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Preprint series, Vol. 44 (2006), 1024

STAR COLORINGS AND ACYCLIC COLORINGS OF LOCALLY PLANAR GRAPHS

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ISSN 1318-4865

November 20, 2006

Ljubljana, November 20, 2006

Star colorings and acyclic colorings of locally planar graphs

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It is proved that every graph embedded in a fixed surface with sufficiently large edge-width is acyclically 7-colorable and that its star chromatic number is at most $2s_0^* + 3$, where $s_0^* \leq 20$ is the maximum star chromatic number for the class of all planar graphs.

Key Words: Acyclic coloring, star coloring, genus, edge-width, locally planar graph.

1. INTRODUCTION

All graphs in this paper are simple. For terms related to graphs embedded in surfaces we refer to [7]. A surface is a compact 2-manifold with-

* Research partly supported by Japan Society for the Promotion of Science, Grant-in-Aid for Scientific Research, Grant number 16740044, by Sumitomo Foundation, by C & C Foundation and by Inoue Research Award for Young Scientists.

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‡ Supported in part by the Research Grant P1–0297 (ARRS, Slovenia) and by Canadian CRC program and NSERC Discovery Grant.

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out boundary. The *Euler genus* of a surface S is the nonnegative integer $g = 2 - \chi(S)$, where $\chi(S)$ is the Euler characteristic of S.

Let c be a coloring of vertices of G. If C is a cycle in G on which only two colors a and b appear, then we say that C is *bi-colored* or (a, b)-*colored* if we need a specific reference to its colors. A coloring of a graph G is *acyclic* if there are no bi-colored cycles. The *acyclic chromatic number* $\chi_{\mathrm{ac}}(G)$ of G is the minimum integer k such that G admits an acyclic k-coloring.

Grünbaum [4] proved that every planar graph has an acyclic 9-coloring and conjectured that five colors suffice. His conjecture was confirmed by Borodin [3].

Theorem 1.1 (Borodin). *Every planar graph is acyclically* 5*-colorable.*

For surfaces other than the plane, Borodin (see [5]) conjectured that the maximum acyclic chromatic number equals the maximum chromatic number of graphs on that surface. Alon, Mohar, and Sanders [2] proved that the acyclic chromatic number of an arbitrary surface with Euler genus g is at most $O(g^{4/7})$. They also proved that this is nearly tight; for every g there are graphs with Euler genus g whose acyclic chromatic number is at least $\Omega(g^{4/7}/(\log g)^{1/7})$. Therefore, the conjecture of Borodin is false for all surfaces with large Euler genus (and may very well be false for all surfaces).

Graphs on surfaces of large genus can have large chromatic number. However, those graphs that are locally planar in the sense that there are no short noncontractible cycles are 5-chromatic as proved by Thomassen [9]. Similar property holds for acyclic colorings. Let us recall that the *edge-width* $\mathtt{ew}(G)$ of a graph G embedded in a surface S of positive Euler genus is the length of a shortest cycle of G that is noncontractible in S. In [6], Mohar proved that every graph on a fixed surface with sufficiently large edge-width is acyclically 8-colorable. In fact, he proved that acyclic 8-coloring exists already under a weaker assumption that all surface nonseparating cycles are sufficiently large. In this paper, we prove the following.

Theorem 1.2. *For every surface S there exists an integer $w = w(S)$ such that every graph embedded in S with edge-width at least w is acyclically* 7*-colorable.*

The proof of Theorem 1.2 is deferred to Section 6.

There exist graphs with arbitrarily large girth and arbitrarily large chromatic number. Since the edge-width (in any embedding) cannot be smaller than the girth, the number w in Theorem 1.2 must depend on the genus of S.

Although we conjecture that Theorem 1.2 can be improved to a 6-coloring result, present techniques do not seem strong enough to prove this strengthening. We also believe that the following conjecture may also be true:

Conjecture 1.1. For every surface S there exists an integer $k = k(S)$ such that every graph G embedded in S contains a vertex set U with $|U| \leq k$ such that $G - U$ is acyclically 5-colorable.

Since $\lfloor \frac{1}{2}(g-1) \rfloor$ copies of K_7 can be embedded into a surface of Euler genus g, the value $k(S)$ in Conjecture 1.1 should be at least linear in the genus of S.

Theorem 1.2 implies a weaker form of the above conjecture.

THEOREM 1.3. *For every surface S there exists an integer $k = k(S)$ such that every graph G embedded in S contains a vertex set U with $|U| \leq k$ such that $G - U$ is acyclically 7-colorable.*

Proof. The proof is by induction on the Euler genus g of S. Let G be a graph embedded in S. If the edge-width of G is at least $w(S)$, where the value of $w(S)$ comes from Theorem 1.2, then that theorem shows that G is acyclically 7-colorable. Otherwise, G contains a set of fewer than $w(S)$ vertices, whose removal gives rise to a graph embedded in a surface of Euler genus smaller than g. Now we just apply the induction hypothesis and the proof is easily completed. ∎

Every graph isomorphic to some $K_{1,t}$ $(t \geq 0)$ is called a *star*. A *star coloring* is a special case of acyclic colorings under which any two color classes induce a subgraph whose components are stars. Equivalently, no subgraph isomorphic to the path P_4 on four vertices is bi-colored. The *star chromatic number* $\chi^*(G)$ of a graph G is the smallest integer k such that G admits a star coloring using k colors.

Let s_0^* be the smallest integer such that every planar graph is star colorable with s_0^* colors. It is known that s_0^* exists. Currently, the best bounds on s_0^* are due to Albertson et al. [1].

THEOREM 1.4 (Albertson, Chappell, Kierstead, Kündgen, Ramamurthi). $10 \leq s_0^* \leq 20$.

Albertson et al. also proved in [1] that for graphs of genus g, $\chi^*(G) \leq 5g + s_0^*$. Albertson (private communication) asked if constant number of colors suffices under some local planarity assumption. An affirmative answer to his question follows from Theorem 1.2. Namely, as shown in [1],

$$\chi^*(G) \leq \chi_{\mathrm{ac}}(G)(2\chi_{\mathrm{ac}}(G) - 1)$$

holds for every graph G, and thus for every graph G embedded on a (fixed) surface with sufficiently large edge-width, Theorem 1.2 implies that $\chi^*(G) \le 91$.

By using techniques developed in order to prove Theorem 1.2, we can improve the above result to the following.

THEOREM 1.5. *For every surface S there exists an integer $w^* = w^*(S)$ such that every graph embedded in S with edge-width at least w^* is star colorable with $2s_0^* + 3$ colors.*

The proof of Theorem 1.5 is deferred to the last section.

Again, Theorem 1.5 implies a result in the spirit of Theorem 1.3.

THEOREM 1.6. *For every surface S there exists an integer $k^* = k^*(S)$ such that every graph G embedded in S contains a vertex set U with $|U| \le k^*$ such that $\chi^*(G-U) \le 2s_0^* + 3$.*

The proof is essentially the same as the proof of Theorem 1.3, so we omit it.

2. PLANARIZING CYLINDERS AND MÖBIUS BANDS

If Q is a plane 2-connected graph with outer cycle C_1 and another facial cycle C_0 disjoint from C_1, then we call Q a *cylinder* with *outer cycle* C_1 and *inner cycle* C_0. If C is a cycle in Q, then we denote by $\mathrm{Int}(C)$ the subgraph of Q consisting of C and all vertices and edges embedded in the disk bounded by C. The *cylinder-width* of G is the largest integer q such that G has $q+1$ pairwise disjoint cycles $R_0, \ldots, R_q$ such that $R_0 = C_0$, $R_q = C_1$, and $C_0 \subseteq \mathrm{Int}(R_1) \subseteq \mathrm{Int}(R_2) \subseteq \cdots \subseteq \mathrm{Int}(R_q)$.

Suppose that a graph H embedded in a surface of Euler genus g has disjoint facial cycles C_0', C_1' of the same lengths as the cycles C_0 and C_1 of Q (respectively), then we can identify C_0 and C_0' into a cycle C_0'' and identify C_1 and C_1' into a cycle C_1''. Let G be the graph obtained from the union of Q and H after these identifications. The embeddings of Q and H determine an embedding of G into a surface of Euler genus $g+2$. We also say that H is obtained from G by *cutting out* the cylinder Q.

Let M be a graph embedded in the projective plane and let C_0 be a cycle of M bounding a face. We say that M together with the distinguished face C_0 is a *Möbius band* and that C_0 is its *outer cycle*. The *depth* of the Möbius band G is the largest integer q such that Q has $q+1$ pairwise disjoint cycles $R_0, \ldots, R_q$ such that $R_0 = C_0$, R_q is a 1-sided cycle, and each of $R_1, \ldots, R_{q-1}$ separates R_0 from R_q.

If H is a graph on a surface of Euler genus g with a facial cycle C_0' of the same length as the outer cycle C_0 of Q, then we can identify C_0 and C_0' into a cycle C_0''. Let G be the graph obtained from the union of Q and H after this identification. The embeddings of Q and H determine an embedding of G into a surface of Euler genus $g+1$. We also say that H is obtained from G by *cutting out* the Möbius band Q.

Robertson and Seymour [8] proved an important result about graph minors in "densely embedded" graphs, whose special case restricted to triangulations can be stated as follows:

THEOREM 2.1 ([8]). *Let S be a surface, and let W be a graph that is embedded in S. Then there is a constant w such that every triangulation of S with edge-width at least w contains a subgraph W' such that W is isomorphic to a minor of W' and its induced embedding is combinatorially the same as the embedding of W.*

Theorem 2.1 implies the following.

COROLLARY 2.1. *For any natural numbers g and r there exists an integer $f(g,r)$ such that every triangulation G of a surface of Euler genus g with edge-width at least $f(g,r)$ contains $h = \lfloor g/2 \rfloor$ pairwise disjoint cylinders $Q_1, \dots, Q_h$ of cylinder-width r, and if g is odd, also contains a Möbius band M of depth r and disjoint from $Q_1, \dots, Q_h$, such that cutting out these cylinders and Möbius band (when g is odd) results in a connected plane graph.*

Moreover, when g is odd, M can be chosen in such a way that it contains a shortest one-sided cycle of G, whose distance from the boundary cycle of M is equal to r.

Proof. The first part of the corollary follows from Theorem 2.1. A simple proof of this part for orientable surfaces was obtained by Thomassen [10, Theorem 9.1]; see also [7].

In order to prove the second part of the corollary, we assume that g is odd. We will first find a "deep" Möbius band with a geodesic one-sided cycle in the middle and prove that after cutting out this Möbius band and contracting its boundary to a point, the resulting graph G' is embedded in a surface of Euler genus $g-1$ and still has large edge-width. This will enable us to apply Theorem 2.1 to G' and get the cylinders $Q_1, \dots, Q_h$.

Let C be a shortest one-sided cycle in G. For $i = 1, \dots, r+1$, there is an induced cycle C_i in G whose vertices are all at distance i from C such that C_i is a surface separating cycle which separates a Möbius band containing C from the rest of the surface. This follows from standard homotopy arguments using the 3-path property of one-sided cycles.

Let M be the Möbius band corresponding to C_r. We claim that the rest of the surface contains h disjoint cylinders of cylinder width r if the edge-width is sufficiently large. To see this, we first cut along C_{r+1} to obtain a surface of Euler genus $g-1$ in which C_{r+1} bounds a face. In order to apply our corollary inductively (for genus $g-1$), we add a vertex x into this face and join it to all vertices of C_{r+1}. Let G' be the resulting graph. If the edge-width of G' is at least $f(g-1, 2r+1)$, the cylinders of width $2r+1$ exist. Each of these cylinders is also a cylinder for G unless it contains x. However, in the latter case, it contains a subcylinder of width r which does not contain x. In this way, all h cyclinders for G are obtained.

It remains to show that G' has edge-width at least $f(g-1, 2r+1)$. To prove this, we assume that the edge-width w of G satisfies:

$$w \geq f(g,r) \geq 2f(g-1, 2r+1) + 4r. \tag{1}$$

Let R be a shortest non-contractible cycle of G'. If R does not contain x, then R is also non-contractible cycle of G and hence it is longer than $f(g-1, 2r+1)$. So suppose that R contains x. Let $x_1, x_2 \in R \cap C_{r+1}$ be the neighbors of x on R. For $i = 1, 2$, let Z_i be a path in G of length $r+1$ from x_i to a vertex y_i on C. Let X_1 and X_2 be the paths from y_1 to y_2, whose union is equal to C, and suppose that $|X_1| \leq |X_2|$. Finally, let R_i be the closed walk composed of paths $R - x$, Z_1, Z_2, and X_i, $i = 1, 2$. It is easy to see that R_i is non-contractible and that precisely one of R_1, R_2 is one-sided.

If R_1 is one-sided, then $|R_1| \geq |C|$ and hence by (1):

$$\begin{aligned} |R| &\geq |R_1| - \frac{1}{2}|C| - 2(r+1) + 2 \geq \frac{1}{2}|C| - 2r \\ &\geq \frac{1}{2}f(g,r) - 2r \geq f(g-1, 2r+1). \end{aligned}$$

If $|R_1| \geq 2|X_1|$, then similarly

$$\begin{aligned} |R| &\geq |R_1| - |X_1| - 2(r+1) + 2 \geq \frac{1}{2}|R_1| - 2r \\ &\geq \frac{1}{2}f(g,r) - 2r \geq f(g-1, 2r+1). \end{aligned}$$

As for the third alternative, let us suppose that $|R_1| \leq 2|X_1|$ and that R_1 is two-sided. Then R_2 is one-sided, and hence $|R_2| \geq |C|$. This implies

$$\begin{aligned} |R| &= |R_2| - (|C| - |X_1|) - 2r \geq |X_1| - 2r \\ &\geq \frac{1}{2}|R_1| - 2r \geq f(g-1, 2r+1). \end{aligned}$$

In all cases we have concluded that $|R| \geq f(g-1, 2r+1)$, as claimed. This completes the proof. ∎

3. CRITICAL GRAPHS AND TRIANGULATIONS

For the purpose of this paper, a graph G embedded in a surface S is *critical* if $\chi_{\text{ac}}(G) > 7$, but every graph embedded in S with fewer vertices than G and with edge-width at least $\mathtt{ew}(G)$ has acyclic chromatic number at most 7.

PROPOSITION 3.1. *Let G be embedded in some surface and suppose that G is critical. Then every contractible 3-cycle in G bounds a face. If C is a contractible 4-cycle of G, then the disk bounded by C contains in its interior at most one vertex of G.*

Proof. The claim about 3-cycles is easy. If C is not facial, let U be the set of in the interior of the disk bounded by C. Note that $\mathtt{ew}(G-U) = \mathtt{ew}(G)$. Let f be an acyclic 7-coloring of $G-U$. By Theorem 1.1, $\text{Int}(C)$ can be acyclically 5-colored, and we may assume that this coloring coincides with f on C. The combination of both colorings now yields an acyclic 7-coloring of G, a contradiction.

Let $C = abcd$ be a contractible 4-cycle and let H be the subgraph of G consisting of C and all vertices and edges in the disk D bounded by C. By the above, we may assume that C is chordless in H and that no vertex of H is incident to all vertices on C. Consequently, we may also assume that there is no vertex of $H - a - c$ is adjacent to both b and d. Trying to get a contradiction, we may also assume that H has at least six vertices.

If H has a vertex t which is adjacent to two opposite vertices of C, then t is adjacent to a, c, and possibly b (say). Let G' be the graph obtained from G by deleting all vertices of $H - V(C)$ and adding a vertex t' adjacent to all vertices of C. Since G is critical, and since $\mathtt{ew}(G') = \mathtt{ew}(G)$, G' has an acyclic 7-coloring $f' : V(G') \to \{1, \ldots, 7\}$. We may assume that $f'(a) = 1$, $f'(b) = 2$, and $f'(t') = 5$. Since f' is acyclic, we have the following possibilities for the remaining two vertices of C:

(i) $f'(c) = 1$ and $f'(d) = 3$,
(ii) $f'(c) = 3$ and $f'(d) = 2$,
(iii) $f'(c) = 3$ and $f'(d) = 4$.

Let H' be the graph obtained from H by identifying vertices a and c into a single vertex (which we also call a). Finally, we replace double edges between a and b, a and d (and a and t if t exists) by single edges.

Since H' is planar, it has an acyclic 5-coloring $h' : V(H') \to \{1, 4, 5, 6, 7\}$. By possibly permuting the colors, we may assume that $h'(a) = 1$. In case (i) we also assume that $h'(t) = 5$ if t exists. In case (iii), we assume that $h'(d) = 4$.

Finally, we extend the coloring f' of G' to a coloring f of G by coloring all vertices inside D the same as under the coloring h'. If the resulting coloring of G has a bi-colored cycle R, then R uses at least one vertex in D. Since colors 2 and 3 are not used in h', one of the two colors on R is distinct from 2 and 3. If the other color were 2 or 3, R would have only one vertex in D and would leave and enter D through vertices of that color. But since b and d have no common neighbors distinct from a and c, this is not possible. Clearly, R cannot enter through a and leave through c since its bi-colored segment in D would give rise to a bi-colored cycle in H', which is not possible under the acyclic coloring h'. It follows that R does not exist and hence we have an acyclic 7-coloring of G. ∎

The following observation, whose easy proof is left to the reader, will enable us to restrict our attention to triangulations.

Proposition 3.2. *Let G be graph embedded in a surface S. Then there is a triangulation G' of S such that G is an induced subgraph of G' and the edge-width of G' is equal to the edge-width of G. Moreover, if every contractible 3-cycle of G is facial and every contractible 4-cycle in G contains at most one vertex in its interior, then the same property holds also for G'.*

4. FIT CYCLES

Let G be a Π-embedded graph, and let $R = x_1x_2 \dots x_n$ be an induced path or a cycle. A 2-*jump* over R is a path $J = x_iyx_j$ with the following properties:

(J1) $y \notin V(R)$,
(J2) $j \geq i + 2$,
(J3) all those neighbors of y, which belong to R, lie on the segment $x_ix_{i+1} \dots x_j$, and
(J4) the cycle $S = x_ix_{i+1} \dots x_jyx_i$ is Π-contractible and none of the vertices $x_1, \dots, x_{i-1}$ and $x_{j+1}, \dots, x_n$ lies inside the disk bounded by S.

If $J = x_iyx_j$ is a 2-jump over R, we say that J is *based* on x_i and x_j and that y is its *apex*. The vertices $x_{i+1}, \dots, x_{j-1}$ are said to be *covered* by the 2-jump J. The *length* of the 2-jump is $|j - i|$; the jump is *short* if its length is 2, and *long* otherwise.

Let Q be a cylinder in a Π-embedded graph with boundary cycles Q_1, Q_2, or a Möbius band with boundary cycle Q_1. A cycle $C \subseteq Q$ is said to be *homotopic to the boundary* of Q if Q can be written as the union $R \cup S$, where R is a cylinder with boundary cycles Q_1 and C, and S is either a cylinder with boundary cycles Q_2 and C (if Q is a cylinder), or a Möbius band with boundary cycle C (when Q is a Möbius band). We extend the above definition in the obvious way to the case when C is not disjoint from the boundary of Q by allowing that R or S is a "degenerate cylinder" in which the boundary cycles are allowed to intersect.

A cycle $C \subseteq Q$ is said to be *fit* if the following properties hold:

(F1) C is an induced cycle, homotopic to the boundary of Q and is disjoint from the boundary of Q.

(F2) No vertex of C has degree smaller than 5.

(F3) If two distinct 2-jumps over C are based on the same pair of vertices of C, then they are short 2-jumps, their apices are adjacent, and one of them is a vertex of degree 4.

(F4) If 2-jumps $x_i y x_j$ and $x_k z x_l$ interlace, i.e., $i < k < j < l$, then these 2-jumps are both short.

(F5) If $x_i y x_j$ and $x_k z x_l$ are distinct long 2-jumps, then either $k \geq j$ or $l \leq i$.

Fit cycles will be important for cutting the surface into planar pieces whose acyclic 5-colorings will afterwards be combined into an acyclic 7-coloring of the original graph. Fit cycles exist and can be found by the method used in the proof given below.

A cylinder (or a Möbius band) Q with boundary cycles C_1 and C_2 is *triangulated* if all its faces except the outer and the inner cycle are triangles. The *edge-width* of Q is the length of a shortest cycle in Q which is homotopic to the boundary of Q.

Lemma 4.1. *Suppose that Q is a triangulated cylinder of cylinder-width at least 6 and edge-width at least 5, in which the interior of every 4-cycle contains at most one vertex. Then Q contains a fit cycle.*

Proof. Let C_1, C_2 be the boundary cycles of Q, and let C be a cycle in $Q' = Q - (C_1 \cup C_2)$ such that the following conditions are satisfied (where each condition is understood as being subject to all previous ones):

(1) C is a shortest cycle in Q' homotopic to the boundary of Q.

(2) C has minimum number of vertices of degree less than 5.

(3) The number of long jumps is minimum.

We claim that C is fit. Since Q has cylinder-width more than 1, such a cycle exists. Since cycles homotopic to the boundary satisfy the 3-path-property (cf., e.g., [7]), a shortest such cycle is an induced cycle. Therefore, C satisfies (F1).

Since Q is a triangulated cylinder, no induced cycle disjoint from the boundary contains a vertex of degree less than 4, and if C contains a vertex u of degree 4, it passes through two opposite neighbors a, c of u. Let b, d be the other two neighbors of u. Since the cylinder-width of Q is more than 2, we may assume that $b \in V(Q')$. Since every 4-cycle contains at most one vertex in its interior, the degree of b is at least 5. Therefore, changing the cycle C by replacing the path auc by abc, contradicts (2). This shows that (F2) holds.

To verify (F3), observe that 2-jumps $x_i y x_j$ and $x_i z x_j$ ($y \neq z$) form a 4-cycle, which is contractible since the edge-width of Q is bigger than 4 and the cylinder-width is at least 6. Therefore, we may assume that z does not belong to the boundary of Q. The 3-path-property and (1) then imply that these 2-jumps are short. Thus, $x_i x_{i+1} x_{i+2} y x_i$ and $x_i x_{i+1} x_{i+2} z x_i$ are also contractible 4-cycles. Since they contain at most one vertex in their interiors, it is easy to see that either y, z, or x_{i+1} is a vertex of degree 4 contained in the interior of one of these 4-cycles. By (F2), this cannot be x_{i+1}, and (F3) follows.

Suppose that $x_i y x_j$ is a long 2-jump. Since C is a shortest cycle homotopic to the boundary, the 3-path property of such cycles implies that the apex y of the 2-jump is not in Q', i.e., $y \in C_1 \cup C_2$. This observation shows that two long 2-jumps cannot interlace. Suppose now that $x_k z x_l$ is a short 2-jump interlacing with $x_i y x_j$. By (F3) we may assume that the degree of z is at least 5. We may also assume that $k = j - 1$ and $l = j + 1$. Let us now replace C by the cycle C' obtained by changing the segment $x_{j-1} x_j x_{j+1}$ to $x_{j-1} z x_{j+1}$. The new cycle satisfies (1), has no small degree vertices and has fewer long jumps than C. This contradicts (3) and proves (F4).

Let us now show that (F5) is satisfied. By (F3) and (F4) we may assume (reductio ad absurdum) that $i \leq k < l < j$. Then z is contained in the disk bounded by the contractible cycle $x_i x_{i+1} \dots x_j y x_i$. Moreover, since y and z are both in $C_1 \cup C_2$, this yields a contradiction. ▮

5. CHANGING COLORS ACROSS A CYLINDER

The following result was essentially proved in [6] but we include a sketch of the proof for completeness.

Lemma 5.1. *Let Q be a cylinder of cylinder-width $d \geq 11$, and let Q' be obtained from Q by contracting the boundary cycles of Q to single vertices*

x_1 and x_2, respectively. Then for every $a, b \in \{1, \dots, 5\}$, there exists an acyclic 6-coloring f of Q' such that the following conditions hold:

(i) *$f(x_1) = a$ and $f(x_2) = b$.*
(ii) *No vertex at distance at most two from x_1 or x_2 is colored 6.*
(iii) *There is no bi-colored path from x_1 to x_2.*

Proof. By Theorem 1.1, there is an acyclic 5-coloring f of Q', and we may assume that $f(x_1) = a$. Let $Q_0, Q_1, \dots, Q_d$ be the homotopic cycles showing that the cylinder-width of Q is at least d. For $1 \le i < j \le d$, let us denote by $R_{i,j} \subseteq Q'$ the subgraph of Q' obtained from the subcylinder of Q between Q_i and Q_j, where Q_d is contracted to the vertex x_2 if $j = d$. Observe that $R_{1,d} \supset R_{2,d} \supset \cdots \supset R_{d-1,d}$.

Let us now change the coloring of Q' as follows. If $f(x_2) = b$, we replace color a with color 6 in $R_{3,4}$. Let us observe that changing certain color with another color t gives rise to a new acyclic coloring if t is not used in the first and the second neighborhood of the vertices where the change has been made. The performed change will assure that (iii) is satisfied, and (i)–(ii) are clear. Thus, we are done.

Suppose now that $f(x_2) = c \ne b$. Let us also suppose that $b \ne a$. Then we change color a to color 6 on the whole $R_{3,d}$. Next, we replace color b with color a on $R_{5,d}$. If $c \ne a$, we next change color c to b on $R_{7,d}$ and finally replace color 6 to c on $R_{9,d}$ (in order to assure (ii)). If $c = a$ then, after the change on $R_{5,d}$, the current color of x_2 is 6, so we can simply replace 6 by color b on $R_{7,d}$.

The only remaining case is when $f(x_2) = c \ne b$ and $b = a$. In this case we first replace color a by color 6 on $R_{3,d}$, and afterwards replace color c by a on $R_{5,d}$. ∎

We shall need another coloring result for Möbius bands. The proof of this statement is implicit in the proof of Theorem 2 in [2] and will not be repeated here.

LEMMA 5.2. *Let M be a triangulated Möbius band and let M' be obtained from M by contracting the boundary cycle of M to a single vertex x_1. Let C be a shortest one-sided cycle in M. If the distance of every vertex of C to the boundary of M is at least four, then for every $a \in \{1, \dots, 5\}$, there exists an acyclic 7-coloring f of M' such that*

(i) *$f(x_1) = a$, and*
(ii) *no vertex at distance at most two from x_1 is colored 6 or 7.*

6. PROOF OF THE MAIN THEOREM

This section is devoted to the proof of Theorem 1.2. Let G be a graph that is embedded in a surface S of Euler genus g with edge-width at least w, where w will be specified below. Since the edge-width cannot be decreased by taking induced subgraphs (and their induced embeddings), we may assume that G is critical. By Proposition 3.1 we may assume that every contractible 4-cycle contains at most one vertex in its interior. By Proposition 3.2, we may assume that G is a triangulation of edge-width at least w whose contractible 4-cycles still contain at most one vertex in their interiors. However, we no longer require G to be critical. We will prove in the sequel that our graph is acyclically 7-colorable, thus completing the proof.

We assume that w is large enough such that Corollary 2.1 can be applied to obtain $h = \lfloor g/2 \rfloor$ pairwise disjoint cylinders $Q_1^\circ, \ldots, Q_h^\circ$ of cylinder-width at least 24, and (if g is odd) a Möbius band M° of depth at least 24 such that cutting out these cylinders and Möbius band (when g is odd) results in a connected plane graph. Moreover, if M° exists, then it satisfies the condition of having a shortest one-sided cycle "deep inside" as required in Lemma 5.2.

Let Q_k° $(1 \leq k \leq h)$ be one of these cylinders, and let $R_0, \ldots, R_{24}$ be disjoint cycles homotopic to the boundary showing that the cylinder width of Q_k° is at least 24. Let $R_{i,j}$ be the subcylinder with boundary cycles R_i and R_j. We apply Lemma 4.1 in $R_{1,7}$ to obtain a fit cycle C_1 in $R_{1,7} - (R_1 \cup R_7)$. Similarly, we can obtain a fit cycle C_2 in $R_{17,23} - (R_{17} \cup R_{23})$. We apply the similar procedure to M° if it exists.

By applying the above procedure, we replace the cylinders (and the Möbius band) by their subgraphs and obtain a collection of $h = \lfloor g/2 \rfloor$ pairwise disjoint cylinders $Q_1, \ldots, Q_h$ of cylinder-width at least 11, and (if g is odd) a Möbius band M of depth at least 4, whose boundary cycles are all fit, and such that cutting out these cylinders and Möbius band (when g is odd) results in a connected plane graph H_0. Moreover, if M° exists, then it contains a shortest one-sided cycle of G and the distance from it to the boundary of M° is at least 4.

Observe that H_0 has precisely g nontriangular faces whose boundaries are disjoint cycles $C_1, \ldots, C_g$ of G, corresponding to the boundary cycles of the removed cylinders and Möbius band. Moreover, distinct boundary cycles are at distance at least 5 from each other since the constructed fit cycles C_1, C_2 (and similarly others) are disjoint from R_0, R_1, R_{23}, and R_{24}.

Let H_1 be the plane graph obtained from H_0 by contracting each cycle C_i into a single vertex c_i, $i = 1, \ldots, g$. By Theorem 1.1, H_1 has an acyclic 5-coloring $f_1 : V(H_1) \to \{1, \ldots, 5\}$.

From the cylinders Q_i and the possible Möbius band M we form graphs Q'_i and M' by contracting their boundary cycles to single vertices. For each of them we apply Lemma 5.1 or 5.2 to get an acyclic 7-coloring under which each contracted boundary cycle C_i has color $f_1(c_i)$, same as the corresponding vertex c_i has under the coloring of H_1. It is important to observe that under all these colorings, the vertices, which are at distance at most 2 from $C_1 \cup \cdots \cup C_g$ in G, use only colors $1, \ldots, 5$.

We shall now define a coloring f of G. This coloring coincides with acyclic colorings obtained above for $H_1, Q'_1, \ldots, Q'_h$ and M' (if applicable). In the sequel we shall describe how to color the vertices on all cycles $C_1, \ldots, C_g$ in such a way that the resulting coloring f will be an acyclic 7-coloring of G. The procedure is the same for all cycles, and we shall describe it only for C_1. We assume that C_1 is the boundary cycle in Q_1 and that $f_1(c_1) = 5$. To color C_1 we shall use colors 5, 6 and 7 (each at least once) such that no consecutive vertices on C_1 receive the same color. Since C_1 is an induced cycle and since colors 5, 6, 7 do not occur on the neighbors of c_1 (neither in H_1 nor in Q'_1), this gives rise to a proper coloring of G. All we have to argue is that there are no bi-colored cycles.

Suppose that C would be a bi-colored cycle. Clearly, C would need to contain a vertex from one of $C_1, \ldots, C_g$ (say from C_1). The two colors on C are therefore $a \in \{5, 6, 7\}$ and $b \in \{1, 2, 3, 4\}$. If $a \in \{6, 7\}$, then every second vertex on C is colored a, and since vertices at distance at most 2 from C_1 are not colored 6 or 7, all these vertices belong to C_1. The same holds if $a = 5$: the cycle C cannot start at one side of C_1 and come back from the other side, and can neither cross any other cylinder Q_j, because of condition (iii) in Lemma 5.1. Therefore, the edges of C would give rise to a $(5, b)$-colored cycle in H_1 if at least one vertex of C colored 5 would not belong to C_1. This proves our assertion that every vertex of C, whose color is a, belongs to C_1.

Before examining the structure of bi-colored cycles in more details, let us describe the requirements that we shall impose on the coloring of C_1. Roughly speaking, we will color vertices of C_1 in order by using colors 5, 6, 7 so that no two vertices on C_1 which are a base of a short 2-jump, are colored the same. Under such assumption, 2-colored cycles would not be able to use short 2-jumps. In fact, we will have to allow a few exceptions to the above condition, but they will be apart from each other, so it will still not be possible to use short 2-jumps to make a bi-colored cycle. Henceforth, any bi-colored cycle would need to use apices of long 2-jumps outside of C_1. Because C_1 is fit, the only possibility for this to happen would be that long 2-jumps form a cycle homotopic to C_1. If such a cycle exists, it is uniquely determined and consists of all long 2-jumps. Therefore, it suffices to arrange that not all vertices which are bases of long 2-jumps over C_1 are colored the same.

Let $C_1 = v_1 v_2 \dots v_k v_1$. Every coloring $c : V(C_1) \to \{5, 6, 7\}$ of C_1 with $c(v_1) = 5$ and $c(v_2) = 6$ is completely determined by specifying the set N of those vertices v_i for which $c(v_{i-1}) = c(v_{i+1})$ (indices considered modulo k). The vertices in N are called *exceptional*, since they may provide exceptions to our goal of coloring base vertices of short 2-jumps by distinct colors.

Let us first characterize sets $N \subseteq V(C_1)$ which are exceptional sets of colorings of C_1 with three colors. The first condition is that $|N|$ is even. To see this, look at the corresponding coloring c as a homomorphism into the 3-cycle and consider its winding around the 3-cycle. Each exceptional vertex represents the change of clockwise direction to anticlockwise (or vice versa). Since the direction is clockwise at the beginning and again at the end, the total number of changes is even.

TABLE 1.

Effect of having two exceptional vertices at a given spread

Coloring	Spread modulo 3
... 567567567567567 ...	–
... 567**5**76576**5**67567 ...	0
... 567**5**76576**7**5675 ...	1
... 567**5**7657657**6**756 ...	2

Let $N = \{v_{i_1}, v_{i_2}, \dots, v_{i_{2r}}\}$ $(r \geq 0)$, where $1 \leq i_1 < i_2 < \cdots < i_{2r} \leq k$. Define the *spread* of $v_{i_j} \in N$ to be the "distance" to the next exceptional vertex, i.e., $i_{j+1} - i_j$ (if $j = 2r$, then the spread is deffined accordingly to be $k - i_{2r} + i_1$). So, the sum of all spreads is equal to k. Table 1 shows that the change of coloring made by two consecutive exceptional vertices v_{i_j} and $v_{i_{j+1}}$, when compared to the coloring without exceptional vertices, has the same effect as "prolonging" the cycle C_1 by the spread of v_{i_j} (modulo 3). The colors of exceptional vertices are shown in bold.

Conclusions of the last two paragraphs imply that a set $N \subseteq V(C_1)$ is the set of exceptional vertices of some 3-coloring of C_1 if and only if $|N|$ is even and (with the above notation) the sum of spreads of $v_{i_1}, v_{i_3}, v_{i_5}, \dots,$ $v_{i_{2r-1}}$ is congruent to $-k$ (mod 3). Observe that the sum of all spreads is equal to k, so the sum of "complementary" spreads is also congruent to $-k$ (mod 3).

To describe a coloring of C_1, we shall only define its exceptional set which will consist of 0, 2, or 4 vertices. If $k \equiv 0 \pmod 3$ and there exists a cycle $\hat{C}$ composed of long 2-jumps, whose lengths are all multiples of 3, then we select four long 2-jumps with disjoint bases $\{j_1, l_1\}, \{j_2, l_2\}, \{j_3, l_3\}, \{j_4, l_4\}$, $j_1 < l_1 < j_2 < l_2 < j_3 < l_3 < j_4 < l_4 < k$. Since the edge-width of G is more than 16, such 2-jumps exist. For $t = 1, 2, 3, 4$, we select i_t, $j_t < i_t < l_t$, such that $i_2 - i_1 \equiv 1 \pmod 3$ and $i_4 - i_3 \equiv 2 \pmod 3$. Since our 2-jumps are long, $i_1, \ldots, i_4$ exist. Finally, we set $N = \{i_1, i_2, i_3, i_4\}$. If $k \equiv 0 \pmod 3$ and the cycle $\hat{C}$ with the properties stated above does not exist, then we set $N = \emptyset$.

For the remaining case, suppose that $k \not\equiv 0 \pmod 3$. If there is a cycle $\hat{C}$ composed of long 2-lumps, then the length of at least one of long 2-jumps is not a multiple of 3. In such a case we may assume that such a 2-jump has basis $\{v_j, v_k\}$. If there is a cycle $\hat{C}$ homotopic to C_1, which is composed of long 2-jumps and one or two short 2-jumps, then we assume that $j = k-2$ and that $\{v_j, v_k\}$ is the basis of one of these two 2-jumps. We define $N = \{v_2, v_{8+t}\}$, where $t \in \{1, 2\}$ is selected so that $t \equiv -k \pmod 3$. (Note that $8 + t < j$ when $\hat{C}$ exists.)

Using the above characterization of exceptional sets, it is easy to see that N defines a coloring c of C_1 with colors $5, 6, 7$. The choices made in various cases assure that possible cycle $\hat{C}$ composed of long 2-jumps is not 2-colored. In the first case this is because v_{i_1} and v_{i_2} guarantee that the base of the long 2-jump following the jump with basis $\{v_{j_2}, v_{l_2}\}$ will be colored differently than the vertex v_{j_1}. In the second case, this is because one of the long 2-jumps has length not divisible by 3. We argue similarly in the third case because of the 2-jump with basis $\{v_j, v_k\}$, for which one can show that $c(v_j) \neq c(v_k)$.

The only remaining possibility for a bi-colored cycle would be to contain some short 2-jumps. However, this cannot occur since such a cycle would either contain two 2-jumps $v_i y v_{i+2}$, $v_i z v_{i+2}$ over the same base, or would be composed of a short 2-jump $v_i y v_{i+2}$ and the apex z of a long 2-jump adjacent to both v_i and v_{i+2}, or would contain two consecutive 2-jumps $v_i y v_{i+2}$ and $v_{i+2} z v_{i+4}$. The latter possibility is excluded by our choice of N not containing an element with spread 2. The former two cases are not possible since y and z are adjacent by Proposition 3.1 and by property (F2) of the fit cycle C_1. This completes the proof of Theorem 1.2.

7. PROOF OF THE STAR COLORING THEOREM

This section is devoted to the proof of Theorem 1.5. As the proof builds on the similar principles as the proof of Theorem 1.2, we shall be vague at those points which are identical to that proof, and will try to be clear about the differences.

One difference here is that we do not have the result corresponding to Proposition 3.1. However, that proposition is needed only to achieve property (F2) of fit cycles, which is used at the very end of the proof in Section 6 to exclude bi-colored 4-cycles when an exceptional vertex is covered by two short 2-jumps. Accordingly, the corresponding Proposition 3.2 has to be understood without the assumption and without the conclusion about contractible 3-cycles and 4-cycles.

The application of Lemma 5.2 will be replaced by the following one:

PROPOSITION 7.1. *Let G be a triangulation of the projective plane and let C be a shortest non-contractible cycle in G. Then G has a star coloring with colors $1, 2, \ldots, s_0^* + 4$ such that the colors $s_0^* + 1, \ldots, s_0^* + 4$ are used only on the vertices of C.*

Proof. Let $C = v_1 \ldots v_k v_1$. The graph $G - \{v_1, \ldots, v_{k-1}\}$ is planar and thus admits a star coloring with colors $1, \ldots, s_0^*$. Now we color $v_1, \ldots, v_{k-2}$ with three new colors in such a way that for every $i = 1, \ldots, k-4$, vertices v_i, v_{i+1}, v_{i+2} receive three distinct colors. Finally, we color v_{k-1} with the fourth color. Since C is a shortest noncontractible cycle in G, the 3-path property implies (if $k \geq 6$) that no two vertices on C which are at distance at most 2 in G have the same color. Obviously, this property holds also for $k \leq 5$ since then all vertices of C are colored differently. This implies that we have a star coloring of the whole graph G. ∎

The definition of a fit cycle has to be changed in order to be used for star colorings. On one hand, Proposition 3.1 does not hold for star colorings, so we will no longer require (F2), and we will have to replace (F3) by another, similar condition stated below as (F3'). We shall need an additional property, henceforth denoted by (F2'):

(F2') If a vertex $x_l \in V(C)$ is adjacent to the apex y of a long 2-jump $J = x_i y x_j$ and if $J' = x_l y' x_m$ or $J' = x_m y' x_l$ is a 2-jump based on x_l, where $i \leq m \leq j$, then y and y' are on the same side of C.

(F3') If two distinct 2-jumps over C are based on the same pair of vertices of C, then they are short 2-jumps, and they are both on the same side of C.

As in the proof of Lemma 4.1 we show that a cycle $C = x_1 x_2 \ldots x_k$ with properties (F1) and (F4)–(F5) exists by imposing conditions (1) and (3) stated in that proof. Now, among all cycles satisfying (1) and (3), we select one for which

(4) the number of pairs J, J' which violate (F3') is minimum, and
(5) the number of pairs J, J' which violate (F2') is minimum.

It is easy to see that (4) implies (F3'). So, let us concentrate on proving (F2'). If C does not fulfil (F2'), let $J = x_i y x_j$ and $J' = x_l y' x_m$ (say) be as in (F2'), where y and y' are on different sides of C.

Clearly, J' is a short 2-jump. We may assume that $m = l + 2$, and then we replace C by the cycle $C' = x_1 \ldots x_l y' x_{l+2} \ldots x_k$. Since y' is on the other side than y, and y being the apex of a long 2-jump is on $C_1 \cup C_2$, we conclude that $y' \notin C_1 \cup C_2$. This implies that C' is an eligible cycle, and it suffices to show that we improve on one of the imposed minimality conditions. We may also assume that y' is not contained in a disk bounded by another 2-jump $J'' = x_l y'' x_{l+2}$. In that case, we would rather consider J'' instead of J'.

Under the above assumptions, the new cycle C' still satisfies (F3') and (4) is not affected. However, (5) is improved. This contradiction to the fact that C was chosen optimally proves that (F2') holds as well.

We are ready to proceed with the conclusion of our proof of Theorem 1.5. We start with a graph G embedded with large edge-width. We may assume G is a triangulation of large edge-width, and we decompose G into a plane graph G_0 and $h = \lfloor g/2 \rfloor$ wide cylinders $Q_1, \ldots, Q_h$ and possibly one deep Möbius band M, whose boundary cycles are all fit (according to the modified definition) and which are at distance at least five from each other. We color $Q_1, \ldots, Q_h$ and M with s_0^* and $s_0^* + 4$ colors, respectively, so that the four additional colors are used only on the shortest one-sided cycle "deep inside" M. Next we use colors $s_0^* + 1, \ldots, 2s_0^*$ to color G_0. Let us observe that, unlike in the proof of Theorem 1.2, we do not contract boundary cycles to single vertices. We shall combine these star colorings into a star coloring of G. In order to achieve this, we will need to specify the coloring of vertices on the boundary cycles $C_1, \ldots, C_g$. To color them, we will either use their color from the coloring of G_0 or from the corresponding cylinder or the Möbius band.

Let us describe how we color C_1. The procedure for other cycles is identical. Let c_0 be the star coloring of G_0 with colors $s_0^* + 1, \ldots, 2s_0^*$ and let c_1 be the star coloring of the cylinder or the Möbius band N whose boundary cycle is C_1. We think of using colors $1, \ldots, s_0^*$ in c_1 (and we may neglect possibly used colors $s_0^* + 1, \ldots, s_0^* + 4$ in the case of the Möbius band M since they appear only at distance three or more from C_1). If a vertex $v \in V(C_1)$ is adjacent to the apex of a long 2-jump from G_0 (or from N, respectively), then we color it by $c_0(v)$ ($c_1(v)$, respectively). Let v be a vertex that is not adjacent to the apex of a long 2-jump. If v is a base of a short 2-jump in G_0 but not of one in N, we color it by $c_0(v)$. If it is a base of a short 2-jump in N but not of one in G_0, we color it by $c_1(v)$. For other cases, let us fix an orientation of the cycle C_1. If v is a base vertex of a 2-jump in G_0 and of one in N, then by (F3') one of the 2-jumps precedes the other one, where precedence is considered with respect to the selected orientation of C_1. We use the color corresponding to the side containing the 2-jump which is earlier with respect to the chosen orientation of C_1. In all other cases, we use the color $c_0(v)$.

Let us consider a 4-path P which is bi-colored under the constructed coloring c of G. Since c_0 and c_1 themselves do not have bi-colored 4-paths,

$P = x_1y_1x_2y_2$ has to use two vertices, say x_1 and x_2, whose color a comes from c_0, i.e., $a = c_0(x_1) = c_0(x_2)$, and two vertices y_1, y_2 whose color b comes from c_1, i.e., $b = c_1(y_1) = c_1(y_2)$. Clearly, at least two of these vertices belong to C_1. In particular, $x_1, x_2 \in V(G_0)$ and $y_1, y_2 \in V(N)$.

Let us first suppose that $y_1 \notin V(C_1)$. Then either $x_1y_1x_2$ is a 2-jump in N or y_1 is an apex of a long 2-jump and x_1, x_2 are both adjacent to it. The colors for x_1, x_2 have been taken from c_0 only because they may be incident with a long 2-jump in G_0 or incident with a short 2-jump in G_0 preceding the 2-jump $x_1y_1x_2$ in N. However, it is easy to conclude that this is not possible having properties (F2')–(F3').

A similar conclusion holds when $x_2 \notin V(C_1)$. So, we may conclude that y_1 and x_2 are both on C_1. For every such bi-colored path P, we say that its vertex y_1 is *problematic for* N, and that x_2 is *problematic for* G_0.

We shall modify c so that the bi-colored 4-paths will all disappear. In order to achieve this, we shall use three additional colors $\alpha = 2s_0^* + 1$, $\beta = 2s_0^* + 2$, and $\gamma = 2s_0^* + 3$ to recolor problematic vertices.

Let us first assume that all long 2-jumps at C_1 have their apices in G_0. Then we will recolor problematic vertices for N. Let

$$Y = \{y_1 \in V(C_1) \mid y_1 \text{ is problematic for } N\}.$$

We now color all vertices in Y by using colors α, β, γ in such a way that no two vertices of Y, which appear consecutively on C_1, receive the same color. Since the cycle C_1 is induced and since every problematic vertex is adjacent to a non-problematic vertex on C_1, this requirement assures that no bi-colored 4-path arises using two colors among α, β, γ. It is also clear that former bi-colored paths are no longer bi-colored. The only possibility for a new bi-colored path to arise would be that it contains two problematic vertices re-colored with the same color (say α). Let y and y' be two such vertices and $P = xyx'y'$ be bi-colored. Since y and y' are not consecutive problematic vertices on C_1, x' is the apex of a long 2-jump. As assumed above, x' is in G_0. But then y and y' are incident to the apex of a long 2-jump in G_0 and would not both be using a color of c_1 at the first place. This contradiction shows that bi-colored paths do not exist. Hence, the constructed coloring is a star coloring.

The same approach works when all long 2-jumps at C_1 have apices in N. In that case we recolor problematic vertices for G_0.

The remaining case needs an additional touch. First of all, it is possible to partition C_1 into segments $C_1^1, \ldots, C_1^{2s}$ (where $s \geq 1$) such that long 2-jumps with apices in G_0 are all based on segments C_1^i where i is odd, and long 2-jumps with apices in N are all based on segments C_1^i where i is even. Now we define

$$\begin{aligned} Y &= \{x \in V(C_1^i) \mid i \text{ is odd, } x \text{ problematic for } G_0\} \cup \\ &\quad \{y \in V(C_1^i) \mid i \text{ is even, } y \text{ problematic for } N\} \end{aligned}$$

and recolor vertices in Y as we did above. Make sure to use distinct three colors when going along C_1 and changing from a segment C_1^i to C_1^{i+1}. This procedure eliminates all bi-colored 4-paths thus yielding a star coloring. The proof is now complete.

REFERENCES

1. M. O. Albertson, G. G. Chappell, H. A. Kierstead, A. Kündgen, and R. Ramamurthi, Coloring with no 2-colored P_4's, Electr. J. Combin. 11 (2004) #R26.
2. N. Alon, B. Mohar, D. P. Sanders, On acyclic colorings of graphs on surfaces, Isr. J. Math. 94 (1996) 273–283.
3. O. V. Borodin, On acyclic colorings of planar graphs, Discrete Math. 25 (1979) 211–236.
4. B. Grünbaum, Acyclic colorings of planar graphs, Israel J. Math. 14 (1973) 390–408.
5. T. Jensen, B. Toft, Graph Coloring Problems, John Wiley, New York, 1995.
6. B. Mohar, Acyclic colorings of locally planar graphs, Europ. J. Combin. 26 (2005) 491–503.
7. B. Mohar, C. Thomassen, Graphs on Surfaces, The Johns Hopkins University Press, Baltimore and London, 2001.
8. N. Robertson and P.D. Seymour, Graph minors VII. Disjoint paths on a surface, J. Combin. Theory Ser. B 45 (1988) 212–254.
9. C. Thomassen, Five-coloring maps on surfaces, J. Combin. Theory Ser. B 59 (1993) 89–105.
10. C. Thomassen, Color-critical graphs on a fixed surface, J. Combin. Theory Ser. B 70 (1997) 67–100.