2-cell embeddings with prescribed face lengths and genus

Bojan Mohar[*][†]

Department of Mathematics
Simon Fraser University
Burnaby, B.C. V5A 1S6
email: mohar@sfu.ca

Abstract

Let n be a positive integer, let $d_1, \dots, d_n$ be a sequence of positive integers, and let $q = \frac{1}{2}\sum_{i=1}^{n} d_i$. It is shown that there exists a connected graph G of order n, whose degree sequence is $d_1, \dots, d_n$ and such that G admits a 2-cell embedding in every closed surface whose Euler characteristic is at least $n - q + 1$, if and only if q is an integer and $q \geq n - 1$. Moreover, the graph G is loopless if and only if $d_i \leq q$ for $i = 1, \dots, n$. This, in particular, answers a question of Arkadiy Skopenkov.

1 Introduction

The following problem was communicated to me by Arkadiy Skopenkov.

Problem 1.1 *Let $g \geq 0$ be an integer and let $d_1, \dots, d_n \geq 1$ be a non-increasing sequence of integers. Is there a dissection of the orientable (respectively, non-orientable) surface of genus g into n 2-cells such that the ith cell has size d_i for $i = 1, \dots, n$?*

It turned out later that this problem is not exactly what Skopenkov was asked by his colleagues Elena Kudrjavtseva and Igor Shnurnikov. Nevertheless, Problem 1.1 is important in the study of integrable Hamiltonian systems and Morse functions [4], see [1, 2].

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[†]On leave from IMFM & FMF, Department of Mathematics, University of Ljubljana, 1000 Ljubljana, Slovenia.

Let G be a graph of order n whose vertices have degrees $d_1, \dots, d_n$. If G has loops, then each loop contributes 2 towards the degree of the corresponding vertex. The sequence $d_1, \dots, d_n$ is called the *degree sequence* of G. It is clear that Problem 1.1 is equivalent to the following one (by considering the geometric dual graph of a 2-cell dissection):

Problem 1.2 *Let $g \geq 0$ be an integer and let $d_1, \dots, d_n$ be a sequence of positive integers. Is there a graph G of order n, which admits a 2-cell embedding in the orientable (resp. non-orientable) surface of genus g, and whose degree sequence is $d_1, \dots, d_n$?*

The solution to Problem 1.2 and hence also to Problem 1.1 turns out to be rather straightforward. The purpose of this note to first present this solution, and then discuss some further, less trivial extensions.

The easy part, solution to Problems 1.1 and 1.2, is presented in Section 2. In fact, we give a stronger result that realizations for all surfaces admissible for the given degree parameters $d_1, \dots, d_n$ can be achieved using the same graph. An extension, where we impose an additional requirement that the graphs should be free of loops, is given in Section 3. This requirement, when imposed dually on 2-cell decomposition with given cell lengths, gives a non-singularity property – every edge appears on the boundary of two distinct 2-cells. Again, the main result, Theorem 3.2, actually shows more. We prove that the same (loopless) graph can be used for all surfaces admissible by the given degree sequence $d_1, \dots, d_n$.

2 2-cell embeddings with prescribed face lengths

First we give a simple answer to Problem 1.2.

Theorem 2.1 *Let $n \geq 2$ be an integer, let $d_1, \dots, d_n$ be a sequence of positive integers, and let $D = \sum_{i=1}^n d_i$. Then there exists a connected (possibly non-simple) graph with degree sequence $d_1, \dots, d_n$ if and only if D is even and $D \geq 2n - 2$.*

Proof. If G is a connected graph with degrees $d_1, \dots, d_n$, then $D = 2|E(G)| \geq 2(n-1)$. This shows that conditions are necessary. To prove sufficiency, assume that $d_1 \geq \cdots \geq d_n \geq 1$. First, we claim that for every $k = 1, \dots, n-1$, we have $\sum_{i=1}^k d_i \geq 2k - 1$. This is clear if $d_k \geq 2$. On the other hand, if $d_k = 1$, then $d_j = 1$ for $j \geq k$, and so $\sum_{i=1}^k d_i \geq D - (n-k) \geq 2n - 2 - n + k \geq 2k - 2$.

The rest of the proof proceeds by induction on n. The proof is trivial for $n = 2$. So, we may assume that $n \geq 3$, and hence $d_1 \geq 2$. Let us now replace the sequence $d_1, \ldots, d_n$ with a sequence of length $n-1$, which is either equal to $d_1 - 1, d_2, d_3, \ldots, d_{n-1}$ or to $d_1 - 1, d_2 - 1, d_3, \ldots, d_{n-1}$, whichever of them has even sum. By the above observation, the new degree sequence has even sum $D' \geq D - d_n - 2$. If $d_n = 1$, then $D' \geq D - 3 \geq 2(n-1) - 3$ and since D' is even, we see that $D' \geq 2(n-1) - 2$. If $d_n \geq 2$, then $d_3 \geq \cdots \geq d_{n-1} \geq 2$, so $D' \geq 2(n-1) - 2$ again. By the induction hypothesis there is a connected graph G' with the corresponding degree sequence. Let $v_1, \ldots, v_{n-1}$ be the vertices of G'. Now, add a new vertex v_n and join it to v_1 (and to v_2 if the degree of v_2 in G' is $d_2 - 1$). In this graph, all vertices have degree as expected, except that the degree of v_n is just 1 or 2. However, by adding appropriate number of loops incident with v_n, we obtain a graph with degree sequence $d_1, \ldots, d_n$. $\square$

A more general construction works as follows. Suppose that $d_1 \geq \cdots \geq d_k \geq 2$ $(k \leq n)$ and that $d_j = 1$ for $j > k$. Then we start with the cycle $C_k = v_1 v_2 \ldots v_k$ of order k and add at each vertex v_i $(1 \leq i \leq k)$ $d_i - 2$ half-edges. Next, attach to $n - k$ of so obtained half-edges a new vertex of degree 1, and split all remaining half-edges into pairs (their number is easily seen to be even). Finally, for each such pair, replace the two half-edges by an edge (possibly a loop) joining corresponding vertices. This construction will succeed if $D = \sum_{i=1}^{n} d_i \geq 2n$. However, if $D = 2n - 2$, we can do the same except that we start with the k-vertex path P_k instead of C_k.

If S is a surface of Euler characteristic $\chi(S)$, then $h = 2 - \chi(S)$ is a non-negative integer called the *Euler genus* of S. In order to describe 2-cell embeddings, we shall use combinatorial description by means of a *rotation system* together with a *signature* as explained in [3].

Theorem 2.2 *Problems 1.1 and 1.2 have affirmative answer if and only if* $q = \frac{1}{2} \sum_{i=1}^{n} d_i$ *is an integer which is greater or equal to* $n - 1 + h$, *where* h *is the Euler genus of the surface in question.*

Proof. Suppose that a graph G is 2-cell embedded in a surface of Euler genus h. If G has n vertices, q edges and f faces, then by Euler's formula, $h = q - n - f + 2$, and hence $1 \leq f = q - n + 2 - h$. So the condition of the theorem is necessary.

To prove sufficiency, observe that when $q = \frac{1}{2} D \geq n - 1 + h$, the conditions of Theorem 2.1 are satisfied, so there exists a connected graph G_0 with degree sequence $d_1, \ldots, d_n$. Let T be a spanning tree of G_0. Let us form a graph

G' with half-edges by taking the tree T, and adding to the ith vertex v_i of T, $d_i - \deg_T(v_i)$ half-edges.

Now choose and fix any rotation system of G'. Since G' is a tree with half-edges, this rotation system determines a 2-cell embedding of G' into the 2-sphere, which has precisely one facial walk. The total number of half-edges is even, $2r = D - 2n + 2$, and let us enumerate them as $h_1, \ldots, h_{2r}$ in the order as they appear on the facial walk. Now we form a new graph G_1 by grouping the half-edges in pairs, and replacing each so determined pair with an edge joining the corresponding vertices and keeping existing local rotations. To determine the pairs, we are using the following three operations:

- Pair up h_i with h_{i+2} and pair up h_{i+1} with h_{i+3}. This operation is called *adding a handle*. It preserves the number of facial walks and hence increases Euler genus by two. Moreover, it does not change the orientability of the corresponding 2-cell embedding.

- Pair up h_i with h_{i+1}. This operation, which is called *capping*, increases the number of facial walks by one, and hence preserves the Euler genus.

- Pair up h_i with h_{i+1} and give negative signature to the new edge thus formed. This operation, called *adding a crosscap*, keeps the number of facial walks unchanged, but changes orientability from orientable to non-orientable if it was not already non-orientable before. Here, the Euler genus increases by one.

Now it is clear that we can either add $g = h/2$ handles or add g crosscaps in order to get the desired surface. After that we cap all the remaining edges. This construction gives the desired graph and its 2-cell embedding into the corresponding surface. $\square$

Actually, for every degree sequence $d_1, \ldots, d_n$, for which q is an integer greater or equal to $n-1$, there is a graph G with this degree sequence which has 2-cell embeddings in all surfaces whose Euler genus is at most $q-n+1$. This can be proved using Lemma 3.1 from the next section and just making sure, when pairing up the half-edges as in the proof of Theorem 2.2 that the connected components formed by the edges in the complement of the spanning tree T contain even number of edges with the possible exception of the component containing the edge added at the end.

3 Loopless realizations

The graphs constructed in the previous section may have multiple edges and loops. We may ask when it is possible to get simple graphs. However, this problem may be much more difficult and may not admit a simple answer. Troubles appear at both sides of the spectrum, when the genus is small and when it is large. For the small genus example, let us take $d_1 = d_2 = \cdots = d_{12} = 5$, and let all other values d_i be equal to 6. Then it is known that there are examples of genus 0 if and only if $n \neq 13$. However, it is not clear why 13 is exceptional.

For an example with large genus, let d_i be either $n-1$ or $n-2$ for $i = 1, \ldots, n$. Then determining the maximum g, for which there exists a solution with a simple graph, amounts to computing the genus of complete graphs from which we remove a perfect matching. This by itself is not straightforward.

Therefore it makes sense to address the question about loopless examples. In this case we will be able to say more. Given a degree sequence $d_1, \ldots, d_n$ satisfying the necessary conditions of Theorem 2.1, there is an obvious obstruction for having a loopless graph with the given degree sequence. Namely, if

$$d_1 > \sum_{i=2}^{n} d_i \tag{1}$$

then any graph with degree sequence $d_1, \ldots, d_n$ will have loops at the vertex with degree d_1. It is interesting that this kind of a problem is the only obstruction for having a loopless realization, as we shall se in Theorem 3.2.

A graph G is said to be *universally embeddable* if it admits 2-cell embeddings in all closed surfaces whose Euler characteristic is at least $|V(G)| - |E(G)| + 1$. Note that, by Euler's formula, G cannot have 2-cell embeddings in surfaces whose Euler characteristic is smaller than $|V(G)| - |E(G)| + 1$. The graph G is said to be *upper embeddable* if it has a 2-cell embedding in an orientable surface S with at most two faces. Note that the Euler characteristic of S is either $|V(G)| - |E(G)| + 1$ or $|V(G)| - |E(G)| + 2$, whichever of these two numbers is even.

The proof of the following lemma just places together several known ingredients from topological graph theory. Let us recall that a graph is *planar* if it has an embedding in the 2-sphere.

Lemma 3.1 *A graph G is universally embeddable if and only if G is a connected planar graph which contains a spanning tree T such that $G - E(T)$ has at most one connected component with an odd number of edges.*

Proof. If G is a graph with an embedding in a surface of Euler characteristic c, then G admits 2-cell embeddings in all non-orientable surfaces whose Euler characteristic is between $c+1$ and $|V(G)|-|E(G)|+1$, see [3, Theorem 4.5.1]. It is also known that orientable 2-cell embeddings satisfy the "interpolation property": If G has orientable embeddings of genera g_1 and g_2, then it has one for every genus g between g_1 and g_2, see [3, Theorem 4.5.3]. This shows that G is universally embeddable if and only if it is connected and planar (has a genus 0 embedding) and is upper embeddable. Finally, Xuong [5] (see also [3, Theorem 4.5.4]) has proved that G is upper embeddable if and only if it has a spanning tree T such that $G - E(T)$ has at most one connected component with an odd number of edges. This completes the proof. □

Theorem 3.2 *Let $n \geq 2$ be an integer, let $d_1, \ldots, d_n$ be a sequence of positive integers, and let $q = \frac{1}{2}\sum_{i=1}^{n} d_i$. Then the following statements are equivalent:*

(a) *There exists a connected loopless graph with degree sequence $d_1, \ldots, d_n$.*

(b) *There exists a universally embeddable connected loopless graph with degree sequence $d_1, \ldots, d_n$.*

(c) *q is an integer, $q \geq n-1$, and $d_i \leq q$ for $i = 1, \ldots, n$.*

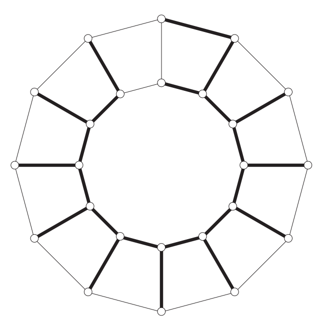

Figure 1: A prism and its spanning tree

Proof. Clearly, (b) ⇒ (a) and (a) ⇒ (c). To prove that (c) ⇒ (b), we shall first construct a loopless graph G with the given degree sequence, and then find its spanning tree T for which the condition of Lemma 3.1 will be verified. In particular, we have to verify that all constructed graphs are planar. Fortunately, this will always be either evident by construction or obvious by applying induction. Therefore, we shall leave this property to be verified by the reader. As a consequence we shall conclude that G is universally embeddable.

We shall need a sligthly stronger statement in order to be able to apply the induction hypothesis in all cases. We will prove that in the case when $d_i \geq 4$ for every $i = 1, \ldots, n$, the constructed graph G has a spanning tree T such that $G - E(T)$ is a spanning connected subgraph of G. We shall assume that d_i are non-increasing, $d_1 \geq d_2 \geq \cdots \geq d_n$.

The proof is by induction on n. If $n = 2$, then $d_1 = d_2$ and we take the 2-vertex graph with d_1 edges in parallel joining the two vertices. Clearly, Lemma 3.1 shows that this graph is universally embeddable.

Suppose now that $n \geq 3$. We will distinguish several possibilities. Let us first consider the case when all degrees are equal, i.e. $d_n = d_1 \geq 2$. Let $\delta = \lfloor \frac{1}{2} d_1 \rfloor$. If d_1 is even, then we get a realization G isomorphic to the cycle of length n with all its edges having multiplicity δ. If d_1 is odd, then n is even, and we take vertices $v_1, \ldots, v_n$ and join v_i and v_{i+1} ($1 \leq i \leq n$, indices modulo n) with δ parallel edges if i is odd, and with $\delta + 1$ parallel edges if i is even. It is clear by Lemma 3.1 that G is universally embeddable if $d_1 \neq 3$ or when $d_1 = 3$ and $n = 4$. Moreover, if $d_n \geq 4$, then a spanning hamiltonian path has a connected complement.

If $d_1 = d_n = 3$, then we take for G the $\frac{n}{2}$-prism instead. See Figure 1 for the 12-prism. Let T be the spanning tree of G consisting of all edges which are drawn thicker in Figure 1. The cotree edges $E(G) \setminus E(T)$ form a connected subgraph (together with some isolated vertices), so Lemma 3.1 implies that G is universally embeddable.

If $d_1 = q$, then G is a "star" graph in which the vertex v_1 is joined to v_i (for $i = 2, \ldots, n$) with d_i parallel edges. By Lemma 3.1, this graph is universally embeddable. For the rest of the proof we may therefore assume that $d_n < d_1$ and $d_1 \leq q - 1$.

If $d_n = 1$, then we consider the degree sequence $d_1 - 1, d_2, \ldots, d_{n-1}$. This sequence satisfies the necessary conditions, and by the induction hypothesis there is a loopless realization G'. By adding a new vertex and joining it to the vertex of degree $d_1 - 1$ in G', we obtain a loopless realization G of the original degree sequence. Clearly, G is universally embeddable if and only G' is (which we may assume by the induction hypothesis).

If $d_n = 2$, then the sequence $d_1, \ldots, d_{n-1}$ satisfies the conditions of the theorem (since $d_1 \leq q - 1$). Let G' be its loopless realization and let T' be the corresponding spanning tree with at most one odd cotree component. By subdividing an edge of T', we get a loopless relization G for the original degree sequence. Its spanning tree corresponding to T' shows that G is universally embeddable.

Suppose now that $d_n \geq 3$. The case when $n = 3$ is easy, so assume that $n \geq 4$. We consider the sequence $d_1 - d_n, d_2, \ldots, d_{n-1}$. The assumptions made so far show that we can apply the induction hypothesis to get a realization G'. Let T' be its spanning tree confirming upper embeddability of G'. We add the new vertex v_n and join it to v_1 by d_n edges. The resulting graph G is clearly planar and realizes the original degree sequence. Let T be the spanning tree of G obtained by adding one of the edges joining v_1 and v_n to the spanning tree T'. Let us first argue about the case when $d_n = 3$. In this case we see that one cotree component of $G' - E(T')$ has gained two more edges in $G - E(T)$. So, the induction hypothesis and Lemma 3.1 show that G is universally embeddable. Suppose next that $d_n \geq 4$ and that $d_1 - d_n \geq 4$. In this case, all degrees in G' are at least 4, and we now apply the stronger induction hypothesis, namely that $G' - E(T')$ is a connected spanning subgraph of G'. Clearly, the same also holds for the spanning tree T of G, so we have proved the theorem also in this case.

For the rest of the proof we may assume that $d_1 > d_n \geq 4$ and that $d_1 - d_n \leq 3$. Let $\delta = \lfloor \frac{1}{2} d_n \rfloor \geq 2$, and let G_0 be the graph which is isomorphic to the cycle C_n with each edge repeated δ times. Let $\delta_i = d_i - 2\delta$. Clearly, $\delta_i \in \{0, 1, 2, 3, 4\}$. In order to realize the degree sequence $d_1, \ldots, d_n$, we will add edges to G_0 in such a way that v_i is incident with precisely δ_i new edges. We start adding edges so that the vertices v_i with smaller indices i become saturated first. That is, we first add $\delta_2 \leq \delta_1$ edges joining v_1 and v_2. If $\delta_1 > \delta_2$, then we add additional edges to v_1 joining it with v_3, etc. Once v_1 is saturated, we take the first unsaturated vertex and continue the process. What we end up is a graph relizing the original sequence (if all necessary edges have been added), or we are left with precisely one unsaturated vertex v_k. Observe that $1 \leq k < n$ since $\delta_n \leq 1$ and the "unsaturated degree" t is even, so t is either 2 or 4. If $k \neq 1$, then let us remove $t/2$ edges joining v_1 and v_n and add $t/2$ edges from v_k to v_1 and $t/2$ edges from v_k to v_n. If $k = 1$, do the same but remove $t/2$ edges joining v_2 and v_3.

Let T be the hamilton path $v_1 v_2 \ldots v_n$ (if $k \neq 1$) or $v_3 \ldots v_n v_1 v_2$ (if $k = 1$). It is clear that $G - E(T)$ is connected. This proves universal embeddability of G and the claimed stronger property for constructed graphs with minimum degree at least 4. Now the proof is complete. $\square$

A 2-cell embedding in a surface is said to be *non-singular* if every edge appears on the boundary of two distinct 2-cells. Clearly, the dissection is non-singular if and only if the dual graph is loopless.

Corollary 3.3 *Let $d_1, \ldots, d_n$ be positive integers and let $q = \frac{1}{2}\sum_{i=1}^{n} d_i$. Then the following statements are equivalent:*

(a) *Some surface has a non-singular decomposition into n 2-cells whose lengths are $d_1, \ldots, d_n$, respectively.*

(b) *Every surface S, whose Euler genus is at most $q - n + 1$, has a non-singular decomposition $\mathcal{D}_S$ into 2-cells $F_1, \ldots, F_n$, whose lengths are $d_1, \ldots, d_n$, respectively. Moreover, there exist non-negative integers a_{ij}, $1 \leq i < j \leq n$, such that for every such decomposition $\mathcal{D}_S$, the 2-cells F_i and F_j have precisely a_{ij} sides in common.*

(c) *q is an integer, $q \geq n - 1$, and $d_i \leq q$ for $i = 1, \ldots, n$.*

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