University of Ljubljana
Institute of Mathematics, Physics and Mechanics
Department of Mathematics
Jadranska 19, 1000 Ljubljana, Slovenia

Preprint series, Vol. 47 (2009), 1072

THE TWO-COLORING NUMBER AND DEGENERATE COLORINGS OF PLANAR GRAPHS

H. A. Kierstead Bojan Mohar
Simon Špacapan Daqing Yang
Xuding Zhu

ISSN 1318-4865

Ljubljana, January 29, 2009

The two-coloring number and degenerate colorings of planar graphs

H. A. Kierstead*
Department of Mathematics and Statistics
Arizona State University
Tempe AZ 85287
kierstead@asu.edu

Bojan Mohar†‡
Department of Mathematics
Simon Fraser University
Burnaby, B.C. V5A 1S6
mohar@sfu.ca

Simon Špacapan§
University of Maribor, FME
Smetanova 17, 2000 Maribor, Slovenia
simon.spacapan@uni-mb.si

Daqing Yang¶
Center for Discrete Mathematics
Fuzhou University
Fuzhou, Fujian 350002, China
daqing85@yahoo.com

Xuding Zhu‖
Department of Applied Mathematics
National Sun Yat-sen University and
National Center for Theoretical Sciences
Taiwan

August 27, 2008

Abstract

The two-coloring number of graphs, which was originally introduced in the study of the game chromatic number, also gives an upper bound on the degenerate chromatic number as introduced by Borodin. It is proved that the two-coloring number of any planar graph is at most nine. As a consequence, the degenerate list chromatic number of any planar graph is at most nine. It is also shown that the degenerate diagonal chromatic number is at most eleven and the degenerate diagonal list chromatic number is at most tweleve for all planar graphs.

1 Introduction

The two-coloring number of a graph was introduced by Chen and Schelp [6] in the study of Ramsey properties of graphs and used later in the study of the game chromatic number [9, 10, 15, 16]. In [10] it was shown that the two-coloring number is related to the acyclic chromatic number. It turns out that the two-coloring number is also related to the notion of degenerate colorings introduced by Borodin [1], which was the starting motivation for the results of this paper.

Let G be a graph, and let L be a linear ordering of $V(G)$. A vertex x is *k-reachable* from y if $x <_L y$ and there is an xy-path P of length at most k such that $y <_L z$ for all interior vertices z of P. Let $R_{L,k}(y)$ be the set of all vertices x that are k-reachable from y with respect to the linear order L. The *k-coloring number* of G is defined as

$$\mathrm{col}_k(G) = 1 + \min_L \max_{y \in V(G)} |R_{L,k}(y)|,$$

where the minimum is taken over all linear orderings L of $V(G)$. If $k = 1$, then $\mathrm{col}_1(G)$ is also known as the *coloring number* of G since it provides an upper bound for the chromatic number of G. The higher k-coloring numbers provide upper bounds for some other coloring parameters [17].

Let k be a positive integer. A graph G is *k-degenerate* if every subgraph of G has a vertex of degree less than k. A coloring of a graph such that for every $k \geq 1$, the union of any k color classes induces a k-degenerate subgraph is a *degenerate coloring*. Note that this strengthens the notion of acyclic colorings, for which it is required that every color class is 1-degenerate and union of any two color classes induces a 2-degenerate graph (a forest). The *degenerate chromatic*

*Supported in part by NSA grant H98230-08-1-0069.

†Supported in part by the ARRS (Slovenia), Research Program P1–0297, by an NSERC Discovery Grant and by the Canada Research Chair program.

‡On leave from Department of Mathematics, IMFM & FMF, University of Ljubljana, Ljubljana, Slovenia.

§Supported in part by the ARRS (Slovenia), Research Program P1–0297

¶Partially supported by NSFC under grant 10771035, Educational Department of Fujian under grant JA05181, and SRF for ROCS of SEM.

‖Grant number: NSC95-2115-M-110-013-MY3.

number of G, denoted as $\chi_d(G)$, is the least n such that G admits a degenerate coloring with n colors. Suppose that for each vertex $v \in V(G)$ we assign a *list* $L(v) \subset \mathbb{N}$ of *admissible colors* which can be used to color the vertex v. A *list coloring* of G is a function $f : V(G) \to \mathbb{N}$, such that $f(v) \in L(v)$ for each $v \in V(G)$ and $f(u) \neq f(v)$ whenever u and v are adjacent in G. If the coloring is also degenerate, we say that f is a *degenerate list coloring*. If for any choice of lists $L(v), v \in V(G)$, such that $|L(v)| \geq k$, there exists a list coloring of G, then we say that G is *k-choosable*. The *list chromatic number* (or the *choice number*) of G, denoted as $\mathrm{ch}(G)$, is the least k, such that G is k-choosable. Analogously we define the *degenerate choice number* and denote it by $\mathrm{ch}_d(G)$.

In 1976 Borodin proved [2, 3] that every planar graph admits an acyclic 5-coloring and thus solved a conjecture proposed by Grünbaum [7]. At the same time, he proposed the following conjecture:

Conjecture 1 (Borodin [1, 3]) *Every planar graph has a 5-coloring such that the union of every k color classes with $1 \leq k \leq 4$ induces a k-degenerate graph.*

Thomassen settled a weakening of Conjecture 1 by proving that the vertex set of every planar graph can be decomposed into two sets that respectively induce a 2-degenerate graph and a 3-degenerate graph [13], and that the vertex set of every planar graph can be decomposed into an independent set and a set that induces a 4-degenerate graph [14]. However, Conjecture 1 remained basically untouched since there are no tools to deal with degenerate colorings. Very recently, the barrier was overriden by Rautenbach [11] who proved that every planar graph admits a degenerate coloring using at most 18 colors.

In this paper we introduce two different approaches for dealing with degenerate colorings. Both are based on the following observation.

Observation 1 *Let G be a graph and let c be a degenerate coloring of a vertex-deleted subgraph $G - v$. If the neighbors of v are colored by pairwise distinct colors and we color v by a color which is different from all of those colors, then the resulting coloring of G is degenerate.*

The above observation, whose easy proof is left to the reader, is a link between the 2-coloring number and the degenerate chromatic number, since it implies the following:

Corollary 1 *For any graph G, $\mathrm{ch}_d(G) \leq \mathrm{col}_2(G)$.*

Proof. Suppose that L is a linear ordering of $V(G)$, and suppose that each vertex y has a list of $|R_{L,2}(y)| + 1$ colors. Then we can color the vertices of G, one by one according to the linear ordering L, so that each vertex y is colored by a color different from the colors of vertices in $R_{L,2}(y)$. This strategy guarantees that all neighbors of y, which are already colored, have pairwise different colors. To see this, let $y_1 <_L y_2$ be any two neighbors of y, where $y_1, y_2 <_L y$. Then $y_1 \in R_{L,2}(y_2)$, so y_2 has been colored differently from y_1. It follows from Observation 1 that the obtained coloring of G is degenerate and hence $\mathrm{ch}_d(G) \leq \mathrm{col}_2(G)$. ∎

In this paper we shall use two-colorings to get an efficient bound on the degenerate chromatic number. It is proved in [9] that planar graphs have 2-coloring number at most 10. Our main result gives currently best upper bound for the two-coloring number of every planar graph G, and henceforth a bound on $\mathrm{ch}_d(G)$.

Theorem 1 *If G is a planar graph, then $\mathrm{col}_2(G) \leq 9$, and therefore also $\mathrm{ch}_d(G) \leq 9$.*

Theorem 1 will be proved in Section 2. In fact, we will prove a slightly stronger statement that there is a linear ordering L of $V(G)$ such that for each vertex x, $|R_{L,1}(x)| \leq 5$ and $|R_{L,2}(x)| \leq 8$. See Theorem 2.

An example given in [9] shows that there are planar graphs whose 2-coloring number is 8. That example is quite complicated. Here we give a much simpler one. Consider a 5-connected triangulation T of the plane in which no two vertices of degree 5 are adjacent. It is well known and easy to see that there are infinitely many such triangulations. Let L be a linear ordering of vertices of T, and let x be the last vertex that has a larger neighbor with respect to L. Suppose that $y_1, \ldots, y_k$ are the neighbors of x such that $x <_L y_i$ for $i = 1, \ldots, k$, and let $N^-(x)$ be the set of neighbors of x distinct from $y_1, \ldots, y_k$. By the choice of x, all neighbors of $y_1, \ldots, y_k$ are 2-reachable from x. Note that each y_i has at most two neighbors in $N^-(x)$. If $k = 1$, it is easy to see that $N^-(x)$ and the neighbors of y_1 contain at least 7 vertices distinct from x. Thus, $|R_{L,2}(x)| \geq 7$. On the other hand, if $k \geq 2$, then y_1 and y_2 have only x and possibly one neighbor of x as common neighbors. If one of them has degree at least 6, then they have at least 7 neighbors distinct from x, hence $|R_{L,2}(x)| \geq 7$. If they both have degree 5, then x has degree at least 6, and it is again easy to see that $|R_{L,2}(x)| \geq 7$. This shows that $\mathrm{col}_2(T) \geq 8$.

The following remains a challenging open problem:

Question 1 *Is it true that every planar graph G satisfies $\mathrm{col}_2(G) \leq 8$?*

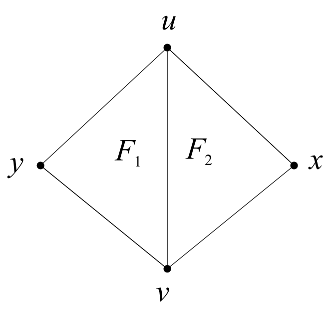

Figure 1: x and y are opposite with respect to $e = uv$

The second tool which can be used to control degeneracy when dealing with graph embedded in surfaces is based on the notion of diagonal colorings defined below. Let G be a plane graph and $e = uv$ be an edge of G. Suppose that the faces F_1 and F_2 incident with e are both triangles. If x, y are the vertices distinct from u, v on the boundary of F_1 and F_2, respectively, then we say that x, y are *opposite* with respect to e (see Fig. 1). Vertices x and y of G are said to be *opposite* if they are opposite with respect to some edge of G. Note that for a vertex of degree k, there are at most k vertices that are opposite to it.

Bouchet et al. [5] introduced the notion of diagonal colorings of plane graphs, for which one requires that any two adjacent or opposite vertices receive different colors. They proposed the following conjecture:

Conjecture 2 (Bouchet, Fouquet, Jolivet, Riviere [5]) *Every plane graph has a diagonal 9-coloring.*

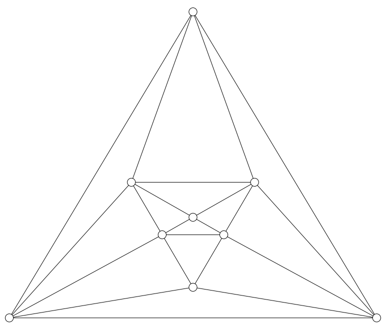

Figure 2: A triangulation which needs 9 colors

The graph shown in Figure 2 is an example which needs 9 colors. Bouchet at al. proved in [5] that 12 colors suffice in general. Borodin [4] showed how to save one color, and Sanders and Zhao [12] proved that 10 colors suffice. See also [8, Problem 2.15].

2 The two-coloring number

First we need some properties of plane triangulations. Suppose $G = (V, E)$ is a triangulation whose vertices are partitioned into two subsets U and C, where C is an independent set and each vertex in C has degree 4. For each vertex x, let $d_U(x)$ be the number of neighbours of x in U and $d_C(x)$ be the number of neighbours of x in C. Observe that

$$\sum_{x \in U} d_C(x) = 4|C|\,.$$

Let $w(x) = d_U(x) + d_C(x)/2$ be the *weight* of x. Then

$$2|E| = \sum_{x \in V} d_G(x) = \sum_{x \in U} d_G(x) + 4|C| = \sum_{x \in U} w(x) + 6|C|\,. \tag{1}$$

Euler's formula implies that $|E| < 3|V| = 3|U| + 3|C|$. This implies that $6|U| > \sum_{x \in U} w(x)$, so there is a vertex $x \in U$ with $w(x) < 6$ (hence $w(x) \leq 5.5$).

For a graph G and $x, y \in V(G)$ we write $x \sim y$ when x is adjacent to y in G and $x \not\sim y$ otherwise. We say that $x \in U$ is a vertex of *type* (a, b) if $d_U(x) = a$ and $d_C(x) = b$. If $x \sim y$, x is of type (5,0) or (5,1) and y is of type (5,0),(5,1) or (5,2), then (x, y) is called a *good pair*. If xyz is a triangle in G, z is of type (5,0) or (5,1) and x, y are both of type (6,0), then (x, y, z) is a *good triple*. If x, y are of types (5,0) or (5,1) and z is of type (6,1) and $x \sim z, y \sim z, x \not\sim y$, then (x, y, z) is also called a *good triple*.

Lemma 1 *Let $G = (V, E)$ be a planar triangulation. Suppose that $C \subset V$ is an independent set where each vertex of C has degree four, let $U = V - C$ and suppose that $d_U(x) \geq 5$ for all $x \in U$. Then G has a good pair or a good triple.*

Proof. Assume that there is no good pair and no good triple. Observe that $U \neq \emptyset$. For each $x \in U$, let $c(x) = w(x) - 6$ be the *initial charge* of x. A vertex x is called a *major vertex* if $c(x) > 0$ and a vertex x with $c(x) < 0$ is called a *minor vertex*. Euler's formula implies that $|E| = 3|V| - 6$ and together with (1) we conclude that

$$\sum_{x \in U} c(x) = \sum_{x \in U} w(x) - 6|U| = 2|E| - 6|C| - 6|U| = -12\,. \qquad (2)$$

If x is a minor vertex of type (5,0), then we let each major neighbour of x send a charge of 1/3 to x. If x is a minor vertex of type (5,1), then each major neighbour of x is asked to send a charge of 1/6 to x. Let us consider the resulting charge $c'(x)$ for all vertices of G. We shall show that $c'(x) \geq 0$ for each x, which will contradict (2) and complete the proof.

Note that every vertex, which is not a major vertex, is of type $(5, 0), (5, 1), (5, 2)$ or $(6, 0)$. If x is a minor vertex then its type is $(5, 0)$ or $(5, 1)$. Since there are no good pairs, x is not adjacent to a vertex of type $(5, 0), (5, 1)$ or $(5, 2)$. Additionally, since there are no good triples, x is not adjacent to two adjacent vertices of type $(6, 0)$. It follows that every minor vertex has at least three major neighbors. Therefore $c'(x) \geq 0$ for all minor vertices x.

Suppose now that x is a major vertex. Since there are no good pairs, no two minor vertices are adjacent and hence x sends charge only to non-neighboring vertices. If a minor U-neighbor y of x is adjacent to a C-neighbor of x then y is of type (5,1) and hence x sends to y only 1/6 charge. A U-neighbor adjacent to two C-neighbors of x receives 0. Therefore the total charge sent out from x is at most $\lfloor d_U(x)/2 \rfloor / 3$. Hence $c'(x) \geq 0$ if $d_U(x) \geq 7$.

Assume now that $d_U(x) = 6$. If $d_C(x) \geq 2$, then it is easy to see that the charge sent out from x is at most 2/3, and hence $c'(x) \geq 0$. So, assume $d_U(x) = 6$ and $d_C(x) = 1$. If x has at least two minor neighbours, then these two minor neighbours are not adjacent (for otherwise we have a good pair). But then we have a good triple, contrary to our assumption. Thus x has at most one minor neighbour, and hence the charge sent out from x is at most 1/3. Since $c(x) = 1/2$ we conclude that $c'(x) \geq 1/2 - 1/3 > 0$.

If $c(x) = 0$, then no charge is sent out from x and hence $c'(x) = 0$. Now every vertex has a non-negative charge, which is a contradiction. ∎

Let $G = (V, E)$ be a graph and $C \subseteq V$. A *C-ordering* of G is a partial ordering L of $V(G)$ such that the following conditions hold:

(i) The restriction of L to C is a linear ordering of C.

(ii) The vertices in $V - C$ are incomparable (that is, neither $x <_L y$ nor $y <_L x$ for $x, y \in V - C$).

(iii) $y <_L x$ for every $x \in C$ and $y \in V - C$.

Let L be a C-ordering of G and $x \in C$. The set $R_{L,2}(x)$ is the set of all $y <_L x$, such that either $y \sim x$ or there exists a $z \in C$ with $x \sim z, y \sim z$ and $x <_L z$. In the latter case we say that y is *two-reachable* from x with respect to L. Suppose that v is a vertex of G and $C = C' - \{v\}$. If L' is a C'-ordering of G and L is a C-ordering of G such that L' and L are equal on C and $v <_{L'} u$ for all $u \in C$, then clearly $|R_{L,2}(x)| = |R_{L',2}(x)|$ for all $x \in C$. Thus, extending the C-ordering to a C'-ordering does not increase the size of any set $R_{L,2}(x)$.

Theorem 2 *If G is a planar graph, then there is a linear ordering L of $V(G)$ such that for each vertex x, $|R_{L,1}(x)| \leq 5$ and $|R_{L,2}(x)| \leq 8$. In particular,* $\mathrm{col}_2(G) \leq 9$.

Proof. We need to find a linear ordering L of $V(G)$ such that for each vertex x, $|R_{L,1}(x)| \leq 5$ and $|R_{L,2}(x)| \leq 8$. Let $C \subseteq V$ and $U = V - C$. We say that a C-ordering L of G is *valid* if each vertex $x \in C$ has at most four neighbours in U and $|R_{L,2}(x)| \leq 8$.

In the following we shall produce a sequence of valid orderings of G so that the ordered set C becomes larger and larger, and eventually produce a linear ordering of $V(G)$. At the beginning $C = \emptyset$ (which is certainly a valid ordering). Suppose we have a valid C-ordering L of G and $U \neq \emptyset$. To produce a larger valid ordering we change the graph G as follows.

First we delete all edges between vertices of C. If $x \in C$ has at most three neighbours in U, then delete x and add edges between each pair of the U-neighbours of x. Observe that this operation preserves planarity of the graph. If $x \in C$ has four neighbours in U, then add edges in the cyclic order (according to the plane embedding) so that the four neighbours form a 4-cycle. The resulting graph G_1 is planar and every vertex of G_1 in C is of degree four and is contained only in triangular faces. Therefore we can add edges only among vertices of U to obtain a plane triangulation G' in which the vertex set $F = C \cap V(G')$ is independent and all its vertices have degree four.

Since G' is a triangulation and F is an independent set of G', we know that for each vertex $x \in U$, $d_U(x) \geq d_F(x)$. If there is a vertex $x \in U$ with $d_U(x) \leq 4$, then extend the ordering L by letting x be the next ordered vertex. The obtained ordering L' is an ordering with $x <_{L'} y$ for all $y \in C$ and $u <_{L'} x$ for all $u \in U - \{x\}$.

Let z be a neighbor of x, which is in C. If $z \in F$, then it has four neighbors in U and at most one of them is not x or a neighbor of x. Consequently at most one new vertex is two-reachable from x through z. On the other hand if $z \in C - F$, then z has at most three neighbors in U and all of them are either x or its neighbors in G'. We infer that $|R_{L,2}(x)| \leq 8$ holds in the original graph G. By assumption, x has at most four neighbors in U, so the extended ordering is valid.

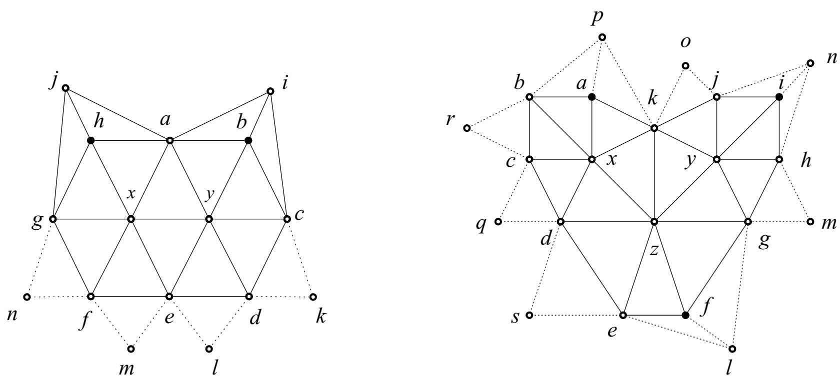

Figure 3: A case of an extension of ordering, where both x and y are of type $(5, 1)$. Diagonal vertices are indicated by broken lines. Vertices of F are full, vertices of U are hollow.

Suppose now that $d_U(x) \geq 5$ in G' for each $x \in U$. We will apply Lemma 1 for G' and the set F playing the role of C. The lemma implies that there is either a good pair (x, y) or a good triple (x, y, z).

If there is a good pair (x, y), then we extend the ordering by requesting that for any $u \in U - \{x, y\}$ and $v \in C$ we have $u <_{L'} x <_{L'} y <_{L'} v$. We claim that the obtained ordering L' is valid in G for the preordered set $C_1 = C \cup \{x, y\}$. We treat the case when x and y are of type $(5, 1)$ in detail, and leave all other cases to the reader. Let h be the neighbor of x in F and let b be the neighbor of y in F (possibly $b = h$). Note that h and b are of degree 4 in G' and that each of them has at most one neighbor that is two-reachable from x (resp. y) and is not a neighbor of x (resp. y). An example of this case is shown in Fig. 3. Note that the type (5,1) and the drawing in Fig. 3 are with respect to G'. The unique neighbors of x and y in F are chosen in Fig. 3 as vertices h and b, respectively, but they can be any other neighbors, including the possibility of being one of the common neihbors of x and y. Since the extension L' of L is defined so that $x <_{L'} y$, we see that the number of neighbors u of x with $u <_{L'} x$ is four (in Fig. 3 these are a, g, f and e). Moreover, the number of vertices that are two-reachable in G from x (and are different from the U-neighbors of x in G') is at most 3, one through h and two more through y (these are j, c and d). On the other hand, y has five neighbors u with $u <_{L'} y$ (namely a, x, e, d and c) and one additional two-reachable vertex through b that is not its neighbor in G', that is i. We proved that $|R_{L',2}(x)|, |R_{L',2}(y)| \leq 7$ and each of x, y has at most four neighbors in $U - \{x, y\}$. Therefore the extended ordering is valid. Other configurations, when (x, y) is a good pair, are treated similarly.

If (x, y, z) is a good triple such that x and y are of types $(5, 0)$ or $(5, 1)$ and z is of type $(6, 1)$, then extend the C-ordering L to L' by finding positions for x, y and z as follows: for any $u \in U - \{x, y, z\}$ and $v \in C$, we set $u <_{L'} x <_{L'} z <_{L'} y <_{L'} v$. If (x, y, z) is a good triple such that x and y are of type $(6, 0)$ and z is of type $(5, 0)$ or $(5, 1)$, then extend the C-ordering L to L' by seting $u <_{L'} x <_{L'} y <_{L'} z <_{L'} v$ for each $u \in U - \{x, y, z\}$ and $v \in C$. As before, it is easy to verify that $|R_{L',2}(x)|, |R_{L',2}(y)|, |R_{L',2}(z)| \leq 8$ and each of x, y, z has at most four neighbours in $U - \{x, y, z\}$. We conclude that L' is a valid ordering. Note also that the newly defined partial order L' satisfies $|R_{L',1}(x)| \leq 5$ for all x. One of the cases is shown in Fig. 3 for the convenience of the reader. ∎

A slight change in the above proof yields an application to degenerate diagonal list colorings using at most twelve colors at each vertex.

Corollary 2 *Every plane graph has a degenerate diagonal list coloring from lists of size at least twelve.*

Proof. Let L be a C-ordering and let $R^d_{L,2}(y)$ be the set of all vertices that are two-reachable from y, together with all vertices x, such that $x <_L y$ and x is opposite y. For the purpose of this proof we say that L is a *valid* ordering if for each vertex $x \in C$, $|R^d_{L,2}(x)| \leq 11$, and x has at most four neighbours in U. We prove the existence of a linear order L so that $|R^d_{L,2}(y)| \leq 11$ for each vertex $y \in V(G)$. We proceed analogously as in the proof of Theorem 1, that is, we construct a sequence of valid orderings until we eventually get a linear ordering of $V(G)$. So suppose that we are given a valid C-ordering L, where $C \neq V(G)$. Then change the graph G as in the proof of Theorem 1 to obtain a plane graph G'. It is easy to see that the construction of G' was done so that if $x, y \in U$ and x is opposite y in G, then either x is adjacent or opposite y in G'. Again we arrive at few cases, where we have to determine how to extend the ordering L.

If there is a vertex x with $d_U(x) \leq 4$ in G', then extend the ordering L to L' so that the next ordered vertex is x and observe that any vertex that is two-reachable from x and is not a neighbor of x is also opposite to x. It follows that $|R^d_{L',2}(y)| \leq 8$ holds in G.

Otherwise G' has either a good pair or a good triple. If (x, y) is a good pair then let $x <_{L'} y$. If x, y are of type (5,1) (see Fig. 3) then observe that x has four adjacent vertices smaller than x (these are a, g, f, e), two vertices which are two reachable and diagonal at the same time (these are j and d), two diagonal vertices which are not 2-reachable (these are m and n) and one other two-reachable vertex (c). This is altogether nine, hence $|R^d_{L',2}(x)| \leq 9$. It is easy to check that $R^d_{L',2}(y) = \{a, x, e, d, c, i, k, l, f\}$, so its size is 9. Other cases of good pairs are treated similarly but $|R^d_{L',2}(x)|$ and $|R^d_{L',2}(y)|$ are always at most ten.

If (x, y, z) is a good triple such that x, y are of types (5,0) or (5,1) and z is of type $(6, 1)$, we extend L to L' so that $z <_{L'} x <_{L'} y$ (see Fig. 3). Then we have $R^d_{L',2}(z) = \{g, k, d, e, l, s, c, j, h, b\}$. Similarly $R^d_{L',2}(x) = \{k, z, d, c, b, e, q, r, p\}$ and $R^d_{L',2}(y) = \{h, g, z, k, j, m, n, k, o\}$, hence in any case the size is less than 10. If (x, y, z) is a good triple where z is of type (5,0) or (5,1) and x, y are of type (6,0), then let $x <_{L'} y <_{L'} z$ and it is easy to check that $|R^d_{L',2}(x)|, |R^d_{L',2}(y)|, |R^d_{L',2}(z)| \leq 11$.

∎

3 Degenerate diagonal 11-colorings

The following theorem is an improvement of Corollary 2 by one color for the non-list version of degenerate and diagonal colorings. We have decided to include it despite the fact that it is slightly longer since the methods used in the proof of this result are different from the methods of the previous section.

Theorem 3 *Every plane graph has a degenerate diagonal coloring with eleven colors.*

Proof. Supppose that the theorem is not true, and let G be a minimum counterexample (with respect to the number of vertices).

Claim 0: G has no vertices of degree three.

Proof: Suppose on the contrary, that $v \in V(G)$ is a vertex of degree three. Since G is a minimum counterexample, there is a degenerate coloring of $G - v$, such that no two opposite vertices are colored with the same color. A desired coloring of G is obtained from the coloring of $G - v$, by coloring the vertex v with a color different from the colors of its neigbors and its opposite vertices. Note that this is always possible, since there are at most six neighbors or opposite vertices of v, therefore in the set of eleven colors there are five possible colors for v. This is a contradiction to the choice of G.

Claim 1: G has no vertices of degree four.

Proof: Suppose on the contrary, that v is a vertex of degree four. See Fig. 4. Without loss of generality assume that ac is not an edge in G. Delete the vertex v, add the edge ac, and call the obtained graph G^*. By the minimality of G, there is a degenerate coloring of G^* such that b and d are colored with distinct colors (since they are opposite). This coloring induces a degenerate coloring of $G - v$. In G there are at most four vertices opposite to v and there are four vertices adjacent to v. We color the vertex v with a color different from colors of the vertices adjacent or opposite to v. Since a, b, c and d are colored by pairwise distinct colors, we infer from Observation 1 that the obtained coloring is degenerate. Moreover, any two opposite vertices are colored with distinct colors.

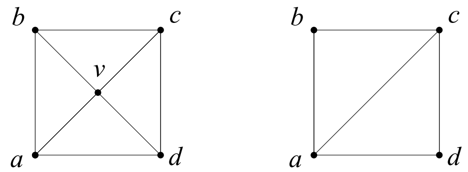

Figure 4: A vertex of degree four

Let us now introduce some notation. If G has a separating 3-cycle, then let C be a separating 3-cycle with the least number of vertices in its interior. Otherwise let C be the 3-cycle on the boundary of the infinite face. Denote by $\overline{C}$ the graph induced by vertices of C and those in the interior of C. If $\overline{C}$ contains a 4-cycle or a 5-cycle with at least two vertices in its interior, then let D be the one with the least number of vertices in its interior; otherwise, let $D = C$. Let $\overline{D}$ be the graph induced by the vertices of D and those in the interior of D. We shall denote by $\mathrm{int}(D)$ the set of internal vertices of $\overline{D}$.

Claim 2: In $\overline{C}$, no two internal vertices of degree five are adjacent.

Proof: Suppose on the contrary, that u, v are adjacent internal vertices of degree five; see Fig. 5 for notation. Since $\overline{C}$ has no separating 3-cycles, ad, bd and bf are not edges of G. Delete the vertices u and v, add the edges ad, bd and bf, and call the obtained graph G^* (see Fig. 5).

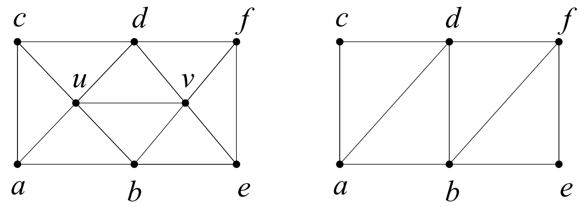

Figure 5: Two internal adjacent vertices of degree five

Observe that b and c, and d and e are opposite in G^*, therefore there exist a degenerate coloring of G^* such that the vertex sets $\{a, b, c, d\}$ and $\{b, d, e, f\}$ are colored by pairwise distinct colors. Since both u and v are vertices of degree 5, there are at most 9 (5 on opposite and 4 on adjacent vertices) colors prohibited for u and v. Therefore, a coloring of G can be obtained from the coloring of G^* by coloring u and v by distinct available colors. Observe that the obtained coloring is degenerate (by Observation 1) and that any two opposite vertices are colored by distinct colors.

Claim 3: An internal vertex u of $\overline{D}$ of degree 5 cannot be adjacent to three internal vertices v, x, z of $\overline{D}$ of degree 6, where z is adjacent to x and v.

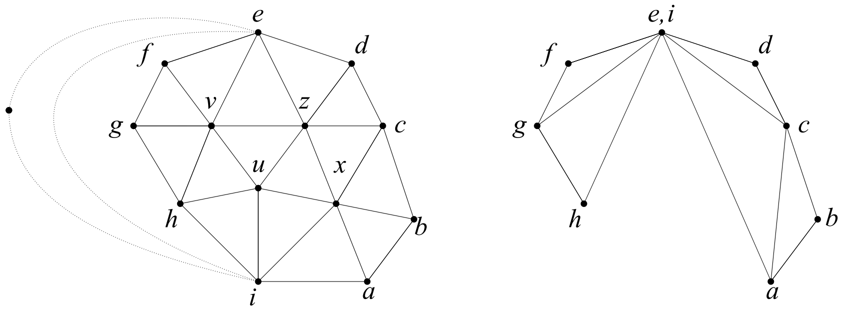

Figure 6: u, v, z and x are vertices of degrees 5, 6, 6 and 6, respectively

Proof: Suppose that u, v, z and x are internal vertices of $\overline{D}$ contradicting the claim (see Fig. 6). We claim that vertices i and e are neither adjacent nor they have a common neighbor.

Assume (for a contradiction) that they are adjacent and that the edge ei is embeded as shown in Fig. 6. Consider the 4-cycle $E = eiuve$ and observe that it contains at least 2 vertices in the interior. By the minimality of $\overline{D}$, the 4-cycle E is not contained in $\overline{D}$. So assume that t is a vertex of D contained in the interior of E. If t is adjacent to both e and i, then $tevuit$ is a 5-cycle contradicting the minimality of D. Otherwise, there are two vertices of D contained in the interior of E and a vertex $t' \in V(D)$ in the exterior of E; in this case t' is adjacent to e and i. Again we conclude that $t'evuit'$ is a 5-cycle, contradicting the minimality of $\overline{D}$. Analogous arguments prove that e and i cannot have a common neighbour. Note also that the vertices $a, b, \ldots, i$ shown in Fig. 6 are pairwise distinct, since otherwise $\overline{D}$ would contain a separating 3-cycle or a 4-cycle (different from D). Moreover, the same arguments show that none of eh, ea, ge, ec, and ca is an edge of G.

Let us delete vertices u, v, z and x and identify i and e. Further, add the edges ge, ec and ca as shown in Fig. 6, and call the obtained graph G^*. As proved above, G^* is a graph without loops or multiple edges. By the minimality of G, there is a degenerate coloring of G^*, such that each of the vertex sets $\{i, a, b, c\}$, $\{c, d, e\}$, $\{e, f, g, h\}$, and $\{h, i\}$ are colored by pairwise distinct colors (note that i and b, and f and h are opposite in G^*). This coloring induces a coloring of (a subgraph of) G, where i and e receive the same color. Since e and i are at distance at least 3 in G, this coloring is degenerate and has opposite vertices colored with distinct colors. We now color the vertices x, z, v and u in this order. Since x is colored before u, v and z and the color of e equals the color of i, we find that there are at most 9 prohibited colors for x; these are the colors used for the vertices a, b, c, d, h, i and additional three opposite vertices distinct from d, v and h. We color the vertex x with one of the two remaining colors not prohibited for x. Next we color z, which has 10 prohibited colors; these are the colors used for the vertices e, d, c, x, f, g, h, b and the two opposite vertices with respect to edges dc and ed. Note that we have requested that color of z be different from g in order to be able to apply Observation 1 when coloring v. So there is a color not prohibited for z, and we use this color to color it. Since i and e have the same color, the color of x is distinct from the color of e. Hence, the colors on the neighbors of z are all distinct. Therefore the coloring is still degenerate. Next we color v, which has at most 10 prohibited colors: it has six neighbours and six opposite vertices, but u is not yet colored, and e, i have the same color. Finally, we color u, which has also at most 9 prohibited colors (the colors of 5 opposite and 5 adjacent vertices, where two of them coincide, namely the colors of e and i).

By applying Observation 1 at each step, we conclude that the resulting coloring of G is degenerate. It was constructed in such a way that all opposite vertices have different colors. This contradiction to non-colorability of G completes the proof of Claim 3.

Claim 4: An internal vertex u of $\overline{D}$ of degree 7 cannot be adjacent to three internal vertices v, x, z of $\overline{D}$ of degree 5 and two vertices y and t of degree 6, such that y is adjacent to v and x and t is adjacent to x and z.

Proof: Suppose that u, v, x, z, y, t are vertices of $\overline{D}$ contradicting the claim. See Fig. 7 for additional notation. Since u, v, z and x are internal vertices of $\overline{D}$, we find that b and g are neither adjacent nor they have a common neighbor (since otherwise one would find a 4 or 5-cycle with fewer vertices in the interior when compared to D).

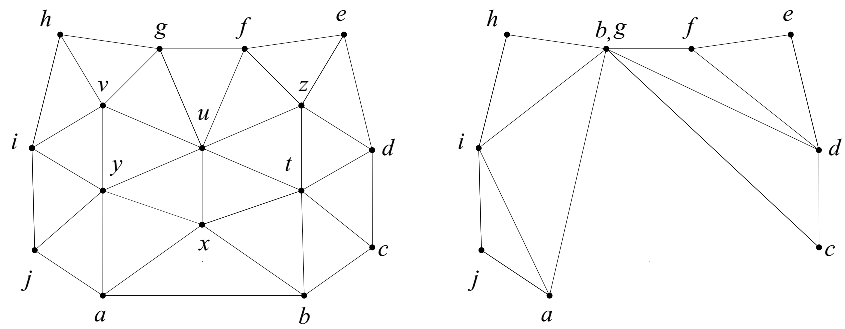

Figure 7: u, v, z and x are vertices of degrees 7, 5, 5 and 5, respectively

Let us remove vertices t, u, v, x, y, z. Then identify vertices b and g, add edges ig, ia, fd and gd (or fc) and call the obtained graph G^*. The degenerate coloring of G^* induces a degenerate coloring of a subgraph of G, where b and g receive the same color. We will color the vertices u, y, t, x, v, z in this order. Observe that for the vertex u, there are at most $10 = 14 + 2 - 5 - 1$ prohibited colors: 14 colors for adjacent and opposite vertices, $+2$ colors for j and c (which we want different from u to use Observation 1 when later coloring y and t), -5 for not yet colored adjacent vertices, and -1 because the color of g is equal to the color of b. Similarly one can argue that for y there are at most 8 restrictions, and for t, v, x, z, there are at most 10 prohibited colors. Since we color with 11 colors and for each vertex u, y, t, x, v, z there are fewer than 11 prohibited colors, we can extend the partial coloring of G induced by the coloring of G^* to a degenerate diagonal coloring of G.

Claim 4 implies that, if u is a vertex of degree 7 with three (non-adjacent) neigbours v, x, z of degree 5 and distributed as shown in Fig. 7, then either y or t has degree at least 7.

We complete the proof of the theorem by applying the discharging method in $\overline{D}$. We define the charge $c(v)$ of a vertex $v \in V(\overline{D})$ as

$$c(v) = \begin{cases} \deg_{\overline{D}}(v) - 6, & \text{if } v \in \mathrm{int}(\overline{\mathrm{D}}), \\ \deg_{\overline{D}}(v) - 3, & \text{if } v \in V(D). \end{cases} \tag{3}$$

It follows from the Euler formula that

$$\sum_{v \in V(\overline{D})} (\deg_{\overline{D}}(v) - 6) = -6 - 2|D|\,,$$

and herefrom

$$\sum_{v \in V(\overline{D})} c(v) = \sum_{v \in V(D)} (\deg_{\overline{D}}(v) - 3) + \sum_{v \in \mathrm{int(D)}} (\deg_{\overline{D}}(v) - 6) = |D| - 6\,. \tag{4}$$

We now perform the discharging procedure as follows:

(i) Every vertex of D gives 1 to each neighbor of degree 5 in $\mathrm{int}(D)$.

(ii) Every vertex in $\mathrm{int}(D)$ of degree at least 8 gives 1/2 to each neighbor of degree 5 in $\mathrm{int}(D)$.

(iii) Every vertex in $\mathrm{int}(D)$ of degree 7 with at most two neighbors of degree 5 in $\mathrm{int}(D)$, gives 1/2 to each neighbor of degree 5 in $\mathrm{int}(D)$.

(iv) A vertex $u \in \mathrm{int}(D)$ of degree 7 with three neighbors of degree 5 in $\mathrm{int}(D)$ has its neighborhood as shown in Fig. 7, except that y and t are not necessarily of degree 6.

   (a) If $\deg(y) \geq 7$ and $\deg(t) \geq 7$, then u gives 1/2 to x and 1/4 to v and z.

   (b) If $\deg(y) \geq 7$ and $\deg(t) = 6$, then u gives 1/2 to z and 1/4 to v and x.

   (c) If $\deg(y) = 6$ and $\deg(t) \geq 7$, then u gives 1/2 to v and 1/4 to z and x.

Note that a vertex of D cannot have degree ≤ 3 in $\overline{D}$, since this would imply that D is not a 3, 4 or 5-cycle with minimum number of vertices in its interior. Moreover, if v is a vertex of D and has $\alpha > 1$ neighbors of degree 5 in $\mathrm{int}(D)$, then the degree of v in $\overline{D}$ is at least $2\alpha + 1$, by Claim 2. This implies that every vertex of D retains (after discharging) at least 0 of its charge. Also, by Claim 2, a vertex of degree $\alpha \geq 7$ has at most $\lfloor \frac{\alpha}{2} \rfloor$ neighbors of degree 5. So we infer from the discharging rules that every vertex of $\mathrm{int}(D)$ of degree ≥ 7 has non-negative charge after the discharging procedure. Since internal vertices of degree 6 keep their charge at zero, and there are no vertices of degree less than 5, the only candidates for having negative charge are internal vertices of degree 5.

Observe that every vertex $v \in \mathrm{int}(D)$ of degree 5 is either adjacent to a vertex of D or it has, by Claims 0, 1, 2 and 3, at least two neighbors of degree ≥ 7. If v has 4 or 5 neighbors of degree ≥ 7, then after discharging, its charge will be at least 0, since every vertex of degree ≥ 7 gives at least $1/4$ to each of its neighbors of degree 5. If v has three neighbors of degree ≥ 7 and one of them is not of degree 7 or does not have three neighbors of degree 5, then this neighbor will give $1/2$ to v and thus v will have charge at least 0. Otherwise v has three neighbors of degree 7, and each of them has three neighbors of degree 5. It follows from the rule (iv) that one of the neighbors of v will give $1/2$ to v, so the final charge of v will be non-negative. In the remaining case, where v has exactly two neighbors of degree ≥ 7, we see that these two neighbors are not adjacent, by Claim 3. Therefore the discharging rule (iv) implies that both of them give $1/2$ to v. Thus, the final charge at v is non-negative.

This discharging process proves that the left side of equation (4) is non-negative, while the right side is negative, a contradiction. This completes the proof of Theorem 3. ∎

Finally we turn to the proof of the diagonal list coloring result. Observe that identification of vertices cannot be done with list colorings, so the proofs of Claims 3 and 4 from the above proof cannot be extended to list colorings. We next give a different proof of Corollary 2.

The proof is similar to the proof of Theorem 3. We start by assuming the contrary, and let G be a minimum counterexample. Define $C, \overline{C}, D$ and $\overline{D}$ the same way as in the proof of Theorem 3. The proofs of Claims 0, 1 and 2 from the proof of Theorem 3 also hold for list colorings.

Claim 5: An internal vertex of $\overline{D}$ of degree 5 cannot be adjacent to two adjacent internal vertices of $\overline{D}$ of degree 6.

Proof: Suppose on the contrary, that u, v, z are vertices of degree 5, 6 and 6, respectively. See Fig. 8 for further notation. We do the reduction as follows.

Delete the vertices u, v and z and add edges be, bd, he and hf. Note that these are not edges of G, since $\overline{D}$ has no separating 3-cycles. Let us call the obtained graph G^*. The lists for vertices of G^* are inherited G. By the minimality of G, there is a degenerate list coloring of G^*, such that b, c, d, e and e, f, g, h (respectively) are colored by pairwise distinct colors (since e and c, and e and g are opposite in G^*). This coloring induces a coloring of a subgraph of G. We next color v, z and u (in this order). Observe that v and z have at most 11 prohibited colors (6 opposite vertices and 4 resp. 5 neighbors, and the color of c is prohibited for v); the vertex u is of degree 5, thus it has at most 10 prohibited colors. Thus, for all three remaining vertices there is a free color so that we

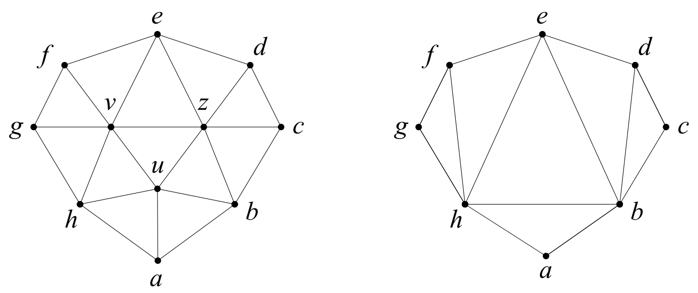

Figure 8: u, v and z are vertices of degree 5, 6 and 6, respectively

can extend the list coloring of G^* to a degenerate list coloring of G; this is a contradiction proving the claim.

Finally, the following dicharging procedure leads to a contradiction.

(i) Every vertex of D gives 1 to each neighbor of degree 5 in $\text{int}(D)$.

(ii) Every vertex of degree at least 7 of $\text{int}(D)$, gives 1/3 to each neighbor of degree 5 in $\text{int}(D)$.

Since every vertex of degree 5 of $\text{int}(D)$ has (by Claims 0, 1, 2, and 5) at least three neighbors of degree ≥ 7 or it has a nighbor in D, and every vertex of degree $\alpha \geq 7$ of $\text{int}(D)$ has at most $\lfloor \frac{\alpha}{2} \rfloor$ neighbors of degree 5 in $\text{int}(D)$, we conclude that the left side of equation (4) is non-negative, while the right side is negative, a contradiction. ∎

References

[1] O. V. Borodin, On decomposition of graphs into degenerate subgraphs, Diskretny analys, Novosibirsk 28 (1976) 3–12 (in Russian).

[2] O. V. Borodin, A proof of Grünbaum's conjecture on the acyclic 5-colorability of planar graphs, Sov. Math. Dokl. 17 (1976) 1499–1502.

[3] O. V. Borodin, On acyclic colorings of planar graphs, Discrete Math. 25 (1979) 211–236.

[4] O. V. Borodin, Diagonal 11-coloring of plane triangulations, J. Graph Theory 14 (1990) 701–704

[5] A. Bouchet, J.-L. Fouquet, J.-L. Jolivet, M. Riviére, On a special face colouring of cubic graphs, Ars Combin. 24 (1987) 67–76.

[6] G. Chen, R. H. Schelp, Graphs with linearly bounded Ramsey numbers, J. Combin. Theory Ser. B 57 (1993) 138–149.

[7] B. Grünbaum, Acyclic colorings of planar graphs, Israel J. Math. 14 (1973) 390–412.

[8] T. R. Jensen, B. Toft, Graph Coloring Problems, Wiley, New York, 1995.

[9] H. A. Kierstead, W. T. Trotter, Planar graph coloring with an uncooperative partner, J. Graph Theory 18 (1994), 569–584.

[10] H. A. Kierstead, D. Yang, Orderings on graphs and game coloring number, Order 20 (2003) 255–264.

[11] D. Rautenbach, A conjecture of Borodin and a coloring of Grünbaum, Cracow conference on Graph Theory, Ustron '06, Electr. Notes Discrete Math. 24 (2006) 187–194.

[12] D. P. Sanders, Y. Zhao, On diagonally 10-coloring plane triangulations, J. Graph Theory 20 (1995) 77–85.

[13] C. Thomassen, Decomposing a planar graph into degenerate graphs, J. Combin. Theory Ser. B 65 (1995) 305–314.

[14] C. Thomassen, Decomposing a planar graph into an independent set and a 3-degenerate graph, J. Combin. Theory Ser. B 83 (2001) 262–271.

[15] X. Zhu, Game coloring number of planar graphs, J. Combin. Theory Ser. B 75 (1999) 245–258.

[16] X. Zhu, Refined activation strategy for the marking game, J. Combin. Theory Ser. B 98 (2008) 1-18.

[17] X. Zhu, Colouring graphs with bounded generalized colouring number, Discrete Mathematics, to appear.