{j+2} is empty, in which case we stop). If $i \equiv 0 \pmod{k}$, let $l \ge 2$ be the smallest integer such that $L_{j+l+1},\dots,L_{j+l+2k-2}$ are all disjoint from F. Since all edges from each face F_t ($1 \le t \le q$) are contained in two consecutive level sets, we have $2 \le l \le 2q(k-1)+2$. Now we add $L_{j+1},\dots,L_{j+l}$ to K_i, and proceed with the next value of i (unless L_{j+l+1} is empty, in which case we stop).

Each K_i consists of at least two and at most $2 + 2q(k-1)q$ consecutive level sets L_j. Next, we let $\tilde{K}_t$ ($1 \le t \le k$) be the set of all K_i, where $i \equiv t \pmod{k}$. Finally, we let $E_t \subseteq E(G)$ be the set of all edges in $\tilde{K}_t$.

Let $1 \le t \le k$, and let us consider a 2-connected component B of G/E_t. It is easy to see that planarity of G implies that $B \subseteq (K_i \cup K_{i+1} \cup \cdots \cup K_{i+k})/E_t$, where $i \equiv t \pmod{k}$. Because $K_0 \cup K_1 \cup \cdots \cup K_i$ is connected, B is of the form $S(a,b)$, where $b - a + 1 \le 2 + 2(k-1)q + 2(k-1) = 2(k-1)(q+1) + 2$. Lemma 3.4 implies that the treewidth of G/E_t is at most $6k(q+1)$. This completes the proof. □

Theorem 3.2 will be extended to other surfaces by applying induction on the Euler genus. The inductive proof will use some geometric surgery, so we recall some definitions.

Let G be embedded in a surface S of Euler genus g, and let $C = v_1v_2\dots v_rv_1$ be a cycle of G. If we traverse C starting at v_1 and going through $v_2,\dots,v_r$ and back to v_1, we can classify the edges incident to C as those on the "left" and those on the "right" side of C. It may happen that, when we come back to v_1 after the traversal, the left and right interchange. In such a case we say that C is a 1-*sided cycle*; otherwise it is 2-*sided*.

The 2-sided cycles can be classified further as those that are *surface-separating* and those that are not. The former ones have the property that no edge incident to C is simultaneously on the left and on the right of C, and every path in G that starts with an edge on the left and ends with an edge on the right contains an intermediate vertex that is in C. If C is a 2-sided cycle in G, then we can *cut* the surface *along* C. When doing so, C is replaced by two copies C', C'' of itself, and edges on the left of C are incident with C', while edges on the right stay incident with C''. The new graph has a natural 2-cell embedding where C' and C'' become additional facial cycles. If C is surface-separating, then the graph obtained after cutting is disconnected and the corresponding embedded graphs have genus g' and g'', respectively, such that $g' + g'' = g$. If $g' = 0$ or $g'' = 0$, then we say that C is *contractible* in S.

We can also define cutting along a 1-sided cycles: replace $C = v_1v_2\dots v_rv_1$ by a single cycle $C' = v_1'\dots v_r'v_1''\dots v_r''v_1'$, which becomes facial in the corresponding embedding. The reader is referred to [MT01] for more details.

If G is 2-cell-embedded in a surface of positive Euler genus, then G contains noncontractible cycles. The minimum number r such that there exist r facial walks $F_1,\dots,F_r$, whose union contains a noncontractible cycle in G, is called the *face-width* or *representativity* of the embedding. In this case, there is a noncontractible simple closed curve γ in the surface S that passes through $F_1,\dots,F_r$ and intersects G precisely in r vertices. We can define the operation of *cutting along* the curve γ in the same way as we did for cutting along a cycle. While doing this, each vertex of $G \cap \gamma$ is replaced by two vertices. The facial walks of the cut graph are the same as those in G except that $F_1,\dots,F_r$ are replaced by two (or one if γ is 1-sided) new facial walks.

THEOREM 3.3. *Given any integers $k \ge 1$, $q \ge 0$, and $g \ge 1$, let $w = 120\,kg(2g+q+2)$. Then the class $\mathcal{C}_{k,q}^w$ contains all graphs embedded in surfaces whose Euler* *genus is at most g.*

Proof. Let G be a graph embedded in a surface S of Euler genus g. By Lemma 3.4 it suffices to show that $E(G)$ can be partitioned into $E_1, \dots, E_k$ such that every G/E_i contains a set of at most $36k(2g-1)(2g+q+2)$ vertices whose removal leaves a graph embedded in the plane such that all vertices in the same 2-connected component B are at face-distance at most $8k(2g-1)(2g+q+2)$ from a reference vertex x_0 in B. The proof is by induction on g. As the base case we shall consider $g = 0$, which is covered by Theorem 3.2 and where each 2-connected component of G/E_i contains a vertex whose face-distance from all other vertices is at most $2k(q+1)$. We assume henceforth that $g \geq 1$.

Let $F_1, \dots, F_q$ be the distinguished faces whose edges are requested to be in E_1, and let r be the face-width of G.

Suppose first that $r < 36k(q+1)$. There is a simple noncontractible curve γ in the surface S that intersects G in precisely r vertices; denote them by $v_1, \dots, v_r$. Let us cut the surface and also the graph G along the curve γ. If γ is a surface-separating curve, then G is split this way into two graphs G' and G'' which are embedded into closed surfaces of Euler genera g' and g'', respectively, where $1 \leq g' \leq g'' < g$ and $g' + g'' = g$. Note that $E(G) = E(G') \cup E(G'')$ is a partition of $E(G)$ and that $V(G') \cap V(G'') = \{v_1, \dots, v_r\}$. If γ is not surface-separating, the graph G' resulting after cutting along γ is connected and is either embedded in a surface of Euler genus $g-1$ (if γ is 1-sided) or $g-2$ (if γ is 2-sided). All vertices that were affected by the cutting along γ belong to precisely two faces F' and F'', if γ is 2-sided, and to precisely one face F' when γ is 1-sided. In the latter case we set $F'' = F'$ so that we can refer to F' and F'' in all cases. The faces F' and F'' in G' (and G'') are not faces of G, while all other faces of G' and G'' coincide with those in G.

In G' (and G'' if applicable), we let F' and F'' become additional distinguished faces that are requested to be contained in E_1. Now, we apply the induction hypothesis to G' (and G'') with the extended collection of at most $q+2$ distinguished faces and with Euler genus $g-1$. Let $E_1, \dots, E_k$ be the corresponding partition of edges of G' (and G''). Observe that some of the original faces F_i may have disappeared, but all edges of those faces would then be contained in F' and F''. Hence, all edges of $F_1, \dots, F_q$ are contained in E_1. Also, let us observe that $E(G) = E(G')$ (or $E(G) = E(G') \cup E(G'')$ when G'' exists), so $E_1, \dots, E_k$ is a partition of $E(G)$.

Let us consider a contraction $G_i = G/E_i$. Let $U \subseteq V(G_i)$ be the set of vertices corresponding to $v_1, \dots, v_r$, so $|U| \leq r$. Now we apply the induction hypothesis. If γ is not surface-separating and the genus of G' is positive, then G'/E_i has a set A' of at most $36k(2g-3)(2(g-1)+(q+2)+2) = 36k(2g-3)(2g+q+2)$ vertices such that $G'/E_i - A'$ is embedded in the plane with 2-connected components having face-distance from one of their vertices at most $8k(2g-3)(2(g-1)+(q+2)+2) = 8k(2g-3)(2g+q+2)$. The same conclusion also holds for G_i with the set $A = A' \cup U$ removed. This completes the proof because $|A| \leq |A'| + |U| \leq 36k(2g-3)(2g+q+2) + 36k(q+1) < 36k(2g-2)(2g+q+2)$. The case when the genus of G' is 0 is similar, the details are omitted. Finally, if γ is surface-separating, we apply induction on G' and G''. Again, the arithmetic works – the removed set $A = A' \cup A'' \cup U$ is smaller than claimed.

From now on, we assume that the face-width r of G is at least $36k(q+1)$. Let C_0 be a shortest noncontractible cycle in G. Let us first assume that C_0 is 2-sided (possibly surface-separating). On the "left" side of C_0, there are disjoint cycles $C'_1, \dots, C'_t$ of G, all homotopic to C_0 and, moreover, all vertices and edges of C'_i are at face-distance precisely i from C_0, for $i = 1, \dots, t$, where $t = 4k(q+1) \leq \lfloor r/8 \rfloor - 1$. Similarly, on the "right" of C_0 there are disjoint homotopic cycles $C''_1, \dots, C''_t$ of G that are all homotopic to C_0 and also at face-distance at least 2 from $C'_1, \dots, C'_t$. These facts are well-known; see, e.g., [MT01] or [BMR96, Moh92].

If C_0 is 1-sided, there are cycles $C'_1, \dots, C'_t$ such that C'_i is at face-distance i from C_0. In this case, $C'_1, \dots, C'_t$ are all homotopic to each other (and homotopic to the "square" of C_0). They separate a Möbius strip containing C_0 from the rest of the surface. For convenience we write $C''_i = C'_i$ for $1 \leq i \leq t$.

Now we cut the surface along C'_t and along C''_t. If C_0 is surface-separating, then we obtain three embedded graphs, G_0, G', and G'', that are embedded into closed surfaces of Euler genera 0, g', and g'', respectively, where $g' \leq g'' < g$ and $g' + g'' = g$. This situation is represented in Figure 1.

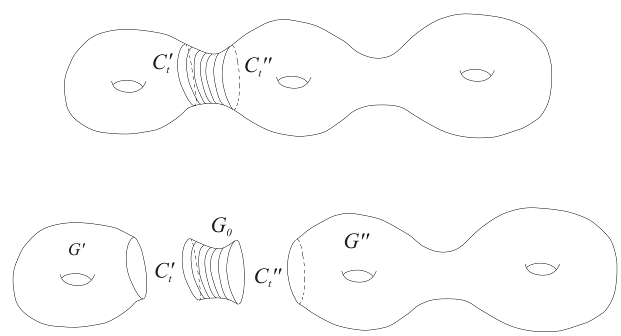

Figure 1: Cutting out a cylinder around C_0.

If C_0 is not surface-separating but is 2-sided, the situation is similar to the above except that G' and G''

coincide and Euler genus of G' is $g-2$. If C_0 is 1-sided, then G_0 is embedded in the projective plane (after we add a disk to C'_t), and $G' = G''$ has Euler genus $g-1$.

Now we apply the induction hypothesis to each of these surfaces in the same way as when considering the case of small face-width. We have new distinguished faces bounded by C'_t and C''_t, two in G_0 and one in each of G', G''. If $g = 1$, then we cannot apply the induction hypothesis directly because G_0 has Euler genus 1. However, the face-width of G_0 is $2t + 1$ and hence the reduction described in the first part of the proof can be used. This gives partitions of the edge sets of G_0, G', G''. Let the subsets (pieces) of these partitions be E_{0i}, E'_i, E''_i (respectively), and let $E_i = E_{0i} \cup E'_i \cup E''_i$ for $i = 1, \ldots, k$. All edges that appear in two of the graphs are part of distinguished faces, hence they all occur in the first subset of the corresponding partition. This shows that $E_1, \ldots, E_k$ is a partition of $E(G)$. The partition of G_0 is made in the same way as described in the proof of Theorem 3.2. We start with the breadth-first search definition of level sets L_j with the face C'_t of G_0 being L_1 and included in K_1. This guarantees that $C'_t, \ldots, C'_1$ are in consecutive level sets $L_1, \ldots, L_t$. Because $t = 4k(q + 1)$, each set E_{0i} $(1 \leq i \leq k)$ contains one of the cycles C'_j.

Let us now consider an arbitrary contraction

$$G_i = G/E_i = G'/E'_i \cup G''/E''_i \cup G_0/E_{0i}.$$

Let A' and A'' be the vertex sets of G'/E'_i and G''/E''_i (respectively) whose removal leaves planar graphs whose 2-connected components have small face-diameter. As mentioned above, E_{0i} contains one of the cycles C'_j, $1 \leq j \leq t$. Let x be the vertex of G_0/E_{0i} corresponding to C'_j. If $A = A' \cup A'' \cup \{x\}$, then $G_i - A$ is embedded in the plane. If $i = 1$, its 2-connected components coincide with those in $G_0/E_{0i} - x$, $G'/E'_i - A'$, and in $G''/E''_i - A''$. The same holds when $i \neq 1$, except that a 2-connected component B of $G'/E'_i - A'$ may be joined with a 2-connected component B_0 of $G_0/E_{0i} - x$ containing C'_t (and similarly, one in $G''/E''_i - A''$ which can be joined with another 2-connected component of $G_0/E_{0i} - x$). Because every vertex in B_0 is at face-distance at most t from some C'_j that is contracted to a point and becomes a cut vertex after the removal of x, the merging of B and B_0 can increase the face-distance from a reference vertex in B by at most $2t = 8k(q+1)$. We now complete the proof by applying induction. □

The derived bounds on the width in Theorems 3.2 and 3.3 are by no means best possible. They can easily be improved at the expense of longer proofs. However, their dependence on k, q, and g cannot be eliminated.

The proofs of Theorems 3.2 and 3.3 yield polynomial-time algorithms that enable us to construct appropriate edge partitions of graphs of bounded Euler genus:

THEOREM 3.4. *Given integers $k \geq 1$, $q \geq 0$, $g \geq 0$, $w \geq 120k(g + 1)(2g + q + 2)$, a graph embedded in a surface of Euler genus at most g, and a collection of q faces, one can find a partition of $E(G)$ showing that $G \in \mathcal{C}^w_{k,q}$ in polynomial time.*

Proof. The construction of the partition follows proofs of Theorems 3.2 and 3.3 by adding the following ingredients. A shortest noncontractible cycle in a graph of n vertices embedded in a surface of Euler genus g can be found in time $O(g^{3/2}n^{3/2} \log n)$ as shown by Cabello and Mohar [CM05]. (If g is considered fixed, the $\log n$ factor can be eliminated and the time complexity becomes $O(n^{3/2})$.) In [CM05] it is also shown how to determine the face-width in time $O(n^{3/2})$. Level sets can be constructed in linear time by a version of a breadth-first search.

In conclusion, the overall computational complexity of our algorithm is $O(g^{5/2}n^{3/2} \log n)$, where an additional factor of g appears because of the depth of the recursion on g. (If g is considered to be fixed, the time complexity becomes $O(n^{3/2})$.) □

This concludes the proof of Theorem 1.1.

4 Toward *H*-Minor-Free Graphs

We conjecture that our contraction decomposition result of Theorem 1.1 extends to H-minor-free graphs for any fixed graph H. Such a result would be quite general, paralleling the deletion decomposition of [DHK05], and would have many implications on algorithmic and structural Graph Minor Theory, in particular leading to generalized PTASs as detailed below. However, as we illustrate, solving this conjecture seems to require significant new insights into structural Graph Minor Theory, beyond the insights already presented in this paper for bounded-genus graphs. On the other hand, we show how our results extend to "h-almost-embeddable graphs", which is one step toward solving the H-minor-free case.

To understand how we might approach H-minor-free graphs, we describe the deep decomposition theorem of Robertson and Seymour. For the relevant background and terminology, refer to Appendix A.

THEOREM 4.1. [RS03, DHK05, DDO+04] *For every graph H, there exists an integer $h \geq 0$ depending only on $|V(H)|$ such that every H-minor-free graph can be obtained by at most h-sums of graphs that are h-almost-embeddable in surfaces of genus at most h. Furthermore, the clique-sum decomposition, written as*

$G_1 \oplus G_2 \oplus \cdots \oplus G_N$, has the additional property that the join set of each clique-sum between $G_1 \oplus G_2 \oplus \cdots \oplus G_{i-1}$ and G_i is a subset of the apices in G_i, and contains at most three vertices from the bounded-genus part of $G_1 \oplus G_2 \oplus \cdots \oplus G_{i-1}$. Furthermore, the decomposition can be found in polynomial time.

To generalize our decomposition result of Theorem 1.1 to H-minor-free graphs, we need to generalize our partition algorithm from bounded-genus graphs to h-almost-embeddable graphs, and then find a way to combine the partitions obtained from each piece of the clique sum. The first part—generalizing to h-almost-embeddable graphs—can be achieved as follows, building on our techniques from Section 3. The apices and their incident edges can increase the treewidth of any (contracted) graph by at most an additive $h = O(1)$, so they can be placed arbitrarily into color classes of the partition without effect. To handle vortices, we contract each vortex subgraph down to a single vertex, then apply the bounded-genus decomposition, and then decontract the vortex subgraphs and assign these edges arbitrarily into color classes. It can be shown that the last decontraction phase increases the treewidth by an additive $O(1)$, following the techniques of [Gro03, DH04b]. Thus we obtain the decomposition result of Theorem 1.1 extended to h-almost-embeddable graphs.

The second part—combining the partitions from each piece of the clique sum—seems difficult. The root cause of difficulty is that some of the edges in the join set of a clique sum are *virtual*: these edges are not in the actual graph, but appear in the individual pieces. If we keep these virtual edges when applying the decomposition to each piece, the partition may assign some of these virtual edges to be contracted in certain cases, but the edges cannot actually be contracted because they do not exist in the actual graph. On the other hand, if we delete these virtual edges before applying the decomposition, we still obtain that the pieces have bounded treewidth after contracting one of the classes, but it becomes impossible to join together these tree decompositions, because the join set no longer forms a clique and thus it is no longer contained in a single bag in each tree decomposition. A naïve combination of these tree decompositions causes a blowup in treewidth proportional to the number of clique-sum operations, which can be large, while intelligent combination with the join sets being cliques causes the treewidth to simply become the maximum treewidth over all the pieces [DHN+04, Lemma 3]. In contrast, this problem does not arise if we only delete edges within a label class as in [DHK05], instead of contracting them, because the virtual edges can be deleted (indeed, they must be deleted, but this can only help), whereas they cannot be contracted. We believe that nonetheless these difficulties can be surmounted, but only via a deep understanding of virtual edges connecting to the bounded-genus part, and how they can be realized by paths of real edges, in the graph minor decomposition of Theorem 4.1.

With the decomposition result in hand, we obtain the same general PTAS result of Theorem 1.2 for the new class of graphs. In particular, this result gives us PTASs for unweighted TSP and minimum-size c-edge-connected submultigraph, because every H-minor-free graph serves as its own unweighted spanner. Such a PTAS for TSP in H-minor-free graphs would solve an open problem of Grohe [Gro03]. However, to obtain such PTASs for weighted graphs as in Corollary 1.1, we also need to generalize the existing spanner results [Gri00] from bounded-genus graphs to H-minor-free graphs. We conjecture that such spanners exist. Some partial progress toward this goal has been made in [GS02].

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References

[AGK+98] Sanjeev Arora, Michelangelo Grigni, David Karger, Philip Klein, and Andrzej Woloszyn. A polynomial-time approximation scheme for weighted planar graph TSP. In *Proceedings of the 9th Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 33–41, 1998.

[Ami01] Eyal Amir. Efficient approximation for triangulation of minimum treewidth. In *Proceedings of the 17th Conference on Uncertainty in Artificial Intelligence (UAI-2001)*, pages 7–15, San Francisco, CA, 2001. Morgan Kaufmann Publishers.

[ARV04] Sanjeev Arora, Satish Rao, and Umesh Vazirani. Expander flows, geometric embeddings and graph partitioning. In *Proceedings of the 36th Annual ACM Symposium on Theory of Computing*, pages 222–231, 2004.

[Bak94] Brenda S. Baker. Approximation algorithms for NP-complete problems on planar graphs. *Journal of the Association for Computing Machinery*, 41(1):153–180, 1994.

[BCGZ05] André Berger, Artur Czumaj, Michelangelo Grigni, and Hairong Zhao. Approximation schemes for minimum 2-connected spanning subgraphs in weighted planar graphs. In *Proceedings of the 13th Annual European Symposium on Algorithms*, volume 3669 of *Lecture Notes in Computer Science*, pages 472–483, Palma de Mallorca, Spain, October 2005.

[BMR96] Richard Brunet, Bojan Mohar, and R. Bruce Richter. Separating and nonseparating disjoint homotopic cycles in graph embeddings. *Journal of Combinatorial Theory. Series B*, 66(2):201–231, 1996.

[Bod05] Hans L. Bodlaender. Discovering treewidth. In *Proceedings of the 31st Conference on Current Trends in Theory and Practice of Computer Science*, volume 3381 of *Lecture Notes in Computer Science*, pages 1–16, Liptovský Ján, Slovakia, January 2005.

[CGSZ04] Artur Czumaj, Michelangelo Grigni, Papa Sissokho, and Hairong Zhao. Approximation schemes for minimum 2-edge-connected and biconnected subgraphs in planar graphs. In *Proceedings of the 15th Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 496–505, Philadelphia, PA, USA, 2004. Society for Industrial and Applied Mathematics.

[CM05] Sergio Cabello and Bojan Mohar. Finding shortest non-separating and non-contractible cycles for topologically embedded graphs. In *Proceedings of the 13th Annual European Symposium on Algorithms*, volume 3669 of *Lecture Notes in Computer Science*, pages 131–142, Palma de Mallorca, Spain, October 2005.

[DDO+04] Matt DeVos, Guoli Ding, Bogdan Oporowski, Daniel P. Sanders, Bruce Reed, Paul Seymour, and Dirk Vertigan. Excluding any graph as a minor allows a low tree-width 2-coloring. *Journal of Combinatorial Theory, Series B*, 91(1):25–41, 2004.

[DFHT04] Erik D. Demaine, Fedor V. Fomin, MohammadTaghi Hajiaghayi, and Dimitrios M. Thilikos. Bidimensional parameters and local treewidth. *SIAM Journal on Discrete Mathematics*, 18(3):501–511, December 2004.

[DFHT05a] Erik D. Demaine, Fedor V. Fomin, MohammadTaghi Hajiaghayi, and Dimitrios M. Thilikos. Fixed-parameter algorithms for (k, r)-center in planar graphs and map graphs. *ACM Transactions on Algorithms*, 1(1):33–47, 2005.

[DFHT05b] Erik D. Demaine, Fedor V. Fomin, MohammadTaghi Hajiaghayi, and Dimitrios M. Thilikos. Subexponential parameterized algorithms on graphs of bounded genus and H-minor-free graphs. *Journal of the ACM*, 52(6):866–893, 2005.

[DFT06] Frederic Dorn, Fedor V. Fomin, and Dimitrios M. Thilikos. Fast subexponential algorithm for non-local problems on graphs of bounded genus. In *Proceedings of the 10th Scandinavian Workshop on Algorithm Theory*, Riga, Latvia, July 2006. To appear.

[DH04a] Erik D. Demaine and MohammadTaghi Hajiaghayi. Diameter and treewidth in minor-closed graph families, revisited. *Algorithmica*, 40(3):211–215, August 2004.

[DH04b] Erik D. Demaine and MohammadTaghi Hajiaghayi. Equivalence of local treewidth and linear local treewidth and its algorithmic applications. In *Proceedings of the 15th ACM-SIAM Symposium on Discrete Algorithms (SODA'04)*, pages 833–842, January 2004.

[DHK05] Erik D. Demaine, MohammadTaghi Hajiaghayi, and Ken-ichi Kawarabayashi. Algorithmic graph minor theory: Decomposition, approximation, and coloring. In *Proceedings of the 46th Annual IEEE Symposium on Foundations of Computer Science*, pages 637–646, Pittsburgh, PA, October 2005.

[DHN+04] Erik D. Demaine, MohammadTaghi Hajiaghayi, Naomi Nishimura, Prabhakar Ragde, and Dimitrios M. Thilikos. Approximation algorithms for classes of graphs excluding single-crossing graphs as minors. *Journal of Computer and System Sciences*, 69(2):166–195, September 2004.

[DHT06] Erik D. Demaine, MohammadTaghi Hajiaghayi, and Dimitrios M. Thilikos. The bidimensional theory of bounded-genus graphs. *SIAM Journal on Discrete Mathematics*, 20(2):357–371, 2006.

[Epp00] David Eppstein. Diameter and treewidth in minor-closed graph families. *Algorithmica*, 27(3-4):275–291, 2000.

[FK98] Uriel Feige and Joe Kilian. Zero knowledge and the chromatic number. *Journal of Computer and System Sciences*, 57(2):187–199, 1998.

[FT04] Fedor V. Fomin and Dimitrios M. Thilikos. Fast parameterized algorithms for graphs on surfaces: Linear kernel and exponential speed-up. In *Proceedings of the 31st International Colloquium on Automata, Languages and Programming (ICALP 2004)*, pages 581–592, Turku, Finland, July 2004.

[GHT84] John R. Gilbert, Joan P. Hutchinson, and Robert Endre Tarjan. A separator theorem for graphs of bounded genus. *J. Algorithms*, 5(3):391–407, 1984.

[GKP95] Michelangelo Grigni, Elias Koutsoupias, and Christos Papadimitriou. An approximation scheme for planar graph TSP. In *Proceedings of the 36th Annual Symposium on Foundations of Computer Science (Milwaukee, WI, 1995)*, pages 640–645, Los Alamitos, CA, 1995.

[Gri00] Michelangelo Grigni. Approximate TSP in graphs with forbidden minors. In *Proceedings of the 27th International Colloquium of Automata, Languages and Programming (Geneva, 2000)*, volume 1853 of *Lecture Notes in Computer Science*, pages 869–877. Springer, Berlin, 2000.

[Gro03] Martin Grohe. Local tree-width, excluded minors, and approximation algorithms. *Combinatorica*, 23(4):613–632, 2003.

[GS02] Michelangelo Grigni and Papa Sissokho. Light spanners and approximate TSP in weighted graphs with forbidden minors. In *Proceedings of the 13th Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 852–857, 2002.

[Kel06] Jonathan A. Kelner. Spectral partitioning, eigenvalue bounds, and circle packings for graphs of bounded genus. *SIAM Journal on Computing*, 35(4):882–902 (electronic), 2006.

[Kle05] Philip N. Klein. A linear-time approximation scheme for TSP for planar weighted graphs. In *Proceedings of the 46th IEEE Symposium on Foundations of Computer Science*, pages 146–155, 2005.

[Kle06] Philip N. Klein. A subset spanner for planar graphs, with application to subset TSP. In *Proceedings of the 38th ACM Symposium on Theory of Computing*, 2006. to appear.

[LR99] Tom Leighton and Satish Rao. Multicommodity max-flow min-cut theorems and their use in designing approximation algorithms. *Journal of the ACM*, 46(6):787–832, 1999.

[LT80] Richard J. Lipton and Robert Endre Tarjan. Applications of a planar separator theorem. *SIAM Journal on Computing*, 9(3):615–627, 1980.

[Moh92] Bojan Mohar. Combinatorial local planarity and the width of graph embeddings. *Canadian Journal of Mathematics*, 44(6):1272–1288, 1992.

[Moh99] Bojan Mohar. A linear time algorithm for embedding graphs in an arbitrary surface. *SIAM Journal on Discrete Mathematics*, 12(1):6–26, 1999.

[Moh01] Bojan Mohar. Graph minors and graphs on surfaces. In *Surveys in Combinatorics*, volume 288 of *London Math. Soc. Lecture Note Ser.*, pages 145–163. Cambridge Univ. Press, Cambridge, 2001.

[MT01] Bojan Mohar and Carsten Thomassen. *Graphs on surfaces*. Johns Hopkins Studies in the Mathematical Sciences. Johns Hopkins University Press, Baltimore, MD, 2001.

[RS86] Neil Robertson and Paul D. Seymour. Graph minors. II. Algorithmic aspects of tree-width. *Journal of Algorithms*, 7(3):309–322, 1986.

[RS94] Neil Robertson and P. D. Seymour. Graph minors. XI. Circuits on a surface. *Journal of Combinatorial Theory, Series B*, 60(1):72–106, 1994.

[RS03] Neil Robertson and P. D. Seymour. Graph minors. XVI. Excluding a non-planar graph. *Journal of Combinatorial Theory, Series B*, 89(1):43–76, 2003.

[Tho95] Robin Thomas. Problem Session of the 3rd Solvene Conference on Graph Theory, Bled, Slovenia, 1995.

[Tho98] Mikkel Thorup. All structured programs have small tree-width and good register allocation. *Information and Computation*, 142(2):159–181, 1998.

[Wag37] K. Wagner. Über eine Eigenschaft der eben Komplexe. *Mathematische Annalen*, 114:570–590, December 1937.

A Graph Minor Decomposition Theorem

In this appendix, we define the terminology necessary for the Graph Minor Decomposition Theorem 4.1 of Robertson and Seymour [RS03]. At a high level, this theorem says that, for every graph H, every H-minor-free graph can be expressed as a "tree structure" of pieces, where each piece is a graph that can be drawn in a surface in which H cannot be drawn, except for a bounded number of "apex" vertices and a bounded number of "local areas of non-planarity" called "vortices". Here the bounds depend only on H. To make this theorem precise, we need to define each of the notions in quotes.

Each piece in the decomposition is "h-almost-embeddable" in a bounded-genus surface where h is a constant depending on the excluded minor H. Roughly speaking, a graph G is *h-almost embeddable* in a surface S if there exists a set X of size at most h of vertices, called *apex vertices* or *apices*, such that $G - X$ can be obtained from a graph G_0 embedded in S by attaching at most h graphs of pathwidth at most h to G_0 within h faces in an orderly way. More precisely, a graph G is *h-almost embeddable* in S if there exists a vertex set X of size at most h (the *apices*) such that $G - X$ can be written as $G_0 \cup G_1 \cup \cdots \cup G_h$, where

1. G_0 has an embedding in S;
2. the graphs G_i, called *vortices*, are pairwise disjoint;
3. there are faces $F_1, \dots, F_h$ of G_0 in S, and there are pairwise disjoint disks $D_1, \dots, D_h$ in S, such that for $i = 1, \dots, h$, $D_i \subset F_i$ and $U_i := V(G_0) \cap V(G_i) = V(G_0) \cap D_i$; and
4. the graph G_i has a path decomposition $(\mathcal{B}_u)_{u \in U_i}$ of width less than h, such that $u \in \mathcal{B}_u$ for all $u \in U_i$. The sets $\mathcal{B}_u$ are ordered by the ordering of their indices u as points along the boundary cycle of face F_i in G_0.

The pieces of the decomposition are combined according to "clique-sum" operations, a notion which goes back to characterizations of $K_{3,3}$-minor-free and K_5-minor-free graphs by Wagner [Wag37] and serves as an important tool in the Graph Minor Theory. Suppose G_1 and G_2 are graphs with disjoint vertex sets and let $k \geq 0$ be an integer. For $i = 1, 2$, let $W_i \subseteq V(G_i)$ form a clique of size k and let G'_i be obtained from G_i by deleting some (possibly no) edges from the induced subgraph $G_i[W_i]$ with both endpoints in W_i. Consider a bijection $h : W_1 \to W_2$. We define a *k-sum* G of G_1 and G_2, denoted by $G = G_1 \oplus_k G_2$ or simply by $G = G_1 \oplus G_2$, to be the graph obtained from the union of G'_1 and G'_2 by identifying w with $h(w)$ for all $w \in W_1$. The images of the vertices of W_1 and W_2 in $G_1 \oplus_k G_2$ form the *join set*. Note that each vertex v of G has a corresponding vertex in G_1 or G_2 or both. Also, $\oplus$ is not a well-defined operator: it can have a set of possible results.