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IMMERSING SMALL COMPLETE GRAPHS

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Immersing small complete graphs

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Abstract

Following in the spirit of the Hadwiger and Hajós conjectures, Abu-Khzam and Langston have conjectured that every k-chromatic graph contains an immersion of K_k. They proved this for $k \leq 4$. Here we make the stronger conjecture that every simple graph of minimum degree $k-1$ contains an immersion of K_k. We prove that this conjecture holds for every $k \leq 7$.

1 Introduction

In this paper, all graphs are finite and may have loops and multiple edges. A graph H is a *minor* of a graph G if (a graph isomorphic to) H can be obtained from a subgraph of G by contracting edges. A graph H is a *topological minor* of a graph G if G contains a subgraph which is isomorphic to a graph that can be obtained from H by subdividing some edges. In such a case, we also say that G contains a *subdivision* of H. The chromatic number of G, denoted $\chi(G)$, is the minimum number of colors required by

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G in any proper coloring of its vertices. The graph G is *k-chromatic* if $\chi(G) = k$.

Our paper is motivated by two famous conjectures concerning the chromatic number and the minor and topological order, namely, Hadwiger's conjecture and Hajós' conjecture.

Hadwiger's Conjecture from 1943 suggests a far-reaching generalization of the Four Color Theorem [2, 3, 12] and is considered to be one of the deepest open problems in graph theory. It states that every loopless graph without a K_k-minor is $(k-1)$-colorable. In 1937, Wagner [17] proved that the case $k = 5$ of the conjecture is, in fact, equivalent to the Four Color Theorem. In 1993, Robertson, Seymour and Thomas [15] proved that the case $k = 6$ also follows from the Four Color Theorem. The cases $k \geq 7$ are open; For the case $k = 7$, a partial result in [11] is best known.

Hajós proposed a stronger conjecture that for all $k \geq 1$, every k-chromatic graph contains a subdivision of the complete graph on k vertices. He already considered the conjecture in the 1940's in connection with attacks on the Four Colour Conjecture (now theorem). For $k \leq 4$, the conjecture is true, for $k = 5, 6$, it still remains open. But for every $k \geq 7$, it was disproved by Catlin [6]. In fact, Erdős and Fajtlowicz [7] proved that the conjecture is false for almost all graphs, see also Bollobás and Catlin [4]. Recently, Thomassen provided a variety of interesting natural examples where Hajós' Conjecture fails [16].

In this paper, we consider a different containment relation – graph immersions. A pair of adjacent edges uv and vw with $u \neq w$ is *lifted* by deleting the edges uv and vw, and adding the edge uw (possibly in parallel to an existing edge). A graph H is said to be *immersed* in a graph G if a graph isomorphic to H can be obtained from a subgraph of G by lifting pairs of edges. If H is immersed in a graph G, then we also say that *G contains an H-immersion.*

Previous investigation on immersions has been mainly conducted from an algorithmic standpoint. We refer the reader to [5, 8]. On the other hand, it would be interesting to consider structural issues, since the notions of an immersion and a minor seem to be quite similar, and structural approach concerning graph minors has been extremely successful. In fact, Robertson and Seymour [14] extended their proof of the famous Wanger's conjecture [13] to prove that graphs are well-quasi-ordered by the immersion relation. This proves a conjecture of Nash-Williams. The proof is based on the whole series of graph minors papers. Hence, we may expect that structural approach concerning immersions is difficult, maybe as difficult as structure results concerning graph minors.

Immersion containment is quite different from that of topological or minor containment. It is immediate that a topological minor of H implies both a minor and an immersion of H, but the converse is not generally true. The existence of an H minor and an H immersion are easily seen to be incomparable. Therefore, it would be interesting to investigate relations between the chromatic number of a graph G and the largest size of a complete graph immersed in G. In fact, Abu-Khzam and Langston conjectured the following in [1].

Conjecture 1 *The complete graph K_k can be immersed in any k-chromatic graph.*

This conjecture, like Hadwiger's conjecture and Hajós' conjecture, is trivially true for $k \leq 4$. In fact, since Hajós' conjecture is true if $k \leq 4$, this immediately implies Conjecture 1 for the cases $k \leq 4$.

On the other hand, the case $k = 5$ does not seem to be trivial. Abu-Khzam and Langston [1] proved a weaker statement that K_5^- is immersed in every 5-chromatic graph. They also pointed out that structural investigations are extensively studied for graphs without a K_4-immersion (see [5]), but almost nothing is done for graphs without a K_5-immersion. Here we would like to pose the following conjecture for complete immersions.

Conjecture 2 *Every simple graph of minimum degree at least $k-1$ contains an immersion of K_k.*

This conjecture immediately implies Conjecture 1 since every k-chromatic graph has a simple subgraph of minimum degree $k-1$. Again, this conjecture is trivial for $k \leq 4$. The purpose of this paper is to prove the following theorem which establishes our conjecture for some higher values of k.

Theorem 3 *Conjecture 2 holds for $k = 5, 6, 7$.*

Theorem 3 also shows that Conjecture 1 holds for $k \leq 7$. Although the methods of our proof may extend to the next case ($k = 8$), it appears that there may be too many cases to make this approach feasible.

2 Immersing complete graphs

In this section we shall prove the main theorem. Actually, for inductive purposes, we shall prove a slightly stronger statement. We begin with a little notation.

Our graphs may in general have parallel edges; If two vertices are joined by more than one edge, the set of edges joining them is called a *proper parallel class*. The *degree* of a vertex v, denoted $deg(v)$, is the number of edges incident with v (counting loops twice). We let $N(v)$ denote the set of vertices adjacent to v and we let $\bar{N}(v) = N(v) \cup \{v\}$.

Theorem 4 *Let $d \in \{4, 5, 6\}$, let $G = (V, E)$ be a loopless graph, and let $u \in V$. Assume further that G satisfies the following properties:*

- *$|V| \geq d$.*
- *$deg(v) \geq d$ for every $v \in V \setminus \{u\}$.*
- *There are at most $d-2$ proper parallel classes, and every edge in such a parallel class is incident with u.*

Then there is an immersion of K_{d+1} in G.

Proof. Suppose (for a contradiction) that G is a counterexample to the theorem with $|V| + |E|$ minimum. We shall prove properties of G in several steps.

(1) $|V| \geq d + 1$.

It follows from the assumptions that there exists a vertex in $V \setminus \{u\}$ not incident with any proper parallel class. Consequently, $|V| \geq d + 1$.

(2) Every edge has an end of degree at most d.

If e has no end of degree $\leq d$, then $G - e$ is a smaller counterexample.

(3) $deg(v) = d$ for every $v \in N(u)$.

If $v \in N(u)$ has degree $> d$, then $G - uv$ is a smaller counterexample.

(4) $deg(v) \leq d + 1$ for every $v \in V \setminus \{u\}$.

If there exists $v \in V \setminus \{u\}$ with $deg(v) > d+1$, then either the neighbors of v form a complete graph (giving us an immersion of K_{d+1} in G) or there exist $w_1, w_2 \in N(v)$ which are nonadjacent, and the graph obtained from G by lifting vw_1 and vw_2 to form the edge w_1w_2 is a smaller counterexample.

(5) $N(u)$ induces a complete graph.

If $v_1, v_2 \in N(u)$ are nonadjacent, then the graph obtained by lifting uv_1 and uv_2 to form the edge v_1v_2 is a smaller counterexample.

(6) $|N(u)| \geq 3$.

If $|N(u)| \leq 1$, then $G - u$ is a smaller counterexample. If $|N(u)| = 2$, then we may let $\{v_1, v_2\} = N(u)$ and assume that there are s edges between u, v_1 and t between u, v_2 with $s \leq t$. Now G immerses the graph G' obtained from G by deleting u and adding s new parallel edges between v_1, v_2, so G' (with the special vertex v_2) is a smaller counterexample.

(7) $|N(u)| \leq d - 2$.

If $|N(u)| \geq d$ then $\bar{N}(u)$ contains a K_{d+1} subgraph, so G immerses K_{d+1}. If $|N(u)| = d - 1$, then G immerses the graph G' obtained from G by identifying $\bar{N}(u)$ to a single new vertex u' and deleting any resulting loops (here we use (3) and (5)). Now G' (with the special vertex u') is a smaller counterexample.

(8) $d = 6$ and $|N(u)| = 3$.

Suppose (for a contradiction) that (8) fails. Then by (6) and (7) we must have $d \in \{5, 6\}$ and $|N(u)| = d - 2$. Again in this case, we form a new graph G' from G by identifying $\bar{N}(u)$ to a single new vertex u' and deleting any resulting loops. By the minimality of G we find that G' immerses K_{d+1}. This immersed K_{d+1} in G' can be used to find an immersion of K_{d+1} in G. The "hardest" case is when $d = 6$, u' has degree 8, and the immersed K_7 uses all of these edges, and has u' as a vertex. Up to symmetries, there are two cases to be considered, and they are conveniently displayed in Figure 1. This contradiction completes the proof of (8).

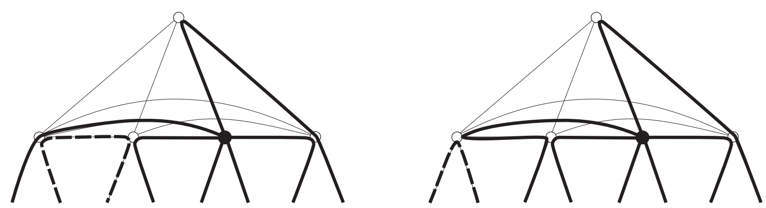

Figure 1: Extending K_7-immersion from G' to G when $|N(u)| = 4$

(9) G does not have exactly one proper parallel class.

Suppose (for a contradiction) that $N(u) = \{v_1, v_2, v_3\}$ and that the edges between u and v_1 form the only proper parallel class in G. Now, form a new graph G' by deleting u and adding edges v_1v_2 and v_1v_3. The graph G' is now immersed in G, and gives us a smaller counterexample (using the special vertex v_1).

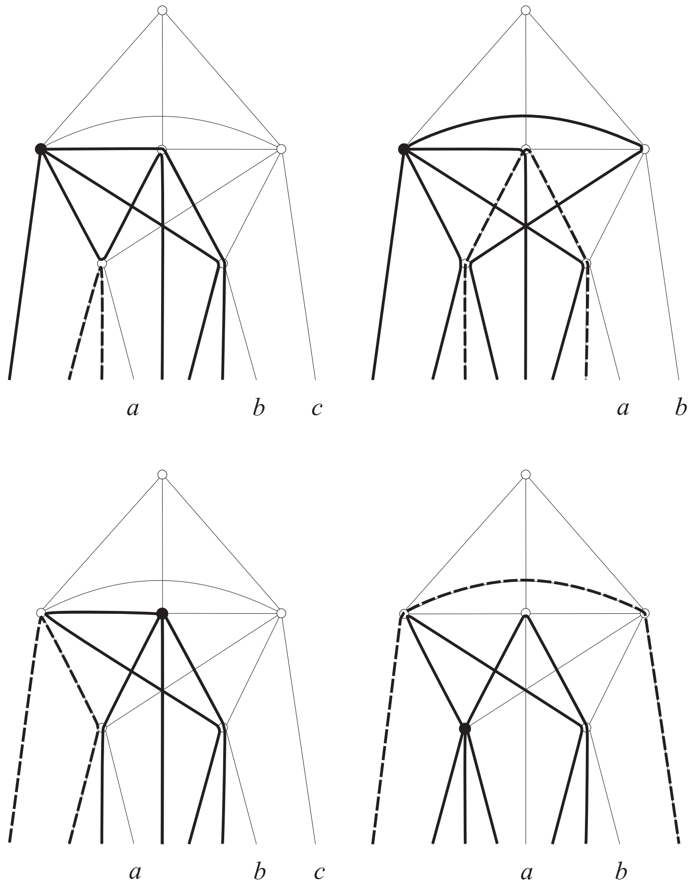

Figure 2: Extending K_7-immersion from G' to G for claim (10)

(10) There do not exist $w_1, w_2 \in \bigcap_{v \in N(u)} N(v) \setminus \{u\}$ so that either $w_1w_2 \in E$ or $deg(w_1) = 6 = deg(w_2)$.

Suppose that such a pair w_1, w_2 exists and let us form the graph G' from G by identifying $\bar{N}(u) \cup \{w_1, w_2\}$ to a single new vertex u'. It follows from (2)–(5) and (8) that u' will have degree ≤ 9 (so G' will have at most four proper parallel classes). It also follows that G has a vertex that is not adjacent to $N(u)$, and this implies that G' has at least 6 vertices. Consequently, G' contains an immersion of K_7. This immersion of K_7 in G' can be used to find an immersion of K_7 in G. The "hardest" case is when eight of the edges incident with u' are used in the K_7 and u' is a vertex of the immersed K_7. As for the most general subcase, we may also assume that $deg(u') = 9$. This case can be divided into subcases depending on which pair of edges incident with u' are lifted to form a new edge (drawn as a broken bold curve in Figure 2). Up to symmetries, there are four subcases as shown in Figure 2: both edges are incident with w_1, one is incident with w_1 and the other one with w_2, one with w_1 and the other with $N(u)$, or both with $N(u)$. In Figure 2, bold paths represent partial routings for each of these four cases, up to symmetries. In each subcase, one or two of the edges incident with the vertex of degree 6 in K_7 (drawn as the black vertex) are still missing. In this way, each of these figures describes two or three additional subcases, corresponding to which of the remaining edges a, b (or c) is not used in the K_7-immersion. It is easy to verify that all of these subcases can be completed to form an immersion of K_7 in G.

(11) $|\bigcap_{v \in N(u)} N(v) \setminus \{u\}| \leq 2$.

Suppose (for a contradiction) that $w_1, w_2, w_3 \in \bigcap_{v \in N(u)} N(v) \setminus \{u\}$ are distinct. It follows from (10) that $\{w_1, w_2, w_3\}$ is an independent set, and that at most one of these vertices has degree 6. Now, form a new graph G' from G by deleting $\bar{N}(u)$ and then adding the new edges w_1w_2, w_2w_3, and w_3w_1. The graph G' is simple with at most one vertex of degree < 6 and has at least 6 vertices, so it immerses K_7. But then G immerses K_7 as well, since G' is immersed in G. This yields a contradiction.

(12) Every $v \in N(u)$ satisfies $|N(v)| \leq 5$.

Let $N(u) = \{v_1, v_2, v_3\}$ and suppose (for a contradiction) that $N(v_3) = \{u, v_2, v_3, w_1, w_2, w_3\}$. First suppose that $v_1w_1, v_2w_2 \notin E$. Then the graph G' obtained from G by deleting v_3 and adding the edges v_1w_1, v_2w_2, and uw_3 is immersed in G, and yields a smaller counterexample, a contradiction. Next suppose that $v_1w_1, v_1w_2 \notin E$. By the previous case, we may assume that $v_2w_1, v_2w_2, v_2w_3 \in E$. It follows from this that v_2 and v_3 have six distinct neighbors, so by (9) the graph G is simple. Now, form the graph G'' from G by deleting u and v_3, then adding the edges v_1w_1 and v_1w_2, and then adding a new edge v_2w_3 in parallel to the existing edge. The graph G'' is immersed in G, but then G'' with the special vertex v_2 is a smaller counterexample, giving us a contradiction.

Using the two properties just demonstrated, we may now assume that $v_1w_2, v_1w_3, v_2w_2, v_2w_3 \in E$. Further, by (11), we may assume that $v_1w_1 \notin E$, and by (10) we may assume that $w_2w_3 \notin E$. Now we form the graph G''' from G by deleting v_3 and then adding the edges v_1w_1, w_2w_3, and uv_2. Now G''' is immersed in G, but then G''' is a smaller counterexample, giving us a contradiction. This proves (12).

Now, (12) implies that every neighbor of u is incident with a proper parallel class. It follows from this and (3), (5), (8) that the graph obtained from G by identifying $\bar{N}(u)$ to a single new vertex and deleting loops is immersed in G. This new graph is a smaller counterexample, thus completing the proof. $\square$

References

[1] F. N. Abu-Khzam and M. A. Langston, Graph coloring and the immersion order, *preprint.*

[2] K. Appel and W. Haken, Every planar map is four colorable, Part I. Discharging, *Illinois J. Math.* **21** (1977), 429–490.

[3] K. Appel, W. Haken and J. Koch, Every planar map is four colorable, Part II. Reducibility, *Illinois J. Math.* **21** (1977), 491–567.

[4] B. Bollobás and P. A. Catlin, Topological cliques of random graphs, *J. Combin. Theory Ser. B* **30** (1981), 224–227.

[5] H. D. Booth, R. Govindan, M. A. Langston, and S. Ramachandramurthis, Sequetial and parallel algorithms for K_4-immersion testing, *J. Algorithms* **30** (1999), 344–378.

[6] P. A. Catlin, A bound on the chromatic number of a graph, *Discrete Math.* **22** (1978), 81–83.

[7] P. Erdős and S. Fajtlowicz, On the conjecture of Hajós, *Combinatorica* **1** (1981), 141–143.

[8] M. R. Fellows and M. A. Langston, Nonconstructive tools for proving polynomial-time decidability, *J. ACM* **35** (1998), 727–738.

[9] H. Hadwiger, Über eine Klassifikation der Streckenkomplexe, *Vierteljahrsschr. Naturforsch. Ges. Zürich* **88** (1943), 133–142.

[10] T. R. Jensen and B. Toft, Graph Coloring Problems, Wiley-Interscience, 1995.

[11] K. Kawarabayashi and B. Toft, Any 7-chromatic graph has K_7 or $K_{4,4}$ as a minor, *Combinatorica* **25** (2005), 327–353.

[12] N. Robertson, D. P. Sanders, P. D. Seymour and R. Thomas, The four-color theorem, *J. Combin. Theory Ser. B* **70** (1997), 2–44.

[13] N. Robertson and P. D. Seymour, Graph minors. XX. Wagner's conjecture, *J. Combin. Theory Ser. B* **92** (2004), 325–357.

[14] N. Robertson and P. D. Seymour, Graph Minors XXIII, Nash-Williams' immersion conjecture, *submitted*.

[15] N. Robertson, P. D. Seymour and R. Thomas, Hadwiger's conjecture for K_6-free graphs, *Combinatorica* **13** (1993), 279–361.

[16] C. Thomassen, Some remarks on Hajós' conjecture, *J. Combin. Theory Ser. B* **93** (2005), 95–105.

[17] K. Wagner, Über eine Eigenschaft der ebenen Komplexe, *Math. Ann.* **114** (1937), 570–590.