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Linkless and Flat Embeddings in 3-space in Quadratic Time

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Abstract

We consider embeddings of graphs in the 3-space $\mathbb{R}^3$ (all embeddings in this paper are assumed to be piecewise linear). An embbeding of a graph in $\mathbb{R}^3$ is *linkless* if every pair of disjoint cycles forms a trivial link (in the sense of knot theory), i.e., each of the two cycles (in $\mathbb{R}^3$) can be embedded in a closed topological 2-disk disjoint from the other cycle. Robertson, Seymour and Thomas [38] showed that a graph has a linkless embedding in $\mathbb{R}^3$ if and only if it does not contain as a minor any of seven graphs in Petersen's family (graphs obtained from K_6 by a series of $Y\Delta$ and ΔY operations). They also showed that a graph is linklessly embeddable in $\mathbb{R}^3$ if and only if it admits a *flat* embedding into $\mathbb{R}^3$, i.e. an embedding such that for every cycle C of G, there exists a closed disk $D \subseteq \mathbb{R}^3$ with $D \cap G = \partial D = C$. Clearly, every flat embeddings is linkless, but the converse is not true.

We consider the following algorithmic problem associated with embeddings of graphs in $\mathbb{R}^3$:

Flat and Linkless Embedding

Input: A graph G.

Output: Either detect one of Petersen's family graphs as a minor in G or return a flat (and linkless) embedding in the 3-space.

The first conclusion is a certificate that the given graph has no linkless and no flat embeddings. In this paper we give an $O(n^2)$ algorithm for this problem. Our algorithm does not depend on minor testing algorithms.

Keywords: Linkless embedding, Petersen family, Flat embedding.

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1 Introduction

1.1 Embedding graphs in the 3-space

A seminal result of Hopcroft and Tarjan [13] from 1974 is that there is a linear time algorithm for testing planarity of graphs. This is just one of a host of results on embedding graphs in surfaces. These problems are of both practical and theoretical interest. The practical issues arise, for instance, in problems concerning VLSI, and also in several other applications of "nearly" planar networks, as planar graphs and graphs embedded in low genus surfaces can be handled more easily. Theoretical interest comes from the importance of the genus parameter of graphs and from the fact that graphs of bounded genus naturally generalize the family of planar graphs and share many important properties with them.

In [32] Robertson and Seymour proved that for any fixed graph H there is a cubic-time algorithm for testing whether H is a minor of a given graph G. This implies that there is an $O(n^3)$ algorithm for deciding membership in any minor-closed family of graphs, because by their seminal result in [33], such a family can be characterized by a finite collection of excluded minors. However, this theorem is not explicit. More precisely, the graph minor theorem is not constructive; in general, we do not know how to obtain an excluded minor characterization of a given minor-closed family of graphs. In addition, Robertson and Seymour's algorithm solves the decision problem but it is not apparent how to construct, e.g., an embedding from their algorithm.

Recently, some apparently new and nontrivial linear time algorithms concerning graph embeddings have appeared: one is concerned with drawing a given graph into the plane with at most k crossings (for fixed k) [15], and another one is concerned with embedding a given graph into a surface with face-width at least k with an application to the graph isomorphism problem [16]. In addition, Mohar [24, 25] gave a linear time algorithm for testing embeddability of graphs in an arbitrary surface and constructing an embedding, if one exists. This is one of the deepest results in this area, and it generalizes linear time algorithms for testing planarity and constructing a planar embedding if one exists [4, 13, 43]. This algorithm is further simplified in [17].

In this paper, we are interested in *embedding a graph in the 3-space*, where all embeddings are piecewise linear. There are many minor-closed families of graphs that arise in the study of topological problems. An illustrative example is the class of likelessly embeddable graphs. We call a pair of vertex disjoint cycles drawn in the 3-space $\mathbb{R}^3$ *un-linked* if there is a 2-dimensional disc in $\mathbb{R}^3$ that contains the first cycle and is disjoint from the other one. Otherwise, the two cycles are *linked*. Intuitively, if two cycles in $\mathbb{R}^3$ are linked, we can not contract one into a single point without cutting the other. By a *linkless embedding* of a graph we mean an embedding of a graph in $\mathbb{R}^3$ in such a way that no two vertex-disjoint cycles are linked. Linkless embeddings are first studied by Conway and Gordon [8]. An algorithmic problem concerning linkless embeddings is studied by Motwani, Raghunathan and Saran [27], who gave a partial result for linkless embeddings and its algorithmic applications.

Robertson, Seymour and Thomas [38] proved that G is linklessly embeddable in the 3-space if and only if G does not have any graph in the Petersen family as a minor. By the *Petersen family* we mean the graphs that can be obtained from K_6 by a series of $Y\Delta$ and ΔY operations. See Figure 1 (in the appendix) for drawings of these graphs on the projective plane; note that the third one, $K_{4,4} - e$, cannot be embedded in the projective plane. In the same paper [38] it is shown that a graph is linklessly embeddable in $\mathbb{R}^3$ if and only if it admits a *flat* embedding into $\mathbb{R}^3$, i.e. an embedding such that for every cycle C of G, there exists a closed disk $D \subseteq \mathbb{R}^3$ with $D \cap G = \partial D = C$. Clearly, every flat embeddings is linkless, but the converse is not true. In many ways flat embeddings are nicer to work with and in this paper we will work with flat rather than linkless embeddings.

Linkless embeddings have drawn attention by many researchers. Besides researchers working on knot theory, many researchers working in discrete mathematics are also interested in this topic. For example, Lovász and Schrijver [22, 23] proved that two well-known invariants given by Colin de Verdière [5, 6, 7] are closely related to linklessly embeddable graphs. The invariants are based on spectral properties of matrices associated with a graph G.

Flat embeddings in the 3-space are a generalization of embeddings in the plane. These two embedding problems share an interesting property. The famous theorem by Whitney which says that every 3-connected planar graph has a unique planar embedding, can be generalized to the flat embedding case, i.e., every flatly embeddable 4-connected graph G has an "essentially unique" flat embedding in $\mathbb{R}^3$, see [38]. Here, "essentially unique" means embeddings up to ambient isotopy, which we define later.

We will study the following algorithmic problem associated with linklessly embeddable graphs. As far as we are aware, this problem was open until our work, and even a solution to the simpler problem of findiung linkless embeddings has only been treated very recently.

Flat and Linkless Embedding

Input: A graph G.

Output: Either (1) detect one of Petersen's family graphs as a minor in G, or (2) return a flat (and hence linkless) embedding of G in $\mathbb{R}^3$.

Let us observe that if the output is (1), then it is one of the excluded minors for linklessly embeddable graphs and hence it is a certificate that the given graph has no linkless (and no flat) embeddings. As mentioned above, by Robertson and Seymour's results [32], we can test whether or not an input graph has one of Petersen's family graphs as a minor, but if it does not contain any of them, Robertson and Seymour's algorithm does not give the second conclusion. As pointed out in [38], the algorithm in [40] together with the result in [38] gives rise to a polynomial time algorithm to test whether or not an embedding in the 3-space is flat, but no polynomial-time algorithm for constructing flat embeddings was known prior to our work. Quite recently (see a more detailed discussion below), an algorithm for finding linkless embeddings was proposed by van der Holst [14].

1.2 Our result

Our contribution in this paper is to give a polynomial-time algorithm for the above mentioned problem.

Theorem 1.1. *There is an $O(n^2)$-time algorithm for the flat and linkless embedding problem.*

Clearly, every flat embedding is linkless (but the converse is false, as a graph that consists of one vertex with two loops shows). However, Robertson, Seymour and Thomas proved that a graph admits a linkless embedding in $\mathbb{R}^3$ if and only if it admits a flat embedding. Let us also remark that our algorithm does not use Robertson and Seymour's algorithm [32] (although we need some lemmas from [32]).

We have learned that van der Holst [14] has a polynomial-time algorithm to construct a linkless embedding if one exists. Our algorithm is an advancement over van der Holst's in several respects:

1. Our algorithm finds flat embeddings in $\mathbb{R}^3$, while the algorithm from [14] only finds linkless embeddings that are not necessarily flat.
2. The time complexity of van der Holst's algorithm is considerably more expensive than ours (it is at least $\Omega(n^5)$).
3. In order to give a polynomial time algorithm for the linkless embedding problem, his algorithm uses deep results of Robertson and Seymour [32], while ours does not.
4. The algorithm and the proof method in [14] are very algebraic and are completely different from our paper, which is more combinatorial and geometric.

There are many generally hard problems which can be solved in polynomial time (often, even linear time) when considering planar graphs or "nearly" planar graphs, e.g., MAXIMUM CLIQUE, SUBGRAPH ISOMORPHISM [11]. Even for problems that remain NP-hard on planar graphs, we often have efficient approximation algorithms, e.g, INDEPENDENT SET, TSP, Weighted TSP, VERTEX COVER, DOMINATION SET etc [2, 12, 19, 21]. We expect that all of these fast algorithms for planar graphs can be generalized to linklessly embeddable graphs as well, using our algorithm in Theorem 1.1 and a flat embeddding.

1.3 Overview of our algorithm

At a high level of description, our algorithm for Theorem 1.1 proceeds as follows: the algorithm first iteratively reduces the size of the input graph until it is 4-connected and reaches a graph of bounded tree-width. Then the algorithm solves the problem on this graph of bounded tree-width.

Bounded tree-width case. This second step needs two deep results in [38]. The first ingredient is that any Kuratowski graph, i.e, a subdivision of K_5 or $K_{3,3}$, has a unique flat embedding in the 3-space $\mathbb{R}^3$. The second ingredient is that if G is 4-connected and has a flat embedding in the 3-space $\mathbb{R}^3$, then G has a unique flat embedding. Thus by combining these two ingredients, we get the following strong fact:

> Fix a flat embedding of a Kuratowski subgraph K of a 4-connected graph G in the 3-space $\mathbb{R}^3$. Then the rest of the graph is uniquely attached to K, if G has a flat embedding.

In general, this fact is not enough to give a polynomial time algorithm to construct a flat embedding, if one exists. However, if the tree-width of a 4-connected graph is bounded, we can construct a flat embedding in polynomial time (even in linear time) using dynamic programming, if one exists.

Reduction step. For the reduction step, the algorithm uses the excluded grid minor theorem [10, 28, 30, 37]: if the tree-with of G is big enough, G contains a huge grid minor. By combining the results in [32] and [18], if an input graph does not contain a K_6-minor, then, after deleting at most one vertex, we can find a subgrid minor which is planarly drawn, i.e, up to 3-separations, the subgrid minor induces a planar embedding. In fact, if there is a separation (A, B) of order at most three in this planarly drawn subgrid, then this gives us a reduction (for details, we refer the reader to Section 3). Otherwise, this subgrid minor induces a 2-cell embedding in a plane (and hence it is a planar subgraph).

A deep theorem in [32] tells us that a vertex "deep inside" this subgrid minor is irrelevant with respect to all excluded minors in the Petersen family. In addition we will prove that such a vertex does not affect our flat embedding in the 3-space. Hence, we can remove this vertex without affecting flat embeddability of G. Note that the difficulty here is only in the proof of the existence of this vertex. Once we know that there is an irrelevant vertex, such a vertex can easily be found in linear time.

1.4 Basic definitions

Before proceeding, we review basic definitions. For basic graph theory notions, we refer the reader to the book by Diestel [9], for topological graph theory we refer to the monograph by Mohar and Thomassen [26].

A *separation* of a graph G is a pair (A, B) of subgraphs of G with $A \cup B = G$ and $E(A \cap B) = \emptyset$. The *order* of the separation is $|V(A) \cap V(B)|$. If (A, B) is a separation of G of order k, we write A^+ for the graph obtained from A by adding edges joining every pair of nonadjacent vertices in $V(A) \cap V(B)$. We define B^+ analogously.

A *tree-decomposition* of a graph G is a pair (T, B), where T is a tree and B is a family $\{B_t \mid t \in V(T)\}$ of vertex sets $B_t \subseteq V(G)$, such that the following two properties hold:

(W1) $\bigcup_{t \in V(T)} B_t = V(G)$, and every edge of G has both ends in some B_t.

(W2) If $t, t', t'' \in V(T)$ and t' lies on the path in T between t and t'', then $B_t \cap B_{t''} \subseteq B_{t'}$.

The *width* of a tree-decomposition (T, B) is defined as $\max\{|B_t| \mid t \in V(T)\}$. The *tree-width* of G is defined as the minimum width taken over all tree-decompositions of G. Let (T, B) be a tree-decomposition of a graph G. By fixing a root r of T we give T an orientation. For $t \in V(T)$ we define T_t to be the subtree of T rooted at t, i.e., the subtree of T induced by the set of nodes $s \in V(T)$ such that the unique path from s to r contains t. We define $B(T_t) := \bigcup_{s \in V(T_t)} B_s$.

One of the most important results about graphs, whose tree-width is large, is the existence of a large grid minor or, equivalently, a large wall. Let us recall that an *r-wall* or a *wall of height* r is a graph which is isomorphic to a subdivision of the graph W_r with vertex set $V(W_r) = \{(i, j) \mid 1 \leq i \leq r,\ 1 \leq j \leq r\}$ in which two vertices (i, j) and (i', j') are adjacent if and only if one of the following possibilities holds: (1) $i' = i$ and $j' \in \{j - 1, j + 1\}$ or (2) $j' = j$ and $i' = i + (-1)^{i+j}$.

We can also define an $(a \times b)$-wall in a natural way, so that the r-wall is the same as the $(r \times r)$-wall. It is easy to see that if a graph G contains an $(a \times b)$-wall as a subgraph, then it has an $(\lfloor \frac{1}{2}a \rfloor \times b)$-grid minor, and conversely, if G has an $(a \times b)$-grid minor, then it contains an $(a \times b)$-wall. Let us recall that the $(a \times b)$-grid is the Cartesian product of paths, $P_a \square P_b$. An (8×5)-wall is shown in Figure 2 (in the appendix).

Theorem 1.2. *For any r, there exists a constant $f(r)$ such that if a graph G has tree-width at least $f(r)$, then G contains an r-wall.*

The best known upper bound for $f(r)$ is given in [10, 28, 37]. It is 20^{2r^5}. The best known lower bound is $\Theta(r^2 logr)$, see [37].

2 Flatly and Linklessly Embeddable Graphs

In this section we recall some results about linklessly embeddable graphs used later on. Due to space restrictions, we can only state the most relevant results and almost no background. For the referees convenience, the appendix contains a more coherent exposition.

2.1 Constructions of Flat Embeddings

Recall that a piecewise-linear embedding of a graph in the 3-space $\mathbb{R}^3$ is *flat* if every cycle of the graph bounds a 2-dimensional disk disjoint from the rest of the graph. If C, C' are disjoint simple closed curves in $\mathbb{R}^3$, then their *linking number* is the number of times (modulo 2) that C crosses C' in a regular projection of $C \cup C'$ onto some hyperplane. It is easy to see that the linking number (modulo 2) is the same for every such projection. Hereafter, all linking numbers discussed in this paper will be mod 2. The proof of the following result is easy and the details are left to the reader.

Lemma 2.1. *Let G be a graph, and σ be an embedding of G in $\mathbb{R}^3$. Let C_1, C_2 be disjoint cycles in G, and let P be a path in G that is disjoint from $C_1 \cup C_2$, except that its endvertices are in $V(C_1)$. Let C_1', C_1'' be the two cycles in $V(C_1 \cup P)$ with $V(C_1') \cap V(C_1'') = V(P)$. If the linking number of C_1, C_2 is nonzero, then also one of the linking numbers of C_1', C_2 or C_1'', C_2 is nonzero.*

Let v be a vertex of degree 3 in a graph G, with its three neighbors v_1, v_2, v_3. Let H be a graph obtained from $G - v$ by adding three new edges v_1v_2, v_2v_3, v_3v_1. We say that H is obtained from G by a $Y\Delta$ change (at v), and G is obtained from H by a ΔY change (at v_1, v_2, v_3). If G' can be obtained from G by a series of $Y\Delta$ or ΔY changes, we say that G and G' are $Y\Delta$*-equivalent*. The *Petersen family* is the set of the seven graphs (up to isomorphism) that are $Y\Delta$-equivalent to K_6. One of these is the Petersen graph. The following result was proved in [38].

Theorem 2.2. *Let G be a graph obtained from H by a ΔY operation. Then G has a flat embedding in $\mathbb{R}^3$ if and only if H has.*

It is easy to see that degree 1 vertices can be deleted and degree 2 vertices can be suppressed without affecting linkless embeddability. Thus it follows from Theorem 2.2 that we may assume that each vertex has degree at least four. Furthermore, concerning vertices of degree four, one can say the following.

Lemma 2.3. *Suppose that a graph G contains an edge uv, where $\deg_G(u) = 4$ and $\deg_G(v) = 5$. Suppose, moreover, that $N(u) = (N(v) \cup \{v\}) - \{u, a\}$ for some vertex a in $N(v)$. Then there is a separation (A, B) of order four such that $B - A$ consists of u and v only, and G has a flat embedding in $\mathbb{R}^3$ if and only if for every vertex $b \in N(v) - \{u, a\}$, the graph G' obtained from G by contracting the edge ub has a flat embedding.*

We sketch the proof of Lemma 2.3, which is essentially given in (6.6) in [38]. If G' has a flat embedding in 3-space $\mathbb{R}^3$, then $A \cap B$ is contained in a disk. In fact, as in the proof of (6.6) in [38], if A is planar with $A \cap B$ appearing on the outer face boundary of A, then the flat embedding of G' extends to A. Thus $A \cap B$ is contained in a single disk, and we can easily put the vertices u, v back to this disk so that the resulting embedding of G (that extends the flat embedding of G') is still flat.

The following is (6.5) in [38]. For completeness, we give a sketch of the proof in the appendix. Recall from above the definition of the graphs A^+, B^+ for a separation (A, B) of a graph G.

Theorem 2.4. *Suppose G has a separation (G_1, G_2) with $|V(G_1) \cap V(G_2)| \leq 4$. Suppose furthermore, G contains both G_1^+ and G_2^+ as a minor. Then G has a flat embedding in 3-space $\mathbb{R}^3$ if and only if both G_1^+ and G_2^+ have one.*

2.2 Spatial Embeddings of 4-connected graphs

The aim of this section is to show that 4-connected linklessly embeddable graphs essentially have a unique flat embedding in 3-space $\mathbb{R}^3$. The following results are proved in [38] along with its companion papers [34, 35]. Readers not familiar with these results may wish to consider the survey [39] which contains many of the results needed below.

We refer to the graphs K_5 and $K_{3,3}$ as the *Kuratowski graphs*. A *Kuratowski subgraph* of a graph G is a subgraph of G isomorphic to a subdivision of a Kuratowski graph. For embeddings ϕ_1, ϕ_2 of a graph G we write $\phi_1 \cong_{a.i.} \phi_2$ to denote that they are ambient isotopic. Let us recall that ϕ_1 and ϕ_2 are *ambient isotopic*, if there exists an orientation preserving homeomorphism $\mathbb{R}^3 \to \mathbb{R}^3$ mapping ϕ_1 onto ϕ_2.

Lemma 2.5.

1. *Any two flat embedings of a planar graph in $\mathbb{R}^3$ are ambient isotopic.*
2. *The graphs K_5 and $K_{3,3}$ have exactly two non-ambient isotopic flat embeddings.*
3. *Let ϕ_1, ϕ_2 be two flat embeddings of a graph G which are not ambient isotopic. Then there is a subgraph H of G isomorphic to a subdivision of a Kuratowski graph for which $\phi_{1|H}$ and $\phi_{2|H}$ are not ambient isotopic. Here $\phi_{i|H}$ denotes the restriction of ϕ_i to H.*

We now define a neighbourhood relation between Kuratowski subgraphs of a graph G. Let $H_1, H_2 \subseteq G$ be Kuratowski subgraphs of G such that $H_1 \neq H_2$. H_1 and H_2 are 1-*adjacent* if there exists a path $P \subseteq G$ and an $i \in \{1, 2\}$ such that P has only its endpoint in common with H_i and such that $H_{3-i} \subseteq H_i \cup P$.

H_1 and H_2 are 2-*adjacent* if there are distinct vertices $v_1, \ldots, v_7 \in V(G)$ and pairwise internally vertex disjoint paths $L_{i,j}$, for $(i,j) \in \{1, \ldots, 4\} \times \{5, 6, 7\} \cup \{(3, 4)\}$ linking v_i and v_j such that $H_1 = \bigcup \{L_{i,j} \mid (i,j) \in \{2, 3, 4\} \times \{5, 6, 7\}\}$ and $H_2 = \bigcup \{L_{i,j} \mid (i,j) \in \{1, 3, 4\} \times \{5, 6, 7\}\}$. The path $L_{3,4}$ is not used here but is required to exist. Note that if H_1, H_2 are 2-adjacent then they are both isomorphic to subdivisions of $K_{3,3}$. H_1 and H_2 are *adjacent* if they are 1- or 2-adjacent. Let H, H' be Kuratowski subgraphs of G. We say that H and H' *communicate* if there are Kuratowski subgraphs $H = H_1, \ldots, H_k = H'$ of G such that H_i and H_{i+1} are adjacent for all $1 \leq i < k$.

Lemma 2.6.

1. *Let ϕ_1, ϕ_2 be flat embeddings of G and let H, H' be adjacent Kuratowski subgraphs of G. If $\phi_{1|H} \cong_{a.i.} \phi_{2|H}$ then $\phi_{1|H'} \cong_{a.i.} \phi_{2|H'}$.*
2. *If G is 4-connected, then all pairs H, H' of Kuratowski subgraphs of G communicate.*
3. *If ϕ, ϕ' are flat embeddings of a 4-connected graph G, then $\phi \cong_{a.i.} \phi'$ or $\phi \cong_{a.i.} -\phi'$.*

Note that (iii) follows easily from (i) and (ii) and the previous lemma. For the purpose of this paper this means that once we have fixed an embedding for a Kuratowski subgraph $H \subseteq G$, there is only one way to extend this embedding to an adjacent Kuratowski subgraph H' (up to ambient isotopy). Furthermore, if G is 4-connected, then by starting from one Kuratowski subgraph whose embedding we fix and iteratively proceeding to adjacent Kuratowski subgraphs, we can embed all Kuratowski subgraphs of G. This embedding then has a unique extension to the complete graph.

3 Bounding the tree-width

In this section we will present the reduction step of the general algorithm for solving the Linkless Embedding problem presented in Section 5 below. Let us observe that by Theorem 2.2 and Lemma 2.3 (and remarks just after Lemma 2.3), we may assume that every vertex in a given graph G has minimum degree at least four, and any vertex of degree four does not satisfy the assumption of Lemma 2.3.

Let us define that the *nails* of a wall are the vertices of degree three within it. The *perimeter* of a wall W, denoted $per(W)$ is the unique face in this embedding which contains more than 6 nails. As walls are 3-connected, Whitney's theorem implies that any wall has a unique planar embedding. For any wall W in H, there is a unique component U of $H - per(W)$ containing $W - per(W)$. The *compass* of W, denoted $comp(W)$, consists of the graph with vertex set $V(U) \cup V(per(W))$ and edge set $E(U) \cup E(per(W)) \cup \{xy | x \in V(U), y \in V(per(W))\}$. A *subwall* of a wall W is a wall which is a subgraph of W. A h-subwall of W is *proper* if it consists of h consecutive bricks from each of h consecutive rows of W. The *pegs* of a proper subwall W' of a wall W are the nails of W on the perimeter of W' which are not nails of W'. The *exterior* of W' is $W - W'$. A proper subwall is *dividing* if its compass is disjoint from its exterior. We say a proper subwall W' is dividing in a subgraph H of F if $W' \subseteq H$ and the compass of W' in H is disjoint from $(W - W') \cap H$.

A wall is *planarly drawn* if its compass does not contain two vertex disjoint paths connecting the diagonally opposite corners. Note that if the compass of W has a planar embedding whose infinite face is bounded by the perimeter of W then W is clearly planarly drawn.

Seymour [41], Thomassen [42], and others have characterized that if the wall W is planarly drawn, then its compass $comp(W)$ can be embedded into a plane, up to 3-separations, such that its perimeter $per(W)$ is the outer face boundary. For the definition of "up to 3-separations", we refer the reader to the appendix (Subsection B.4).

It is easy to see that any subwall of a planarly drawn wall must be both planarly drawn and dividing. Furthermore, if x and y are two vertices of a planarly drawn wall W and there is a path between them which is internally disjoint from W then either x and y are both on $per(W)$ or some brick contains both of them. Robertson and Seymour [32] proved:

Theorem 3.1. *For every pair of integers l and t there exist integers $w(l,t) > w'(l,t) > max(l,t)$ such that the following holds. Let K be a graph of order t. If the tree-width of a graph H is at least $w(l,t)$, and H has no K-minor, then there is a wall W of height $w'(l,t)$, and for some subset X of less than $\binom{t}{2}$ vertices of H there are t^{10} disjoint proper subwalls W' (of the wall W) of height l, which are disjoint from X and are planarly drawn and dividing in $H - X$. In* *addition, any of these disjoint proper subwalls of height l has face-distance at least t^{10} from any other in the wall W.*

In fact, we can give the explicit bound for $w(l,t)$. Combining the best known bound for the grid theorem in [37], the proof of Theorem 3.1 in [32] implies that $w(l,t) \leq 10^{10^{10^q}}$, where $q = l^t$.

A vertex v in G is called *irrelevant* with respect to a given minor M in G if G has an M-minor if and only if $G - v$ has. The following result was proved in [32].

Theorem 3.2. *There is a computable constant $f(t)$ satisfying the following: Let $H, W, X, K, w(l,t)$, $w'(l,t)$ be as in Theorem 3.1 and $l \geq f(t)$. Let W' be one of the proper subwalls (of the wall W) of height l which is disjoint from X, and is planarly drawn and dividing in $H - X$. Suppose furthermore that the comp(W') has a 2-cell embedding in a plane with per(W') in the outer face boundary. Then the unique vertex v which has distance exactly $l/2$ from the per(W') in the wall W' is irrelevant with respect to a K-minor.*

We now give an algorithmic result of Theorems 3.1 and 3.2 in [32].

Theorem 3.3. *Let $H, W, K, f(t), w'(l,t)$ be as in Theorem 3.1. Suppose the wall W of height $w'(t,l)$ is given. Then there is an $O(m)$ time algorithm, where m is the number of edges, to construct one of the following:*

1. *a K-minor in H, or*
2. *for some subset X of less than $\binom{t}{2}$ vertices of H there are t^{10} disjoint proper subwalls W' (of the wall W) of height l, which are disjoint from X, and are dividing in $H - X$. In addition, any of these disjoint proper subwalls of height l has face-distance at least t^{10} from any other in the wall W.*

It is easy to see that if G has at least $2^t|V(G)|$ edges, then one can easily find a K_t-minor in linear time (see [29]). If we can find a K_6-minor in a given graph G in linear time, we are done. So a graph G has at most $2^6|V(G)|$ edges, which, hereafter, we assume. Thus the time complexity of Theorem 3.3 can be improved to $O(n)$, where n is the number of vertices of an input graph.

If $K = K_6$ in Theorem 3.1, then the following stronger version of Theorem 3.2 is proved in [18].

Theorem 3.4. *Let $f(t)$ be as in Theorem 3.1 with $t = 6$. For any $l \geq f(6)$, there are integers $w(l) > w'(l)$ satisfying the following: If the tree-width of a graph H is at least $w(l)$, then there is a wall W of height $w'(l)$, and one of the following holds:*

1. *a K_6-minor in H, or*
2. *for some subset X of at most one vertex of H, there are at least 10 disjoint proper subwalls W' (of the wall W) of height l, which are disjoint from X*

and dividing in $H - X$. In addition, any of these disjoint proper subwalls of height l has face-distance at least 2 from any other in the wall W. Moreover,

(a) if $|X| = 0$ and three of the disjoint proper subwalls of height l are not planarly drawn, then H has a K_6-minor, and

(b) if $|X| = 1$ and there is a proper subwall of height l that is not planarly drawn, then H has a K_6-minor.

Furthermore, given the wall W, there is a linear time algorithm to compute one of 1 and 2 above.

We now prove the following result for the linkless embedding case. Hereafter, we may assume that a graph G has minimum degree at least 4 by Theorem 2.2 and the remarks just after Theorem 2.2.

Theorem 3.5. *There is a computable constant f satisfying the following: Let G be a given input graph with minimum degree 4, and $W, w(l), w'(l)$ be as in Theorem 3.4 with $l \geq f$. Then one of the following holds:*

1. *G has a K_6-minor, or*
2. *for some subset X of at most one vertex of G, there is a proper subwall W' (of the wall W) of height l, which is disjoint from X and dividing in $G-X$, and has a 2-cell embedding with per(W') in the outer face boundary in $G - X$. Moreover, the unique vertex v which has distance exactly $l/2$ from the per(W') in the wall W' is irrelevant with respect to a K-minor, where K is any graph in the Petersen family. Furthermore, no matter how we give a flat embedding of the graph $G - v$ in 3-space $\mathbb{R}^3$, the embedding of $G - v$ can be changed so that, after putting the vertex v back to the resulting embedding, the resulting embedding of G is flat, or*
3. *either there is a reduction as in Lemma 2.3, or there is a separation (A, B) of order at most three in $G - X$ such that $V(A) \cap V(B)$ only involves the vertices in the comp(W) in $G - X$, and B contains all the vertices of the per(W'). Moreover, G contains both $(A \cup X)^+$ and $(B \cup X)^+$ as minors, and hence this separation $(A \cup X, B \cup X)$ is a reduction as in Theorem 2.4. Note that it is a separation of order at most 4 in G.*

A proof of Theorem 3.5 will be given in Section C. We now state our main algorithmic result.

Theorem 3.6. *Let $G, W, f, l, w(l), w'(l)$ be as in Theorem 3.5. Suppose the wall W is given. Then there is an $O(n)$ time algorithm, where n is the number of vertices of G, to construct one of the following:*

1. *one of the graphs in the Petersen family in G as a minor, or*

2. *for some subset X of at most one vertex of G there is a proper subwall W' (of the wall W) of height l, which is disjoint from X, dividing in $G-X$, and has a 2-cell embedding with $per(W')$ in the outer face boundary in $G-X$. Moreover, we can find an irrelevant vertex as in the second conclusion of Theorem 3.5, or*

3. *either there is a reduction as in Lemma 2.3, or there is a separation (A, B) of order at most three in $G-X$ such that $V(A) \cap V(B)$ only involves the vertices in the $comp(W)$ in $G-X$, and B contains all the vertices of the $per(W')$. Moreover, G contains both $(A \cup X)^+$ and $(B \cup X)^+$ as minors, and hence this separation $(A \cup X, B \cup X)$ is a reduction as in Theorem 2.4. Note that it is a separation of order at most 4 in G.*

Again, a proof of Theorem 3.6 will be given in Section C.

4 Linkless Embeddings on Graph Classes of Bounded Tree-Width

In this section we present a linear time algorithm for the linkless embedding problem on graph classes of bounded tree-width. The algorithm is based on various results about flat embeddings presented in Section 2.2. We first consider the case of 4-connected graphs.

Lemma 4.1. *There is an algorithm which for a given 4-connected graph $G \in \mathcal{C}$ either returns a flat embedding of G or produces a minor $H \preceq G$ of the Petersen family, in time $f(\mathrm{tw}(G)) \cdot |G|$, for some computable function $f : \mathbb{N} \to \mathbb{N}$.*

Proof. Due to space restrictions we only prove the existence of an $f(\mathrm{tw}(G)) \cdot |G|^2$-algorithm here (which is enough for the application in Section 5). With some further effort this can be improved to linear time. Details of the linear time algorithm can be found in Section D in the appendix.

Given G, we first test in linear time if any graph in the Petersen family is a minor of G. If no such minor is found, we compute a flat embeding of G. The lemmas outlined in Section 2.2 suggest the following simple algorithm for constructing a flat embedding of a 4-connected graph in $\mathbb{R}^3$.

1. If G is planar, compute the unique embedding of G into $\mathbb{R}^3$. Otherwise, choose a Kuratowski subgraph of G and embed it flat into $\mathbb{R}^3$.
2. While there is a Kuratowski subgraph K of G that is not yet embedded, compute two adjacent Kuratowski subgraphs H, H' so that H is already embedded but at least one edge of H' is not yet embedded. By Lemma 2.6, there is a unique way of extending the embedding of H to an embedding of $H \cup H'$ which can easily be computed.
3. Once all Kuratowski subgraphs are embedded, the rest of the graph is planar and, by Lemma 3, there is essentially unique extension to an embedding of G.

We claim that this algorithm can be implemented to run in time $f(\text{tw}(G)) \cdot |G|^2$, where f is a computable function $f : \mathbb{N} \to \mathbb{N}$. Given a 4-connected graph G, first use Bodlaender's algorithm [3] to compute an optimal tree-decomposition of G in time $g(\text{tw}(G)) \cdot |G|$, where $g : \mathbb{N} \to \mathbb{N}$ is some computable function. Using a planarity algorithm [4, 13, 43], we can find the Kuratowski subgraph K of G required in step 1 in linear time or conclude that G is planar, in which case we can easily compute the unique embedding into $\mathbb{R}^3$. Towards the second step, note that the while loop can take at most $||G||$ iterations as each iteration embeds at least one new vertex or edge. In each iteration we have to find the two Kuratowski subgraphs H and H'. This can easily be done in linear time, either by dynamic programming or by realizing that the condition "H, H' are Kuratowski subgraphs of which H is already embedded and H' is not" has a straightforward definition in monadic second-order logic (MSO). It follows from a result by Arnborg, Lagergren and Seese [1] that given the MSO-definition, the graphs H, H' can be computed in time $h(\text{tw}(G)) \cdot ||G||$. As for all graphs $||G|| \leq \text{tw}(G) \cdot |G|$, the result follows. □

We are now ready to present the complete linear time algorithm for computing flat embeddings of linklessly embeddable graphs of bounded tree-width.

Lemma 4.2. *There is an algorithm which, on input G, solves the* Flat and Linkless Embedding *problem for G in time $f(\text{tw}(G)) \cdot |G|$, for some computable function $f : \mathbb{N} \to \mathbb{N}$.*

Due to space restrictions, we only sketch the proof here. A complete proof appears in the appendix. Let G be given. Using Bodlaender's algorithm [3], we first compute a tree-decomposition of G of width $\text{tw}(G)$ in time $g(\text{tw}(G)) \cdot |G|$. The next step is to test if any graph of the Petersen family is a minor of G. This can be done in time $g'(\text{tw}(G)) \cdot |G|$. If G contains a Petersen family minor, we can conclude that G has no linkless embeddings and return the minor. Otherwise, we split the graph into its 4-connected components using dynamic programming over the tree-decomposition. After some preprocessing in which we use $Y\Delta$-transformations to ensure that 3-separations are joined at triangles, we then use Lemma 4.1 to find flat embeddings of 4-connected components. The individual embeddings can then be glued together along the disks bounding 3-separations.

5 Algorithm

Finally, we are ready to present the complete algorithm.

Flat and Linkless Embedding

Input: A graph G.

Output: Either detect one of Petersen's family graphs as a minor in G or return a flat (and hence also linkless) embedding of G in $\mathbb{R}^3$.

Running time: $O(n^2)$.

Description: Initially, we delete all vertices of degree at most 1. Also, if there is a vertex v of degree 2, then we just contract vu, where u is one of the two neighbors of v. Thus we may assume that minimum degree is at least 3.

Step 1. If there is a vertex v of degree 3, then we perform $Y\Delta$ operation at v. We repeat doing this as long as there are some vertices of degree 3. Hereafter, we may assume that the minimum degree of the resulting graph G is at least 4.

Step 2. Test if the tree-width of the current graph G is small or not, say smaller than some value f, where f comes from Theorem 3.5. This can be done in linear time by the algorithm of Bodlaender [3]. If the tree-width is at least f, then go to Step 3. Otherwise we use the algorithm described in Lemma 4.2 to compute a flat embedding in linear time or certify that no such embedding exist by computing a minor of G in the Petersen family.

Step 3. It is easy to see that if the current graph G has at least $2^6|V(G)|$ edges, then one can easily find a K_6-minor in linear time (see [29]). So we may assume that the current graph G has at most $2^6|V(G)|$ edges.

At this moment, the tree-width of the current graph G is at least f. Use the algorithm of Bodlaender [3] (or the algorithm of Robertson and Seymour [32]) to construct a wall W of height at least $w'(l)$, where $w'(l)$ is as in Theorem 3.4. Perform the algorithm of Theorem 3.6 to find a separation or reduction as in the third conclusion of Theorem 3.6, or an irrelevant vertex v, or a minor of a graph in the Petersen family. If the third outcome occurs, then output the minor. If the second one happens, then we recurse this algorithm to $G - v$. If the first one happens, then we reduce the size of the the current graph G as in the proof of Theorem 3.6. This completes the description of the algorithm. $\square$

The correctness of Steps 2 and 3 follow from Sections 3 and 4. It is easy to see that degree 1 vertices can be deleted and degree 2 vertices can be contracted. Thus the correctness of Step 1 follows.

Now we shall estimate time complexity of the algorithm. All steps except in the above algorithm can be done in linear time. Another factor of n pops up because of applying the recursion. Thus the time complexity is $O(n^2)$.

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Appendix

A Figures explaining introduced concepts

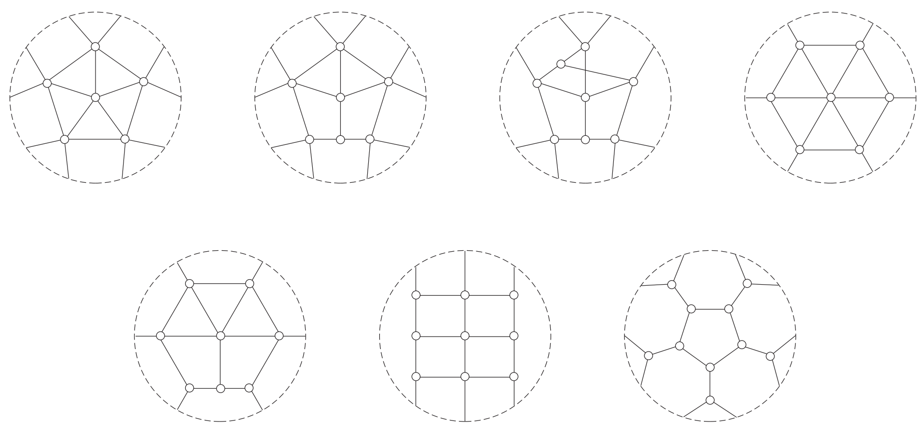

Figure 1: Petersen's family

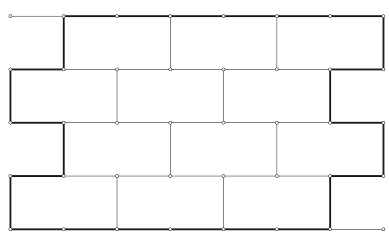

Figure 2: The (8×5)-wall and its outer cycle

B Flatly and Linklessly Embeddable Graphs

In this section we recall some results about linklessly embeddable graphs used in the paper. To provide a coherent exposition, we repeat some lemmas and definitins already contained in Section 2.

B.1 Constructions for Flat Embeddings

Recall that a piecewise-linear embedding of a graph in 3-space $\mathbb{R}^3$ is *flat* if every cycle of the graph bounds a disk disjoint from the rest of the graph.

If C, C' are disjoint simple closed curves in 3-space $\mathbb{R}^3$, then their *linking number* is the number of times that C crosses C' in a regular projection of

$C \cup C'$. Hereafter, all linking numbers discussed in this paper will be mod 2. The proof of the following result is easy and the details are left to the reader.

Lemma B.1. *Let G be a graph, and σ be an embedding of G in 3-space $\mathbb{R}^3$. Suppose the linking number of two disjoint cycles C_1, C_2 in G is nonzero. Let P be a path with both endvertices in $V(C_1)$ such that C_2 and P are disjoint, and $|V(C_1) \cap V(P)| = 2$. Then there are two cycles C_1', C_1'' in $V(C_1 \cup P)$ with $V(C_1') \cap V(C_1'') = V(P)$ such that one of the linking numbers of C_1', C_2 and C_1'', C_2 is nonzero.*

Let v a vertex of degree 3 in a graph G, with its three neighbors v_1, v_2, v_3. Let H be a graph obtained from $G - v$ by adding three new edges with ends in v_1v_2, v_2v_3, v_3v_1. We say that H is obtained from G by a $Y\Delta$ change (at v), and G is obtained from H by a ΔY change (at v_1, v_2, v_3). If G' can be obtained from G by a series of $Y\Delta$ or ΔY changes, we say that G and G' are *$Y\Delta$-equivalent*. The Petersen family is the set of the seven graphs (up to isomorphism) that are $Y\Delta$ equivalent to K_6. One of these is the Petersen graph. The following result was proved in [38].

Theorem B.2. *Let G be a graph obtained from H by a ΔY operation. Then G has a flat embedding in $\mathbb{R}^3$ if and only if H has.*

It is easy to see that degree 1 vertices can be deleted and degree 2 vertices can be suppressed without affecting linkless embeddability. Thus it follows from Theorem B.2 that we may assume that each vertex has degree at least four. Furthermore, concerning vertices of degree four, one can say the following.

Lemma B.3. *Suppose that a graph G contains an edge uv, where $\deg_G(u) = 4$ and $\deg_G(v) = 5$. Suppose, moreover, that $N(u) = (N(v) \cup \{v\}) - \{u, a\}$ for some vertex a in $N(v)$. Then there is a separation (A, B) of order four such that $B - A$ consists of u and v only, and G has a flat embedding in $\mathbb{R}^3$ if and only if for every vertex $b \in N(v) - \{u, a\}$, the graph G' obtained from G by contracting the edge ub has a flat embedding.*

We sketch the proof of Lemma B.3, which is essentially given in (6.6) in [38]. If G' has a flat embedding in 3-space $\mathbb{R}^3$, then $A \cap B$ is contained in a disk. In fact, as in the proof of (6.6) in [38], if A is planar with $A \cap B$ appearing on the outer face boundary of A, then the flat embedding of G' extends to A. Thus $A \cap B$ is contained in a single disk, and we can easily put the vertices u, v back to this disk so that the resulting embedding of G (that extends the flat embedding of G') is still flat.

We now prove the following result, which is important for our application. A proof is given in (6.5) in [38], but for the completeness, we shall give a sketch of the proof. Recall from above the definition of the graphs A^+, B^+ for separations (A, B) of a graph G.

Theorem B.4. *Suppose G has a separation (G_1, G_2) with $|V(G_1) \cap V(G_2)| \leq 4$. Suppose furthermore, G contains both G_1^+ and G_2^+ as a minor. Then G has a flat embedding in 3-space $\mathbb{R}^3$ if and only if both G_1^+ and G_2^+ have one.*

Sketch of the proof. Clearly if one of G_1^+ and G_2^+ does not have a flat embedding in 3-space $\mathbb{R}^3$, then the original graph G does not, because both G_1^+ and G_2^+ are minors of G. Suppose both G_1^+ and G_2^+ have flat embeddings σ_1, σ_2 in 3-space $\mathbb{R}^3$, respectively. Let W_i be the graph of G_i^+ in $V(G_1) \cap V(G_2)$ for $i = 1, 2$. Since any cycle in the complete graphs W_i bounds a disk in the embedding σ_i for $i = 1, 2$, we can glue the embeddings of σ_1, σ_2 together at the disks bounded by the cycles of the complete graph W_i, for $i = 1, 2$, so that any cycle in A and any cycle in B are not linked. We claim that every cycle bounds a disk in the resulting embedding of the whole graph G. Suppose there are two cycles C_1, C_2 in G such that the linking number of C_1, C_2 is nonzero. By our construction of G_1^+, G_2^+, $V(C_i) \cap V(A - B) \neq \emptyset$ and $V(C_i) \cap V(B - A) \neq \emptyset$ for $i = 1, 2$. Thus both the cycle C_1 and C_2 have chords e_1, e_2 in W_1 (or W_2). In particular, this implies $k = 4$. Let the cycles C_i' in $C_1 \cup e$ be such that C_1' is contained in A and C_2' is contained in B for $i = 1, 2$. Similarly, let the cycles C_i'' in $C_2 \cup e$ be such that C_1'' is contained in A and C_2'' is contained in B for $i = 1, 2$. By our construction, the linking number of C_i' and C_i'' is zero for $i = 1, 2$. By Lemma B.1, this implies that one of the linking number of C_1', C_2'' and C_1'', C_2' is nonzero, which is a contradiction. Thus the linking number of any two disjoint cycles in the resulting embedding is zero. Since any cycle in A and any cycle in B are not linked, thus every cycle bounds a closed disk that is disjoint from the rest of the embedding of G. This completes the proof of Theorem B.4. □

B.2 Embeddings of 4-connected graphs

We first need a classic result by Kuratowski.

Lemma B.5 (Kuratowski, [20]). *A graph is planar if and only if it has no subgraph isomorphic to a subdivision of K_5 or $K_{3,3}$.*

Here, K_5 is the complete graph on 5 vertices and $K_{3,3}$ is the complete bipartite graph with 3 vertices per partition. We refer to the graphs K_5 and $K_{3,3}$ as the *Kuratowski graphs*. A *Kuratowski subgraph* of a graph G is a subgraph of G isomorphic to a subdivision of a Kuratowski graph.

An embedding ϕ of a graph in $\mathbb{R}^3$ is *spherical*, if there exists a surface $\mathcal{S} \subseteq \mathbb{R}^3$ homeomorphic to the plane such that $\phi(G) \subseteq \mathcal{S}$.

Lemma B.6 (Wu, [44]). *Let σ be an embedding of a planar graph in 3-space $\mathbb{R}^3$. Then σ is flat if and only if it is spherical.*

The following lemmas were proved in [38] along with its companion papers [34, 35]. Readers not familiar with these results may wish to consider the survey [39] which contains many of the results needed below. Let ϕ_1, ϕ_2 be embeddings of a graph in $\mathbb{R}^3$. ϕ_1 and ϕ_2 are *ambient isotopic*, written as $\phi_1 \cong_{a.i.} \phi_2$, if there exists an orientation preserving homeomorphism from $\mathbb{R}^3$ to $\mathbb{R}^3$ mapping ϕ_1 onto ϕ_2. We will generaly not distinguish between two ambient isotopic embeddings and consider them to be equivalent.

An important fact we will be using below is that 4-connected linklessly embeddable graphs have (essentially) a unique flat embedding into 3-space (up to ambient isotopy). Precisely, if ϕ_1, ϕ_2 are flat embeddings of a 4-connected graph in $\mathbb{R}^3$, then either $\phi_1 \cong_{a.i.} \phi_2$ or $\phi_1 \cong_{a.i.} -\phi_2$, where $-\phi_2$ is the composition of ϕ_2 and the antipodal map. This is a consequence of the following lemmas.

Lemma B.7. *Any two flat embedings of a planar graph are ambient isotopic.*

In fact, a graph has a unique flat embedding if and only if it is planar. The following is (7.7) from [38] (see also (3.4) in [39]).

Lemma B.8. *The graphs K_5 and $K_{3,3}$ have exactly two non-ambient isotopic flat embeddings.*

Sketch of proof. Let G be a Kuratowski graph and let $e \in E(G)$. Then $G - e$ is planar and, by Lemma B.7, has a unique flat embedding ϕ into $\mathbb{R}^3$. In fact, by Lemma B.6, $\phi(G) \subseteq \mathcal{S}$ for a surface $\mathcal{S} \subseteq \mathbb{R}^3$ homeomorphic to the plane. Essentially, we have two choices where to draw the edge e, namely the two components of $\mathbb{R}^3 - \mathcal{S}$, as any two flat embeddings of the edge in the same component yield ambient isotopic embeddings. □

The next lemma is (3.5) in [39]. If ϕ is an embedding of a graph G and $H \subseteq G$, we write $\phi_{|H}$ for the restriction of ϕ to H.

Lemma B.9. *Let ϕ_1, ϕ_2 be two flat embeddings of a graph G which are not ambient isotopic. Then there is a subgraph H of G isomorphic to a subdivision of a Kuratowski graph for which $\phi_{1|H}$ and $\phi_{2|H}$ are not ambient isotopic.*

We now define a neighbourhood relation between Kuratowski subgraphs of a graph G. Let $H_1, H_2 \subseteq G$ be Kuratowski subgraphs of G such that $H_1 \neq H_2$. H_1 and H_2 are 1-*adjacent* if there exists a path $P \subseteq G$ and an $i \in \{1, 2\}$ such that P has only its endpoint in common with H_i and such that $H_{3-i} \subseteq H_i \cup P$.

H_1 and H_2 are 2-*adjacent* if there are distinct vertices $v_1, \ldots, v_7 \in V(G)$ an pairwise internally vertex disjoint paths $L_{i,j}$, for $(i, j) \in \{1, \ldots, 4\} \times \{5, 6, 7\} \cup \{(3, 4)\}$ linking v_i and v_j such that $H_1 = \bigcup \{L_{i,j} \mid (i, j) \in \{2, 3, 4\} \times \{5, 6, 7\}\}$ and $H_2 = \bigcup \{L_{i,j} \mid (i, j) \in \{1, 3, 4\} \times \{5, 6, 7\}\}$. The path $L_{3,4}$ is not used here but is required to exist. Note that if H_1, H_2 are 2-adjacent then they are both isomorphic to subdivisions of $K_{3,3}$. H_1 and H_2 are *adjacent* if they are 1- or 2-adjacent.

Lemma B.10. *Let ϕ_1, ϕ_2 be flat embeddings of G and let H, H' be adjacent Kuratowski subgraphs of G. If $\phi_{1|H} \cong_{a.i.} \phi_{2|H}$ then $\phi_{1|H'} \cong_{a.i.} \phi_{2|H'}$.*

For the purpose of our paper this means that once we have fixed an embedding for the Kuratowski subgraph $H \subseteq G$, there is only one way to extend this embedding to the adjacent subgraph H' (up to ambient isotopy). The following lemma, which is key to our algorithm, says that if G is 4-connected, then by starting from one Kuratowski subgraph whose embedding we fix and iteratively proceeding to adjacent Kuratowski subgraphs, we can embed all Kuratowski subgraphs of G.

Let H, H' be Kuratowski subgraphs of G. We say that H and H' *communicate* if there are Kuratowski subgraphs $H = H_1, \ldots, H_k = H'$ of G such that for all $1 \leq i < k$, H_i and H_{i+1} are adjacent. The following is proved in [34].

Lemma B.11. *If G is 4-connected, then all pairs H, H' of Kuratowski subgraphs of G communicate.*

In fact, the stronger statement holds that Kuratowski subgraphs communicate for all graphs which are *Kuratowski 4-connected*, which essentially says that no two Kuratowski subgraphs can be separated by a separation of order 3. For our purposes it suffices to note that any 4-connected graph is also Kuratowski 4-connected. The previous lemmas imply the following result about uniqueness of embeddings for 4-connected graphs.

Lemma B.12. *If ϕ, ϕ' are flat embeddings of a 4-connected graph G, then $\phi \cong_{a.i.} \phi'$ or $\phi \cong_{a.i.} -\phi'$.*

Proof. If G is planar then by Lemma B.7 there is only one flat embedding (up to ambient isotopy). Otherwise, G contains a Kuratowski subgraph K. By Lemma B.8 either $\phi_{|K} \cong_{a.i.} \phi'_{|K}$ or $\phi_{|K} \cong_{a.i.} -\phi'_{|K}$. By going from ϕ' to $-\phi'$ if necessary, we can assume that $\phi_{|K} \cong_{a.i.} \phi'_{|K}$. By Lemma B.10 and B.11 $\phi_{|H} \cong_{a.i.} \phi'_{|H}$ for all Kuratowski suubgraphs $H \subseteq G$. Hence, by Lemma B.9, $\phi \cong_{a.i.} \phi'$. $\square$

B.3 Some useful graph theoretical lemmas

Finally, we give purely graph theoretic results in [36], which are necessary for our proof.

Theorem B.13. *Let G be a connected graph and v_1, v_2, v_3 be three vertices in G. Suppose G has no separation (A, B) of order at most two such that B contains all the vertices of v_1, v_2, v_3 and $A - B$ is not empty. Then either*

1. *G has three disjoint trees T_i such that T_i contains v_i for $i = 1, 2, 3$, and for any two $i < j$, there is an edge between T_i and T_j, or*

2. *$A - B$ consists of a single vertex v, and $N(v) = \{v_1, v_2, v_3\}$. Moreover, $v_1.v_2, v_3$ are independent.*

Theorem B.14. *Let G be a connected graph and v_1, v_2, v_3, v_4 be four vertices in G. Suppose G has no separation (A, B) of order at most three such that B contains all the vertices of v_1, v_2, v_3, v_4 and $A - B$ is not empty. Then either*

1. *G has four disjoint trees T_i such that T_i contains v_i for $i = 1, 2, 3, 4$, and for any two $i < j$, there is an edge between T_i and T_j, or*

2. *$A - B$ consists of a single vertex v, and $N(v) = \{v_1, v_2, v_3, v_4\}$, or*

3. *$A - B$ consists of two vertices v, v' with $vv' \in E(G)$ such that both v and v' have degree at least four.*

4. *G can be drawn in a plane such that all of v_1, v_2, v_3, v_4 are in the outer face boundary.*

Theorem B.13, together with Theorem B.4, implies the following:

Theorem B.15. *Suppose G has minimum degree 4. If G does not have a separation (G_1, G_2) with $|V(G_1) \cap V(G_2)| \leq 3$ such that G contains both G_1^+ and G_2^+ as a minor, then G is 4-connected.*

Proof. Suppose G has a separation (G_1, G_2) with $|V(G_1) \cap V(G_2)| \leq 3$. We take such a separation with minimum order. If $|V(G_1) \cap V(G_2)| \leq 2$, then we can easily find both G_1^+ and G_2^+ as a minor. Suppose $|V(G_1) \cap V(G_2)| = 3$. By Theorem B.13, if G does not contain G_1^+ as a minor, then $G_2 - G_1$ consists of a single vertex, which is impossible, because minimum degree of G is at least 4. Similarly, G contains G_2^+ as a minor. Thus G is 4-connected. □

We use the idea of a "society" introduced by Robertson and Seymour in [31]. Let Ω be a cyclic permutation of the elements of some set; we denote this set by $V(\Omega)$. A *society* is a pair (G, Ω), where G is a graph and Ω is a cyclic permutation with $V(\Omega) \subseteq V(G)$. A society (G, Ω) is *rural* if G can be drawn in a disk with $V(\Omega)$ drawn on the boundary of the disk in the order given by Ω. A *cross* in a society (G, Ω) is a pair of disjoint paths P_1 and P_2 such that each P_i has its endpoints in $V(\Omega)$ and no other vertex in $V(\Omega)$, and further if we let s_i and t_i be the ends of P_i for $i = 1, 2$ then $s_1 \leq s_2 \leq t_1 \leq t_2$ in the order Ω. Clearly, a rural society has no cross; a theorem of Robertson and Seymour says that this is essentially the only obstruction. First we give a definition. A society (G, Ω) is *rurally 4-connected* if for every separation (A, B) of order at most three with $V(\Omega) \subseteq A$, the graph B can be drawn in a disk with the vertices of $A \cap B$ drawn on the border of the disk. The following follows from [31].

Theorem B.16. *Every cross-free rurally 4-connected society is rural.*

B.4 Definition of planarly drawn: up to 3-separations

We now mention the characterization of planarly drawn. We need to define an embedding *up to 3-separations*. We say that *G can be embedded into a plane, up to 3-separations*, if, for some $k \geq 0$, there are pairwise disjoint sets $A_1, \dots, A_k \subseteq V(G)$ such that

(1) for $1 \leq i, j \leq k$ with $i \neq j$, $N(A_i) \cap A_j = \emptyset$,

(2) for $1 \leq i \leq k$, $|N(A_i)| \leq 3$, and

(3) if G' is the graph obtained from G by (for each i) deleting A_i and adding new edges joining every pair of distinct vertices in $N(A_i)$, then G' may be drawn in a plane.

Let us observe the following. Let $\mathbf{A} = \{A_1, \dots, A_k\}$. Then we can choose $\mathbf{A}$ so that, subject to (1), (2) and (3), the following property holds:

If $N(A_i) = 3$, then $N(A_i)$ induces a facial triangle.

To see this, we may choose $\mathbf{A}$ such that, subject to (1), (2) and (3), the number of non-facial triangles in G' induced by members of $\mathbf{A}$ is minimum. Suppose, without loss of generality, that $|N(A_1)| = 3$ and $N(A_1)$ induces a triangle T_1 in G', which is not facial. Let $D_1 \subseteq V(G')$ be such that, for each $x \in V(G)$, $x \in D_1$ if and only if x is contained in the closed disk bounded by T_1. Define $A'_1 \subseteq V(G)$ such that, for each $x \in V(G)$, $x \in A'_1$ if and only if $x \in D_1 - N(A_1)$ or $x \in A_j$ for some A_j with $N(A_j) \subseteq D_1$. Let $\mathbf{A}' = (\mathbf{A} - \{A_j : N(A_j) \subseteq D_1\}) \cup \{A'_1\}$. Then (1), (2) and (3) hold for $\mathbf{A}'$, but the number of non-facial triangles in G' is smaller, a contradiction.

Let C be the outer face boundary of the planar graph G'. We may also assume that

(4) If $|N(A_i)| = j$, then there is no separation (A, B) of order at most $j - 1$ in G' such that A contains $N(A_i)$ and B contains all the vertices of C.

For otherwise, we can replace A_i by $A \cup A_i$.

C Proofs omitted from Section 3

Theorem 3.5. *There is a computable constant f satisfying the following: Let G be a given input graph with minimum degree 4, and $W, w(l), w'(l)$ be as in Theorem 3.4 with $l \geq f$. Then one of the following holds:*

1. *G has a K_6-minor, or*

2. *for some subset X of at most one vertex of G, there is a proper subwall W' (of the wall W) of height l, which is disjoint from X and dividing in $G-X$, and has a 2-cell embedding with $per(W')$ in the outer face boundary in $G - X$. Moreover, the unique vertex v which has distance exactly $l/2$ from the $per(W')$ in the wall W' is irrelevant with respect to a K-minor, where K is any graph in the Petersen family. Furthermore, no matter how we give a flat embedding of the graph $G - v$ in 3-space $\mathbb{R}^3$, the embedding of $G - v$ can be changed so that, after putting the vertex v back to the resulting embedding, the resulting embedding of G is flat, or*

3. *either there is a reduction as in Lemma 2.3, or there is a separation (A, B) of order at most three in $G - X$ such that $V(A) \cap V(B)$ only involves the vertices in the $comp(W)$ in $G - X$, and B contains all the vertices of the $per(W')$. Moreover, G contains both $(A \cup X)^+$ and $(B \cup X)^+$ as minors,*

and hence this separation $(A \cup X, B \cup X)$ is a reduction as in Theorem 2.4. Note that it is a separation of order at most 4 in G.

Proof. By Theorem 3.4, if a given graph G does not contain a K_6-minor, then there is a wall W of height $w'(l)$, and for some subset X of at most one vertex of G, there is a proper subwall W' of height l, which is disjoint from X and is planarly drawn and dividing in $G - X$.

Suppose first that the *comp*(W') does not have a 2-cell embedding in a plane with *per*(W') in the outer face boundary. Let us observe that the pair (*comp*(W'), *per*(W')) is a society in $G - X$, because we can impose a cyclic permutation of the *per*(W'). Since the *comp*(W') is planarly drawn and dividing in $G - X$, thus the society (*comp*(W'), *per*(W')) is rurally 4-connected by Theorem B.16. We may assume that there is a separation (A, B) of order at most three in the society (*comp*(W'), *per*(W')) such that B contains all the vertices of the *per*(W'), and in addition, A does not have a 2-cell embedding in a plane with $V(A \cap B)$ in the outer face boundary. For otherwise, the *comp*(W') has a 2-cell embedding in a plane in $G - X$. B has all but at most one nail of the subwall W'. By Theorem B.15 and since minimum degree of G is at least 4, this implies that G is 4-connected, for otherwise, we could get the third conclusion. If $|A \cap B| \leq 2$, then $(A \cup X, B \cup X)$ would be a separation of order at most three in the original graph G. Furthermore, if $X \not\subset N(A)$ or $X = \emptyset$, again, (A, B) would be a separation of order at most three in the original graph G. Thus we may assume that $|A \cap B| = 3$, and $X \subset N(A)$ and $|X| = 1$. By the 4-connectivity of G and the rurally 4-connectivity of the society (*comp*(W'), *per*(W')), we can clearly find the graph $(A \cup X)^+$ as a minor in G. If $A \cup X$ can be drawn in a plane in such a way that $V(A \cap B) \cup X$ appears in the outer face boundary, clearly A has a 2-cell embedding in a plane with $V(A \cap B)$ in the outer face boundary, and hence we can embed $A - B$ in the plane without creating any edge crossings. So this would not happen by our assumption.

Thus since G is 4-connected, if we cannot find the graph B^+, then by Theorem B.14, $A - B$ consists of exactly two vertices v, v'. If degree of v is 5 and degree of v' is 4 in the original graph G, then we can reduce to Lemma 2.3. Thus $vv' \in E(G)$ and $(V(A) \cap V(B)) \cup X \subseteq N(v)$, $(V(A) \cap V(B)) \cup X \subseteq N(v')$. On the other hand, then $(A \cup X)^+$ gives rise to a K_6-minor. Thus we may assume that the *comp*(W') in $G - X$ has a 2-cell embedding in a plane with *per*(W') in the outer face boundary. The vertex v in the second conclusion of Theorem 3.5 is irrelevant with respect to *any* minor of the graphs in the Petersen family, as proved in [32].

We need to prove that no matter how we give a flat embedding of the graph $G - v$ in 3-space $\mathbb{R}^3$, the embedding of $G - v$ can be changed so that, after putting the vertex v back to the resulting embedding, the resulting embedding of G is flat. If $X = \emptyset$, then it follows from Lemma B.6 that the planar graph *comp*$(W) - v$ has a unique spherical embedding. Thus the unique face F that has all the neighbors of v bounds a disk D in the flat embedding of $G - v$. Thus we can put the vertex v back to the disk D so that the resulting embedding of G is still flat. It is easy to see that each cycle in the resulting embedding of G

bounds a disk.

Finally suppose $|X| = 1$. Again, it follows from Lemma B.6 that the planar graph $comp(W) - X - v$ has a unique spherical embedding in a flat embedding of $G - v$. The unique face F in $comp(W) - X - v$ that has all the neighbors of v bounds a disk D in the flat embedding of $G - v$. Thus we can put the vertex v back to the disk D so that the resulting embedding of G is still flat. It is easy to see that each cycle in the resulting embedding of G bounds a disk. $\square$

Theorem 3.6. *Let $G, W, f, l, w(l), w'(l)$ be as in Theorem 3.5. Suppose the wall W is given. Then there is an $O(n)$ time algorithm, where n is the number of vertices of G, to construct one of the following:*

1. *one of the graphs in the Petersen family in G as a minor, or*

2. *for some subset X of at most one vertex of G there is a proper subwall W' (of the wall W) of height l, which is disjoint from X, dividing in $G-X$, and has a 2-cell embedding with $per(W')$ in the outer face boundary. Moreover, we can find an irrelevant vertex as in the second conclusion of Theorem 3.5, or*

3. *either there is a reduction as in Lemma 2.3, or there is a separation (A, B) of order at most three in $G - X$ such that $V(A) \cap V(B)$ only involves the vertices in the $comp(W)$ in $G - X$, and B contains all the vertices of the $per(W')$. Moreover, G contains both $(A \cup X)^+$ and $(B \cup X)^+$ as minors, and hence this separation $(A \cup X, B \cup X)$ is a reduction as in Theorem 2.4. Note that it is a separation of order at most 4 in G.*

Proof. We can get one of the conclusions in Theorem 3.4 in linear time. If the first outcome happens, then we are done. Suppose the second happens. At the moment, the compass $comp(W')$ in $G - X$ is dividing for any proper subwall W' (of the wall W) as in the second conclusion of Theorem 3.4, but it may not be planarly drawn.

By applying the planarity algorithm [4, 13, 43], for any proper subwall W' in the second conclusion of Theorem 3.4, we can test whether or not the compass $comp(W')$ is planar in $G - X$ with $per(W')$ in the outer face boundary. Suppose there is a Kuratowski subgraph, i.e, either a K_5-subdivision or a $K_{3,3}$-subdivision. Let us observe that the pair $(comp(W'), per(W'))$ is a society in $G - X$, because we can impose a cyclic permutation of the $per(W')$. If there is a separation (A, B) of order at most three in the society $(comp(W'), per(W'))$ in $G - X$ such that B contains all the vertices of the $per(W')$ and at least two nodes of the Kuratowski subgraph K are contained in $A - B$, then, as in the proof of Theorem 3.5, we can easily get the third conclusion of Theorem 3.6, since $(A \cup X, B \cup X)$ is the separation as discussed in the proof of Theorem 3.5. Let us observe that A does not have a 2-cell embedding in a plane with $V(A \cap B)$ in the outer face boundary. Suppose such a separation does not exist. This implies that the society $(comp(W'), per(W'))$ in $G - X$ is not rurally 4-connected. So by Theorem B.16, the compass $comp(W')$ is not planarly drawn, because we can find two vertex disjoint paths connecting the diagonally opposite corners in the wall W'.

Let us observe that, for each proper subwall W' in the second conclusion of Theorem 3.4, once we are given a Kuratowski graph K, we can test whether or not such a separation (A, B) exist in the *comp*(W') in $G - X$, in linear time. If such a separation does not exist, by using the standard technique in [32], we can find two vertex disjoint paths connecting the diagonally opposite corners in the wall W' in linear time, using the Kuratowski graph K.

By Theorem 3.4, if $|X| = 1$ and there is a proper subwall W' that is not planarly drawn, then it has a K_6-minor (and such a K_6-minor can be found in linear time using this Kuratowski graph). While if $|X| = 0$ and there are three disjoint proper subwalls that are not planarly drawn, then there is a K_6-minor by Theorem 3.4 (and such a K_6-minor can be found in linear time using the three Kuratowski graphs in these three proper subwalls).

Thus we may assume that there is a proper subwall W' in the second conclusion of Theorem 3.4 such that the *comp*(W') in $G - X$ has a 2-cell embedding with *per*(W') in the outer face boundary (and such a 2-cell embedding can be found in linear time by the planarity algorithm [4, 13, 43]). In this case, by Theorem 3.6, we can find the vertex in the second conclusion of Theorem 3.5 in linear time. □

D Proofs omitted from Section 4

D.1 A linear-time algorithm

The aim of this section is to prove Lemma 4.1.

Lemma 4.1. *There is an algorithm which, given a 4-connected graph $G \in \mathcal{C}$, either returns a flat embedding of G or produces a minor $H \preceq G$ in the Petersen family, in time $f(\mathrm{tw}(G)) \cdot |G|$, for certain computable function $f : \mathbb{N} \to \mathbb{N}$.*

The key observation is that, on graph classes of bounded tree-width, we can list in linear time a sequence $K_1, \ldots, K_l$, where $l \leq c \cdot |G|$ and c is a constant depending only on the bound on the tree-width, of Kuratowski subgraphs of G such that $K \subseteq \bigcup_{i=1}^{l} K_i$ for every Kuratowski subgraph $K \subseteq G$. We can then inductively compute the (unique) flat embedding of $K_1 \cup \cdots \cup K_l$ which has a unique extension to a flat embedding of G, if such an embedding exists.

Let $\mathcal{T} := (T, B)$ be a tree-decomposition of a graph G of tree-width $k := \mathrm{tw}(G)$. W.l.o.g. we assume that the tree T is sub-cubic, i.e. of maximum degree at most 3. By fixing a root of T we impose an orientation on T and use usual terminology, such as children of nodes, etc. To distinguish between vertices of T and vertices of G we will refer to the former as *nodes* and to the latter as *vertices*.

Let P be a path in G with endvertices u, v, oriented from u to v. The *trace* of P on a node $t \in V(T)$ is defined as the sequence $(w; u_1, \ldots u_r; w')$ defined as follows. Let $(u_1, \dots, u_l)$ be the vertices in $V(P) \cap B_t$, listed in the order as they appear on P.

- If $u, v \notin B(T_t)$ we let $w = w' :=\uparrow$ and define the trace of P as $(\uparrow; u_1, \dots, u_l; \uparrow)$.
- If $u \in B(T_t)$ but $v \notin B(T_t)$ we define the trace as $(u_1; u_2, \dots, u_l; \uparrow)$ and, conversely, if $u \notin B(T_t)$ but $v \in B(T_t)$ we define the trace of P as $(\uparrow; u_1, \dots, u_{l-1}; u_l)$.
- Finally, if $u, v \in B(T_t)$ we define the trace of P as $(u_1; u_2, \dots, u_{l-1}; u_l)$.

We write $\mathit{trace}(P, t)$ for the trace of P on t.

Let $t \in V(T)$ be a node and $K \subseteq G$ be a Kuratowski subgraph of G isomorphic to a subdivision of K_5. Let $v_1, \dots, v_5$ be the vertices of degree 4 in K and $P_{i,j}$ be the paths connecting v_i, v_j, for $1 \leq i < j \leq 5$. The *impact* of K on t, written as $\mathit{impact}(K, t)$, is defined as

- $\big(w_1, \dots, w_5; \big(\mathit{trace}(P_{i,j}, t) \mid 1 \leq i < j \leq 5\big)\big)$ if $V(K) \cap B_t \neq \emptyset$, where $w_i := v_i$ if $v_i \in B_t$, $w_i :=\downarrow$ if $v_i \in B(T_t) \setminus B_t$ and $w_i :=\uparrow$ otherwise, and
- $\emptyset$, if $V(K) \cap B_t = \emptyset$.

The impact of K on t captures all relevant information about the subgraph K with respect to the node t of the tree-decomposition. Similarly, if $K \subseteq G$ is isomorphic to a subdivision of $K_{3,3}$, with $V(K) := \{v_1, \dots, v_6\}$ and $P_{i,j}$ being the paths between v_i, v_j $(1 \leq i \leq 3, 4 \leq j \leq 6)$, we define $\mathit{impact}(K, t)$ as

- $\big(w_1, \dots, w_6; \big(\mathit{trace}(P_{i,j}, t) \mid 1 \leq i \leq 3, 4 \leq j \leq 6\big)\big)$ if $V(K) \cap B_t \neq \emptyset$, where $w_i := v_i$ if $v_i \in B_t$, $w_i :=\downarrow$ if $v_i \in B(T_t) \setminus B_t$ and $w_i :=\uparrow$ otherwise, and
- $\emptyset$, if $V(K) \cap B_t = \emptyset$.

By construction, the size of a *trace* is at most $k + 2$ and hence the size of an *impact* is at most $20 \cdot k + 46 = O(k)$. Furthermore, there are at most $O(k!)$ distinct impacts and hence for each node there are at most $2^{O(k!)}$ possible sets of impacts. We denote by $\mathcal{I}$ the set of possible impacts.

If K is a Kuratowski subgraph of G we write $\mathit{impact}(K, \mathcal{T})$ for the set $\{\mathit{impact}(K, t) \mid t \in V(T)\}$. Clearly, given $\mathit{impact}(K, \mathcal{T})$, we can reconstruct the graph K in time in $|G|$.

Let $I := \mathit{impact}(K, \mathcal{T})$ be the impact of a Kuratowski subgraph on G and $\mathcal{T}$. Obviously, the set I satisfies certain consistency criteria. For instance, if t is a node with children s_1, s_2 and $\mathit{impact}(K, t) := (\downarrow, w_2, \dots, w_5; (\mathit{trace}(P_{i,j}, t) : 1 \leq i < j \leq 5)\}$ is the impact of K on t, then exactly one of s_1, s_2 must have $\downarrow$ or $v \in B_{s_1}$ in its first component of $\mathit{impact}(K, s_1)$ and the other child must have $\uparrow$. Similarly, if the trace of a path contains the pair u_i, u_{i+1} saying that there must be a path between u_i and u_{i+1} in $B(T_t)$, then this fact must be reflected in the corresponding traces of the children. We refrain from listing all consistency criteria here and only note that, given potential impacts for t, s_1, s_2, it can be decided in constant time if they are consistent with each other, i.e. can together belong to the impact of a Kuratowski subgraph K on G.

Given a graph G and a tree-decomposition $\mathcal{T} := (T, (B_t))$ of G we define the *Kuratowski impact* of G (with respect to $\mathcal{T}$) as the set $\{impact(K, \mathcal{T}) \mid K$ a Kuratowski subgraph of $G\}$.

Lemma D.1. *There is an algorithm which, given a graph G computes a Kuratowski impact of G in time $f(k) \cdot |G|$, for some computable function $f : \mathbb{N} \to \mathbb{N}$.*

Proof. Using Bodlaender's algorithm [3], we first compute a tree-decomposition $\mathcal{T} := (T, B)$ of G in time $g(k) \cdot |G|$. W.l.o.g. we assume that T is a rooted tree of degree at most 3.

The Kuratowski impact of G and $\mathcal{T}$ can be computed by dynamic programming as follows.

- If $t \in V(T)$ is a leaf, we can compute all potential impacts consistent with the leaf in the obvious way.
- Now suppose t is an internal node with children t_1, t_2. We first compute all potential impacts for t. For each potential impact I for t we check whether s_1 and s_2 contain impacts consistent with I. If not, then I is eliminated from the set of impacts for t.

After this has been done for all nodes, we need a second sweep through the tree-decomposition, this time from the root to the leaves, to remove "dangling" impacts, i.e. potential impacts for a node that can not consistently be extended to its parent.

After the second step we are left with the Kuratowski impact for G and $\mathcal{T}$. It is easily seen that both steps can be performed in linear time. $\square$

Lemma D.2. *There is an algorithm which, given a graph G computes a linear order $I_1 < \cdots < I_l$, where $l < c \cdot \mathrm{tw}(G) \cdot |G|$ for some fixed constant c, in time $f(k) \cdot |G|$, for some computable function $f : \mathbb{N} \to \mathbb{N}$, so that*

- *each I_i is the impact of a Kuratowski subgraph $K_i \subseteq G$ and*
- *$K \subseteq \bigcup_{i=1}^{l} K_i$ for each Kuratowski subgraph $K \subseteq G$.*

Proof. We first use Lemma D.1 to compute the Kuratowski impact of G and a tree-decomposition $\mathcal{T}$. We can then use dynamic programming again to compute a small set of representative impacts satisfying the conditions of the lemma. Once this set is computed, we can impose an arbitrary order on it, for instance an ordering induced by the tree-ordering of the tree-decomposition.

We refrain from giving the technical details of this step. A linear size set of representative impacts can also be computed by formalising the property in monadic second-order logic and using the result proved in [1] that satisfying assignments for MSO formulas can be computed in linear time on graph classes of bounded tree-width. $\square$

We now have all the ingredients needed for a linear time algorith for computing flat embeddings of 4-connected graphs of small tree-width.

Lemma D.3. *Let $\mathcal{C}$ be a class of graphs of bounded tree-width. There is an algorithm which, given a 4-connected graph $G \in \mathcal{C}$, either computes a flat embedding of G or produces a minor $H \preceq G$ of the Petersen family, in time $f(\mathrm{tw}(G)) \cdot |G|$, for some computable function $f : \mathbb{N} \to \mathbb{N}$.*

Proof. Let $G \in \mathcal{C}$ be 4-connected. Using Bodlaender's algorithm [3] we first compute a tree-decomposition of G of width $\mathrm{tw}(G)$ in time $g(\mathrm{tw}(G)) \cdot |G|$. The next step is to test, again in linear time, if G contains any member of the Petersen family as a minor. If it does, we return the minor and thereby certify that G has no linkless embedding.

Otherwise, we use Lemma D.2 to compute a linear order $I_1 < \cdots < I_l$ representing the Kuratowski impact on G. Starting with I_1 and proceeding along the ordering, we embed all Kuratowski subgraphs of G. As explained in Section B.2, there is essentially only one way of embedding the Kuratowski subgraphs of G. Hence there is no backtracking involved and the procedure runs in linear time.

Once an embedding ϕ' of all Kuratowski subgraphs of G has been constructed, there is a unique extension ϕ of ϕ' to an embedding of the whole graph G. $\square$

Note that the test for a Petersen family minor is not exactly necessary. Instead we could modify the algorithm above to stop when embeddings for different Kuratowski subgraphs do not match and then compute a Petersen family minor. For simplicitly, we perform the test explicitly.

D.2 Flat embeddings in linear time on graph classes of bounded tree-width

In this section we prove Lemma 4.2.

Lemma 4.2. *There is an algorithm which, on input G, solves the* Flat and Linkless Embedding *problem for G in time $f(\mathrm{tw}(G)) \cdot |G|$, where $f : \mathbb{N} \to \mathbb{N}$ is certain computable function.*

Proof. Let G be given. Using Bodlaender's algorithm [3], we first compute a tree-decomposition of G of width $\mathrm{tw}(G)$ in time $g(\mathrm{tw}(G)) \cdot |G|$. The next step is to test if any graph of the Petersen family is a minor of G. Again, this can be done in time $g'(\mathrm{tw}(G)) \cdot |G|$. If G contains a Petersen minor, we can conclude that G has no linkless embeddings and return the minor. Otherwise, we perform a preprocessing step to break the graph into 4-connected components.

Step 1. We first compute the 2-connected components of G and perform the algorithm recursively on each component. Once flat embeddings of all 2-connected components have been computed, we can combine them to a flat embedding of G in an obvious way.

Step 2. Now suppose there are no 1-separations of G. The next step is to eliminate 2-separations. Using dynamic programming, we can easily find the 3-connected components of G in linear time. Let $S := \{u, v\}$ be a 2-separator in G. As there are no 1-separations in G, u and v are connected in every component C of $G - S$. Hence, if G is linklessly embeddable then so is $G[C \cup S] + uv$. For every component C of $G - S$, we define $G_C := G[C \cup S] + uv$, if $uv \notin E(G)$, and $G_C := G[C \cup S]$ otherwise and call the algorithm recursively on G_C to obtain a flat embedding of G_C. We can then combine the embeddings as follows. Pick a component C and the flat embedding of G_C. As uv is contained in a cycle C' of G_C there is a 2-cell whose boundary is C'. We can therefore merge the embeddings of the graphs $G_{C'}$ along the edge uv.

Step 3. Now suppose there are no 1- or 2-separations left in G. The final step is to decompose the graph into 4-connected components. Again, using dynamic programming we can find 4-connected components of G easily. Now let $S := \{u, v, w\}$ be a 3-separation of G. As there are no 2-separations in G we can assume that $\{u, v, w\}$ are connected in each component of $G - S$. Hence, if C is a component of $G - S$, then the graph G'_C obtained from $G[C \cup S]$ by adding a new vertex z adjacent to each vertex in S is a minor of G and therefore has a flat embedding. Employing a $Y - \Delta$ transformation if necessary, we can conclude that the graph $G_C := G[C \cup S] \cup \{uv, uw, vw\}$, obtained from G'_C by turning S into a triangle, can be obtained as a $Y - \Delta$ transform of a minor of G and hence has a flat embedding. We can now use Lemma 4.1 to obtain a flat embedding of each G_C. Analogous to the proof of Lemma 2.4, these can then be glued together along the triangles. □