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RIGIDITY AND SEPARATION INDICES OF GRAPHS IN SURFACES

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Rigidity and separation indices of graphs in surfaces

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Abstract

Let Σ be a surface. We prove that rigidity indices of graphs which admit a polyhedral embedding in Σ and 5-connected graphs admitting an embedding in Σ are bounded by a constant depending on Σ. Moreover if the Euler characteristic of Σ is negative, then the separation index of graphs admitting a polyhedral embedding in Σ is also bounded. As a side result we show that distinguishing number of both Σ-polyhedral and 5-connected graphs which admit and embedding in Σ is also bounded.

1 Introduction and results

Let $\mathcal{L}_n$ be the lattice of partitions of the n-set $[n] = \{1, 2, \ldots, n\}$, with minimal element 0 (partition into singletons), and maximal element 1 (having one block only).

Let G be a graph of order n with vertex set $V(G) = [n]$, and let $\Gamma = \mathrm{Aut}(G)$ be the group of automorphisms of G with its natural action on $V(G)$. For $v \in V(G)$ let $\Gamma_v \leq \Gamma$ be the *stabilizer* of v in Γ, and let P_v be the corresponding partition of $V(G)$ into the orbits of Γ_v.

The *separation index* of G, denoted by $\mathrm{sep}(G)$ is the minimum k such that there exists a vertex set $U \subseteq V(G)$ of cardinality k so that

$$\bigwedge_{u \in U} P_u = 0 \qquad (1)$$

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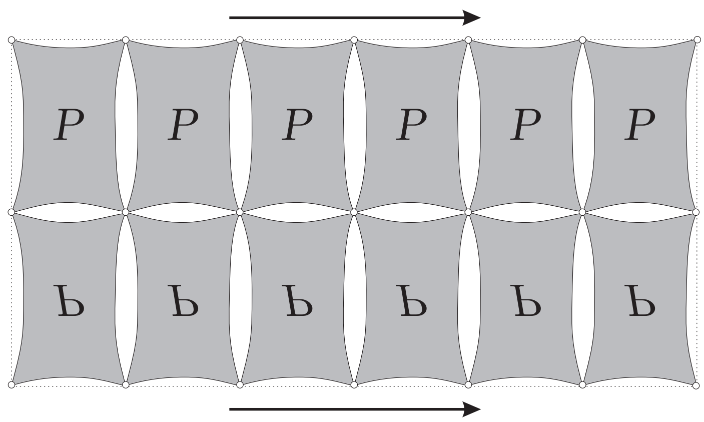

Figure 1: 4-connected graphs in torus (or Klein bottle) can have arbitrarily large rigidity index.

in $\mathcal{L}_n$. We also say that the vertices of U (as in (1)) *separate* G.

Clearly (1) implies that $\bigcap_{w \in W} \Gamma_w = \{\mathrm{id}\}$, though the converse may not hold. This motivates us to define the *rigidity index* of G, denoted by $\mathrm{rig}(G)$ as the minimum k, such that there exists a vertex set $W \subseteq V(G)$ of cardinality k so that

$$\bigcap_{w \in W} \Gamma_w = \{\mathrm{id}\}. \tag{2}$$

The vertices of W are said to *fix* graph G. For example: $\mathrm{sep}(K_n) = \mathrm{rig}(K_n) = n-1$, $\mathrm{sep}(K_{m,n}) = \mathrm{rig}(K_{m,n}) = m+n-2$, $\mathrm{sep}(\overline{G}) = \mathrm{sep}(G)$, and also $\mathrm{rig}(\overline{G}) = \mathrm{rig}(G)$. As a subset separating vertices of G also fixes G we have

$$\mathrm{sep}(G) \leq \mathrm{rig}(G). \tag{3}$$

Inequality may occur in (3), and Paley graphs [4] may serve as the extreme cases.

Vince [13] proved:

Theorem 1.1 *Let G be a 3-connected planar graph. Then* $\mathrm{sep}(G) \leq 3$.

There exist 3-connected planar graphs whose rigidity and separation indices are equal to 3, K_4 is an example. On the other hand, one cannot relax the 3-connectivity condition since the graph $K_{2,n}$ is planar and 2-connected, yet its rigidity (and also separation) index equals n.

Theorem 1.1 cannot be generalized to other surfaces. Consider an example in Figure 1, showing a toroidal graph G whose connectivity is (at most) 4 and whose toroidal embedding has face-width 2. Yet its rigidity index, and consequently also its separation index, is large. Both connectivity at most 4 and a lack of a more representative embedding are natural obstacles when trying to bound separation and rigidity indices of graphs.

A combinatorial embedding of G on Σ is described by a collection of faces (more precisely facial walks). In case when a graph G has a unique combinatorial embedding on Σ (the faces of G on Σ are uniquely defined) the set of three consecutive vertices on an arbitrary face fixes G. Therefore:

Proposition 1.2 *If G has a combinatorially unique embedding on Σ then* $\mathrm{rig}(G) \leq 3$.

Let $\mathcal{P}_\Sigma$ denote the class of graphs which admit a polyhedral embedding (to be defined later) on surface Σ. The main results of our paper are the following two theorems.

Theorem 1.3 *For every surface Σ with negative Euler characteristic there exists a constant s_Σ so that for every $G \in \mathcal{P}_\Sigma$ the separation index of G satisfies* $\mathrm{sep}(G) \leq s_\Sigma$.

Theorem 1.4 *For every surface Σ there exists a constant p_Σ so that every graph G which either belongs to P_Σ or is 5-connected and admits an embedding in Σ the rigidity index of G satisfies* $\mathrm{rig}(G) \leq p_\Sigma$.

We shall devote the entire Section 2 for the proof of the above two theorems.

The example in Figure 1 cannot be embedded on the projective plane. However there exist 3-connected projective-planar graphs with arbitrarily large rigidity indices, see Figure 2. In view of this the following theorem is best possible.

Theorem 1.5 *There exists a constant p so that every 4-connected graph G embeddable in the projective plane satisfies* $\mathrm{rig}(G) \leq p$.

We will postpone the proof until Section 3.

In the remainder of this first section we shall make a digression towards a graph-symmetry measure defined by Albertson and Collins [1]: a graph is said to be *d-distinguishable* if there exists a labelling $\ell : V(G) \to \{1, \ldots, d\}$, so that no automorphism other than the identity preserves the labels assigned by ℓ. The smallest number d so that G is d-distinguishable is called the *distinguishing number* of G, and is denoted by $D(G)$.

Choose a vertex set $U = \{v_1, v_2, \ldots, v_k\}$, for every $i = 1, \ldots, k$ let $\ell(v_i) = i$, and let $\ell(v) = k+1$ for every $v \in V(G) \setminus U$. We have described a $k+1$-labelling of G and every automorphism which preserves this labelling belongs to

$$\Gamma_{v_1} \cap \Gamma_{v_2} \cap \ldots \cap \Gamma_{v_k}.$$

Therefore:

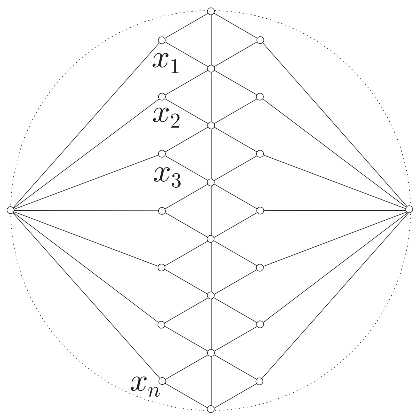

Figure 2: A 3-connected graph in the projective plane with rigidity index equal to n.

Proposition 1.6 *For every graph G we have $D(G) \leq \mathrm{rig}(G) + 1$.*

Negami [11] investigated distinguishing numbers of triangulations and was able to show that for every surface Σ there exists a constant t_Σ so that $D(G) \leq t_\Sigma$ for every graph G which triangulates Σ. A direct consequence of Theorem 1.4 and Proposition 1.6 is an extension to polyhedral and 5-connected graphs which admit an embedding on Σ.

Theorem 1.7 *For every surface Σ there exists a constant d_Σ so that for every graph G which either belongs to P_Σ or is 5-connected and admits an embedding in Σ the distinguishing number of G satisfies $D(G) \leq d_\Sigma$.*

In this paper we shall use standard graph terminology as in [3], and take notation and definitions concerning embeddings of graphs from [8].

2 Polyhedral and 5-connected graphs

This entire section is devoted to the proof of Theorems 1.3 and 1.4.

Choose a surface Σ. We say that G is Σ-*polyhedral* if G is 3-connected and admits an embedding $\Pi : G \to \Sigma$ so that $\mathrm{fw}(G, \Pi) \geq 3$. Both conditions imply that every facial walk of a face of $\Pi(G)$ is a cycle and that every pair of faces intersect in either a vertex, an edge, or do not meet at all. By $\mathcal{P}_\Sigma$ we denote the class of Σ-polyhedral graphs.

Proof.(of Theorem 1.3) Let Σ be a surface with negative Euler characteristic, ie. Σ is neither the sphere, projective plane, torus, nor Klein bottle. Let Γ_Σ be a finite group of homeomorphisms acting on Σ. Hurwitz' Theorem [6] (see also [5]) states that every finite group of homeomorphisms of Σ is of bounded order: $|\Gamma_\Sigma| \leq b_\Sigma = 168(g-1)$, where g denotes the genus of Σ.

Choose an arbitrary graph $G \in \mathcal{P}_\Sigma$ and let Π_0 be a fixed polyhedral embedding of G, so that $\mathrm{fw}(G, \Pi_0) \geq 3$.

Let φ be an automorphism of G. The composition $\Pi_\varphi := \Pi_0 \circ \varphi$ is also a polyhedral embedding of G in Σ. Let $\Gamma_\Sigma \leq \mathrm{Aut}(G)$ denote the subgroup of automorphisms φ so that Π_φ and Π_0 induce the same collection of faces. Every automorphism in Γ_Σ can be extended to a homeomorphism of Σ. We can think of Γ_Σ as a finite group of homeomorphisms acting on Σ, hence $|\Gamma_\Sigma| \leq b_\Sigma$.

Let Φ be the set of left coset representatives of Γ_Σ in $\mathrm{Aut}(G)$. The embeddings in Φ form a maximal collection of nonequivalent polyhedral embeddings of G. By a theorem of Mohar and Robertson [9] there exists a constant p_Σ, so that $|\Phi| \leq p_\Sigma$.

This bounds the order of the automorphism group as

$$|\mathrm{Aut}(G)| = |\Phi| \cdot |\Gamma_\Sigma| \leq b_\Sigma \cdot p_\Sigma \tag{4}$$

Finally let $U \subseteq V(G)$ be an inclusionwise minimal vertex set which separates G. Clearly if u_1 and u_2 are distinct vertices from U then also the corresponding vertex stabilizers Γ_{u_1} and Γ_{u_2} are incomparable as subgroups/subsets of $\mathrm{Aut}(G)$. But a set/group of size n can have at most $\binom{n}{\lfloor n/2 \rfloor}$ pairwise incomparable subsets/subgroups, which finishes the proof of Theorem 1.3. $\square$

Now let us turn to proving Theorem 1.4. We shall first introduce a tool that turns a graph G from one class of graphs into another by a small perturbation.

Let $\mathcal{G}$ and $\mathcal{G}'$ be classes of graphs. We say that $\mathcal{G}$ is *adaptable* to $\mathcal{G}'$, $\mathcal{G} \rightsquigarrow \mathcal{G}'$, if there exists a constant $n \in \mathbb{N}$ so that for every graph $G \in \mathcal{G}$ there exists a graph $G' \in \mathcal{G}'$ which can be obtained from G by performing at most n of the following operations:

(A1) deleting an isolated vertex,

(A2) adding an isolated vertex,

(A3) deleting an edge,

(A4) adding an edge.

Observe that being adaptable to is not a symmetric relation and also that subdivision of an edge with one additional vertex can be modeled by exactly 4 of the above operations.

Lemma 2.1 *Let $\mathcal{G}$ and $\mathcal{G}'$ be classes of graphs and assume that $\mathcal{G}$ is adaptable to $\mathcal{G}'$, $\mathcal{G} \rightsquigarrow \mathcal{G}'$. Assume that there exists a constant c', so that $\mathrm{rig}(G') \leq c'$ for every $G' \in \mathcal{G}'$. Then there exists a constant c so that $\mathrm{rig}(G) \leq c$ for every $G \in \mathcal{G}$.*

Proof. It is enough to see that rigidity indices of graphs $G-w, G+w, G-uv, G+uv$ are not much bigger that $\mathrm{rig}(G)$ for every (potential) edge uv and an isolated (in G or $G+w$) vertex w.

Let $U = \{v_1, v_2, \ldots, v_k\} \subseteq V(G)$ that fixes G for which $\mathrm{rig}(G) = |U|$ and let $G' = G + w$ where w is isolated in G'. By Γ'_v we denote the stabilizer of v in the automorphism group of G', and if $v \neq w$ then Γ_v denotes the stabilizer of v in $\mathrm{Aut}(G)$.

$$\begin{aligned}
&\Gamma'_w \cap \left(\Gamma'_{v_1} \cap \Gamma'_{v_2} \cap \ldots \cap \Gamma'_{v_k}\right) \\
&= \left(\Gamma'_u \cap \Gamma'_{v_1}\right) \cap \left(\Gamma'_u \cap \Gamma'_{v_2}\right) \cap \ldots \cap \left(\Gamma'_u \cap \Gamma'_{v_k}\right) \\
&\leq \Gamma_{v_1} \cap \Gamma_{v_2} \cap \ldots \cap \Gamma_{v_k} = \{\mathrm{id}\}
\end{aligned}$$

By similar arguments we also show that

- $U \setminus \{w\}$ fixes $G - w$, hence $\mathrm{rig}(G - w) \leq \mathrm{rig}(G)$,
- $U \cup \{w\}$ fixes $G + w$, hence $\mathrm{rig}(G + w) \leq \mathrm{rig}(G) + 1$,
- $U \cup \{u, v\}$ fixes $G - uv$, hence $\mathrm{rig}(G - uv) \leq \mathrm{rig}(G) + 2$,
- $U \cup \{u, v\}$ fixes $G + uv$, hence $\mathrm{rig}(G + uv) \leq \mathrm{rig}(G) + 2$.

□

The above lemma is not true in the case of separation index. Paley graphs, as shown in [4], can have arbitrarily large separation indices. But deletion of a single edge in a Paley graph results in a graph with just one nontrivial automorphism [10].

Lemma 2.2 *Let Σ_0 be a fixed surface. If Σ and Σ' are surfaces so that Σ' is connected sum of surfaces Σ and Σ_0, then $\mathcal{P}_\Sigma \rightsquigarrow \mathcal{P}_{\Sigma'}$.*

Proof. There is nothing to prove if Σ_0 is a 2-sphere. By induction it is enough to consider cases where Σ_0 is either projective plane or torus. Note that K_6 triangulates the projective plane $\mathbb{N}_1$, hence $K_6 \in \mathcal{P}_{\mathbb{N}_1}$. On the other hand $K_7 \in \mathcal{P}_{\mathbb{S}_1}$ as K_7 triangulates the torus $\mathbb{S}_1$.

Let Π be a polyhedral embedding of G in Σ and let f be an arbitrary face of Π. Let v_1, v_2, v_3 be three arbitrary vertices lying on f and let v_4, v_5, v_6 (and v_7 in case Σ_0 is the torus) be the *additional vertices*. By forming a clique of size 6 (or 7) on v_1, v_2, v_3 and the additional vertices we obtain a graph G' which admits a polyhedral embedding in a surface which is a connected sum of Σ and either projective plane or torus. And G' is obtained from G by adding at most $4 + 15$ new vertices and edges. □

Let Π be an embedding of G on surface Σ. A *k-curve* is an essential simple closed curve on Σ that intersects $\Pi(G)$ in exactly k vertices (if an essential closed curve intersects $\Pi(G)$ in k points we may homotopically shift its image to obtain a curve that (i) intersects $\Pi(G)$ in vertices and (ii) does so in at most k vertices.) Every such curve corresponds to a cycle in the vertex-face graph of (G, Π), see [8].

We say that a k-curve is *short* if $k \leq 2$. If Π is a polyhedral embedding then no short curves exist.

Lemma 2.3 *Let $\mathcal{C}_{k,\Sigma}$ denote the class of all k-connected graphs that admit a minimum genus embedding in surface Σ. If $k \geq 5$ then $\mathcal{C}_{k,\Sigma} \rightsquigarrow \mathcal{P}_\Sigma$.*

Proof. Let G be a 5-connected graph and assume that Π is a minimum genus embedding of G on Σ ie. G does not embed on a surface of smaller genus and the same orientability type. This implies that no 0-curves exist.

We shall argue that we can by applying a bounded number of operations (A1),. . . , (A4) transform the embedding of G into a polyhedral embedding of a slightly perturbed graph.

Let C be a 2-curve. We can assign to C a face $f = f_C$ that has the following property: C runs through f and intersects vertices u_C and v_C which do not lie consecutively along f. This also implies that f_C is not a triangular face. Also if C is a 1-curve and runs through face f then the facial walk along f is of length at least 6 and f contains at least 5 different vertices.

Let $\mathcal{C}$ be the collection of all short curves. Every pair of curves $C_1, C_2 \in \mathcal{C}$ can intersect in at most 3 vertices and/or faces. By a result of Juvan, Malnič, and Mohar [7] there exists a number $n = n_\Sigma$, so that the number of pairwise nonhomotopic curves in $\mathcal{C}$ is at most n.

On the other hand a collection of pairwise homotopic curves from $\mathcal{C}$ can intersect G in at most 4 vertices. Namely, let $C_1, \ldots, C_m$ be a maximal collection of pairwise homotopic short curves (we may assume that the indices are selected according to their relative position): if $1 \leq i < j < k \leq m$ then C_j lies in the annulus A (can be a degenerate one) which has C_i and C_k as its boundary components. Now either a set of 4 vertices $V(C_i) \cup V(C_k)$ separates $V(C_j)$ from $V(G) \setminus A$ or $V(C_j) \subseteq V(C_i) \cup V(C_k)$.

The above observations imply that $|\mathcal{C}|$ is bounded.

Now for every 1-curve C put a vertex v_C in the interior of f_C making it adjacent to 5 different vertices along f so that at least one pair interlaces the run of C. We have thus obtained an embedding of a 5-connected graph $G_1 \supseteq G$ that allows no 1-curves.

Let $\mathcal{C}_1$ be the collection of 2-curves of the embedding of G_1 and let $\mathcal{F}_1 = \{f_C\,; C \in \mathcal{C}_1\}$, which, by above arguments, contains a bounded number of faces.

Now for every $f \in \mathcal{F}_1$ we put a vertex v_f in the interior of f and for every curve C so that $f_C = f$ make v_f adjacent to a pair of vertices which are interlaced with v_C and u_C along f. Finally if at the end of this process v_f has degree 2 we may suppress it in order to keep our newly obtained graph G_2 3-connected. The obtained embedding of G_2 is polyhedral and the proof is complete. □

We shall finish this section by giving the proof of Theorem 1.4.

Proof.(of Theorem 1.4) Assume first that G admits a polyhedral embedding in Σ. If $\chi(\Sigma) < 0$ then $\text{rig}(G)$ is bounded by (3) and theorem 1.3. If $\chi(\Sigma) \geq 0$, we are forced to apply Lemmas 2.1 and 2.2 beforehand to obtain a polyhedral embedding on a surface of negative Euler characteristic. Hence rigidity index is bounded on the class of Σ-polyhedral graphs.

Let G be a 5-connected graph that admits an embedding on Σ. Inductively we may assume that an embedding of G on Σ is a minimal genus embedding. An application of Lemma 2.3 together with Lemma 2.1 finishes the proof. □

3 4-connected graphs in the projective plane

In this section we shall give a proof of Theorem 1.5. The proof itself will share some flavor with the proof of Theorem 1.4 given in the previous section.

The basic tool is the following theorem by Negami [12].

Theorem 3.1 *If G is 4-connected and admits an embedding Π on projective plane with face width ≥ 4 then G has a combinatorially unique embedding.*

It is a fundamental property of the projective plane that every two essential curves are homotopic an are not disjoint. Further if G admits an embedding on projective plane $\mathbb{N}_1$ with face-width 1 then G is planar. Hence we may limit ourselves to embeddings with face-width ≥ 2.

On the other hand 4-connected graphs embedded on $\mathbb{N}_1$ may allow an arbitrary number of 2- or 3-curves. In view of Theorem 3.1 let us call both 2- and 3-curves *short* in this section.

Let us construct a graph H which has a combinatorially unique embedding in the projective plane, see [12, Lemma 2,2]: $V(H) = \{v_0, v_1, \ldots, v_{12}\}$. Start with a 12-cycle on vertices $\{v_1, \ldots, v_{12}\}$, and add four edges $v_1v_8, v_3v_9, v_5v_{11}$, and v_6v_{12}. We have now obtained a subdivision of the *Möbius 4-ladder* M_4. Finally we connect v_0 with v_1, v_4, v_7, and v_{10}.

Given a 4-connected graph G, which is embedded in the projective plane we shall show that G can be perturbed slightly so that either (i) the newly obtained graph contains a subdivision of H or (ii) the newly obtained graph (or rather embedding) is 4-connected and has face-width ≥ 4. In both cases the slightly perturbed graph would have bounded rigidity index by Proposition 1.2.

Let $\mathcal{C}_2$ be a collection of 2-curves and fix a 2-curve C_0. By above observation every other 2-curve crosses C_0 in either a common vertex or a common face. No more than 7 curves from $\mathcal{C}_2$ run throug the same vertex v, as otherwise we would obtain a contradicition to 4-connectivity of G (note than two curves from $\mathcal{C}_2$ can share both their vertices as hey can still run throug different faces). If at least 7 curves from $\mathcal{C}_2$ run through the same face f, our graph G contains as a subgraph a subdivision of M_4. By adding a vertex and four edges in the interior of f we obtain a subdivion of H. This implies that G admits a bounded number (≤ 24) of distinct 2-curves. As every 2-curve C runs through a nontriangular face f_C we can for every curve C plant a vertex v_C in the interior of f_C and connect it to a suitable set of four vertices in order to eliminate the 2-curve C. The newly obtained graph G_1 is 4-connected by construction and is embedded with face width at least 3.

Let $\mathcal{C}_3$ be a collection of 3-curves of G_1 and let us, as above, fix a 3-curve C_1. Every other 3-curve crosses C_1 in either a common vertex or a common face. Assume first that more than 28 3-curves share a vertex v. The same collection of curves would yield at least 7 2-curves in $G - v$ (a single 2-curve in $G - v$ can be routed as ≤ 4 different 3-curves in G), and hence a subdivision of M_4 in $G - v$. By subdividing at most 4 edges and adding at most 4 edges to G we obtain a subdivision of H.

Let f be a face where a collection of 3-curves all cross C_1. Then f is not a triangle and C_1 does not intersect two consecutive vertices of f. By adding a vertex v in the interior of f and connecting it to a suitable set of four vertices from f we eliminate both C_1 and all 3-curves that cross it in the interior of f.

Hence we may assume that G_1 admits a bounded number of 3-curves $C_1, C_2, C_3, \dots, C_k$. For each C_i in turn we do one of the following:

(a) If the 3-curve C_i runs parallel to an essential triangle $v_1v_2v_3$ then let u and w be two vertices of two different faces that share the edge v_1v_2. Let us subdivide the edge v_1v_2 by a single vertex v_{12} and make it adjacent also to vertices u and w.

(b) Otherwise let f_i be the face so that C_i contains nonconsecutive vertices u_i, v_i along f. Put a new vertex w_i in the interior of f and make it adjacent to u_i, v_i and two additional vertices so that u_i and v_i are not consecutive neighbors around w_i.

Applying (a) or (b) eliminates C_i, preserves 4-connectivity, and may, much to our favour, also eliminate another 3-curve C_j, for some $j > i$. Let us call the resulting graph G_2.

As the number of 3-curves in G_1 is bounded we have obtained a 4-connected graph G_2 embedded with face-width ≥ 4 by using a bounded number of operations (A1),...,(A4). By Theorem 3.1 and Proposition 1.2 G_2 has bounded rigidity index. By Lemma 2.1 also $\mathrm{rig}(G)$ is bounded and the proof of Theorem 1.5 is complete.

4 Conclusions

Finally it is worth mentioning that Theorems 1.3 and 1.4 do not extend to more general minor closed families of graphs. Let $\mathcal{F}_n$ denote the family of all graphs of tree-width at most n. It is well-known that $\mathcal{F}_n$ is a minor closed family. Consider the strong product $K_k \boxtimes T$ of a complete graph K_k and a tree T. It is not too difficult to argue that $K_k \boxtimes T$ is a k-connected graph, its tree width is at most $2k-1$, yet its rigidity index is at least $(k-1)|V(T)|$.

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