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Strong embeddings of minimum genus

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ABSTRACT

A "folklore conjecture, probably due to Tutte" (as described in [P.D. Seymour, Sums of circuits, in: Graph Theory and Related Topics (Proc. Conf., Univ. Waterloo, 1977), Academic Press, 1979, pp. 341–355]) asserts that every bridgeless cubic graph can be embedded on a surface of its own genus in such a way that the face boundaries are cycles of the graph. Sporadic counterexamples to this conjecture have been known since the late 1970s. In this paper we consider closed 2-cell embeddings of graphs and show that certain (cubic) graphs (of any fixed genus) have closed 2-cell embedding only in surfaces whose genus is very large (proportional to the order of these graphs), thus providing a plethora of strong counterexamples to the above conjecture. The main result yielding such counterexamples may be of independent interest.

1. Introduction

In his seminal work *Sums of circuits* [12], Paul Seymour stated the following conjecture, which he has addressed as a "folklore conjecture, probably due to Tutte".

Conjecture 1. *Any bridgeless cubic graph can be embedded on a surface of its own genus in such a way that the perimeters of all regions are circuits.*

As much as this conjecture was folklore in the nineteen seventies, today's methods of topological graph theory enable rather easy constructions of counterexamples. There is evidence that several people were aware that Conjecture 1 is false for toroidal graphs. Apparently, an example of a toroidal cubic graph disproving Conjecture 1 appears in the Ph.D. Thesis of Xuong [13]. Richter [10] found further examples of 2-connected (but not 3-connected) cubic graphs of genus one for which the conjecture fails. Zha [14] constructed graphs G_g, for each integer $g \geq 1$, whose genus is g, but every embedding in the orientable surface of genus g has a face that is not bounded by a cycle of the graph. (The examples of Zha are not cubic graphs, though.) It is also stated by Zha in [14] that Archdeacon and Stahl, and Huneke, Richter, and Younger (respectively) informed him of having constructed further examples of toroidal cubic graphs disproving Conjecture 1. One purpose of this paper is to bring a rich new family of counterexamples to the attention of interested graph theorists.

Our real goal is to provide simple examples, yet powerful enough to exhibit some additional extremal properties. In particular, we shall consider (cubic) graphs of genus one. If a cubic graph G has an embedding in the torus with a face whose boundary is not a cycle, then G contains an edge e whose removal yields a planar graph. We call such a graph *near-planar* and refer to the edge e as a *planarizing edge* (see [1,2]). Our main result, Theorem 6, gives a simple recipe for constructing near-planar (cubic) graphs, all of whose embeddings of small genus have facial walks that are not cycles. See Corollary 7 for more details. Theorem 8 generalizes Theorem 6 to arbitrary surfaces.

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An embedding of a graph in a surface is *strong* (sometimes also referred to as a *closed 2-cell embedding* or a *circular embedding* [11]) if each face boundary is a cycle in the graph. We denote by $\mathbf{g}(G)$ the (orientable) *genus* of G. By $\overline{\mathbf{g}}(G)$ we denote the *strong genus* of G, which is defined as the smallest genus of an orientable surface in which G has a strong embedding. If G has no strong orientable embeddings, then $\overline{\mathbf{g}}(G) = \infty$. In this notation, Conjecture 1 claims that $\overline{\mathbf{g}}(G) = \mathbf{g}(G)$ for every bridgeless cubic graph G.

The well-known *Cycle Double Cover Conjecture* claims that every 2-edge-connected graph admits a collection of cycles such that every edge is contained in precisely two of the cycles from the collection. Such a collection is called a *cycle double cover* of the graph. If G is cubic, then every cycle double cover of G determines a strong embedding of G in some surface (possibly non-orientable). There is also an orientable version of the cycle double cover conjecture:

Conjecture 2 (*Jaeger [7]*). *Every 2-connected graph G has a strong embedding in some orientable surface, i.e. $\overline{\mathbf{g}}(G) < \infty$.*

We follow standard graph theory terminology (see, e.g. [3]). For the notions of topological graph theory we refer to [9]. All embeddings of graphs in surfaces are assumed to be 2-*cell embeddings*. If S is a closed surface, whose Euler characteristic is $c = \chi(S)$, then the *genus* of S is equal to $\frac{1}{2}(2 - c)$ if S is orientable, and is equal to $2 - c$ if S is non-orientable.

2. Facial distance and nonseparating cycles

Let G be a graph embedded in a surface S and let $x, y \in V(G)$, $x \neq y$. We define the *facial distance* $d'(x, y)$ between x and y as the minimum integer r such that there exist facial walks $F_1, \ldots, F_r$ where $x \in V(F_1)$, $y \in V(F_r)$, and $V(F_i) \cap V(F_{i+1}) \neq \emptyset$ for every i, $1 \leq i < r$. The following dual expression for $d'(x, y)$, see [2], can be viewed as a surface version of Menger's Theorem.

Proposition 3. *Let G be a plane graph and $x, y \in V(G)$, where y lies on the boundary of the exterior face. Let r be the maximum number of vertex-disjoint cycles, $Q_1, \ldots, Q_r$, contained in $G - x - y$, such that for $i = 1, \ldots, r$, $x \in int(Q_i)$ and $y \in ext(Q_i)$. Then $d'(x, y) = r + 1$.*

Let $\mathcal{C}$ be a non-empty set of disjoint cycles of a graph G that is embedded in some surface, and let $C \subseteq G$ be the union of all cycles from $\mathcal{C}$. Then $\mathcal{C}$ is *surface-separating* if there is a set of facial walks whose *sum* (i.e., the symmetric difference of their edge-sets) is equal to C. A set $\mathcal{C}$ of disjoint cycles in an embedded graph G is *homologically independent* if no non-empty subset of $\mathcal{C}$ is surface-separating. We say that $\mathcal{C}$ is *induced and nonseparating* in the graph G if C is an induced subgraph of G and $G - V(C)$ is connected.

Induced and nonseparating cycles play a special role in topological graph theory. In particular, if G is a 3-connected planar graph, then the induced and nonseparating cycles of G are precisely those cycles that form face boundaries. The following property generalizes this fact.

Lemma 4. *Let $\mathcal{C}$ be a family of disjoint cycles in a graph G such that $C = \bigcup \mathcal{C}$ is an induced and nonseparating subgraph of G. If G is embedded in some surface and no cycle in $\mathcal{C}$ is facial, then $\mathcal{C}$ is homologically independent.*

The next lemma is taken from [8]. For completeness we include its proof.

Lemma 5. *Let G be a graph embedded in a surface of genus g (either orientable, or non-orientable). If $Q_1, \ldots, Q_k$ are pairwise disjoint cycles in G that are homologically independent, then $k \leq g$.*

Proof. Since the cycles Q_i are homologically independent, it follows that after cutting the surface Σ along these cycles, we obtain a connected surface with boundary, having $2k$ boundary components if Σ is orientable and having at least k boundary components in the non-orientable case. Denote their number by b. If we paste a disc on each of the boundary components we get a closed surface Σ' whose Euler characteristic is equal to $\chi(\Sigma') = \chi(\Sigma) + b \leq 2$. If Σ is orientable then $2k = b \leq 2 - \chi(\Sigma) = 2 - (2 - 2g) = 2g$. In the non-orientable case we have $k \leq b \leq 2 - \chi(\Sigma) = g$, which proves the claimed inequality in either case. $\square$

3. Strong embeddings of near-planar graphs

Conjecture 1 holds for planar graphs. Namely, every bridgeless cubic graph is 2-connected, and every embedding of a 2-connected graph in the plane is strong. This is no longer true on the torus. Fig. 1 shows two embeddings of a non-planar cubic graph in the torus. One is strong, and the other one is not.

As mentioned in the introduction, a cubic toroidal graph admits a non-strong embedding in the torus if and only if it is near-planar. The following result gives rise to a variety of counterexamples to Conjecture 1.

Theorem 6. *Let G be a near-planar graph with $xy \in E(G)$ being a planarizing edge. Suppose that $G - xy$ is a subdivision of a 3-connected graph and let $q = d'(x, y)$ be the facial distance between x and y in the (unique) planar embedding of $G - xy$. Then every strong embedding of G (either in an orientable or in a non-orientable surface) has genus at least $\lfloor \frac{1}{3} q \rfloor$.*

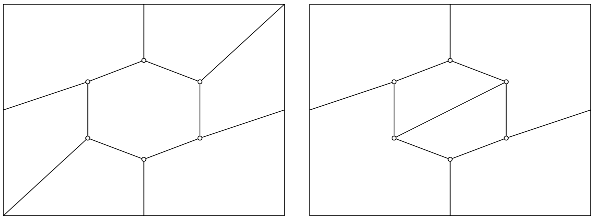

Fig. 1. $K_{3,3}$ has strong and non-strong embeddings in the torus.

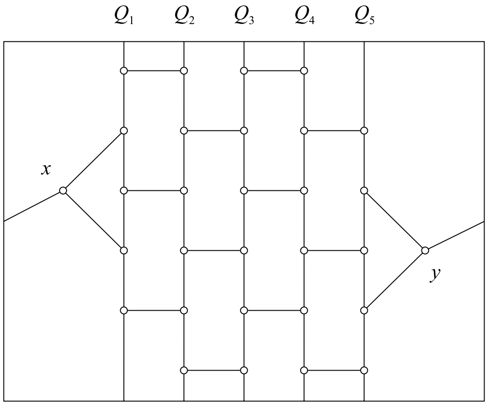

Fig. 2. A near-planar graph G_2.

Proof. Let $Q_1, \ldots, Q_r$ $(r = q - 1)$ be the cycles guaranteed by Proposition 3, enumerated such that Q_i lies in the interior of Q_{i+1} for $1 \le i < r$. Suppose that G has a strong embedding Π in a surface of genus g, and consider a facial cycle F containing the edge xy. Note that F intersects all cycles $Q_1, \ldots, Q_r$ and hence contains, for each $i \in \{1, \ldots, r-1\}$, a path R_i joining Q_i and Q_{i+1}. Since every edge belongs to two facial cycles only, this implies that there is a facial cycle F_i in the planar embedding of $G - xy$ that contains an edge in R_i and is not Π-facial. Since $G - xy$ is a subdivided 3-connected graph, the cycle F_i is induced and nonseparating. By Lemma 4, F_i is surface-non-separating under the embedding Π.

Let us now consider the cycles $F_1, F_4, F_7, \ldots, F_{3k-2}$, where $k = \lfloor (r+1)/3 \rfloor$. Since each F_i is induced and $F_1, F_4, F_7, \ldots$ are at distance at least two from each other, the union $C = F_1 \cup F_4 \cup \cdots \cup F_{3k-2}$ of these cycles is an induced subgraph of G. Next, we argue that C is nonseparating in G. To see that, let $u \in V(G - C)$. There is an index $i \in \{0, 1, \ldots, r\}$ such that $u \in int(Q_{i+1}) \cap ext(Q_i)$ (where $Q_0 = \{x\}$, $Q_{r+1} = \{y\}$, and $ext(Q_0) = int(Q_{r+1}) = G$). Since F_i is nonseparating in G, there is a path from u to Q_i in $G - C$. Similarly, for each $j = 0, 1, \ldots, r$, there is a path in $G - C$ from Q_j to Q_{j+1}. This easily implies that $G - C$ is connected and proves that C is nonseparating.

Since C is an induced and nonseparating subgraph of G and none of the cycles forming C is Π-facial, Lemma 4 shows that the cycles forming C are homologically independent. By Lemma 5, we conclude that $k \le g$, which we were to prove. □

Corollary 7. *For every integer n, there exists a near-planar cubic graph G_n of order $18n - 6$ with $\overline{\mathbf{g}}(G) \ge n$.*

Proof. Let G_n be the near-planar graph whose toroidal embedding is shown in Fig. 2 (for $n = 2$). The graph has $r = 3n - 1$ disjoint cycles that show that $d'(x, y) = r + 1$ in the planar embedding of $G_n - xy$. Now, Theorem 6 applies. □

Let us observe that Euler's formula implies that *every* orientable (2-cell) embedding of the graphs G_n from Corollary 7 has genus at most $6n - 2$. The largest possible genus of a strong embedding of G_n is $6n - 3$, and this bound is attained if and only if G_n has a cycle double cover consisting of three Hamilton cycles of G_n.

4. Examples of higher genus

For embeddings in surfaces of higher genus, we need an additional notion. An embedding of a graph is *polyhedral* if every facial walk is an induced and nonseparating cycle of the graph.

Theorem 8. *Let G be a graph and $xy \in E(G)$. Suppose that $G - xy$ has a polyhedral embedding in some surface, and let $q = d'(x, y)$ be the facial distance between x and y under this embedding. Then every strong embedding of G has genus at least $\lfloor \frac{1}{3} q \rfloor$.*

Proof. The proof is essentially the same as the proof of Theorem 6. Let us only remark the main differences. First of all, there are pairwise disjoint subgraphs $Q_1, \ldots, Q_r$ $(r = q - 1)$, where each Q_i is a union of cycles (i.e. an Eulerian subgraph) of

$G - x - y$ that separates the surface so that one part (called the *interior* of Q_i, $int(Q_i)$) contains x and the previous cycles $Q_1, \ldots, Q_{i-1}$, and the other part of the surface (the *exterior* $ext(Q_i)$) contains y and $Q_{i+1}, \ldots, Q_r$. These subgraphs Q_i are obtained as follows. We set $Q_0 = \{x\}$. For $i \leq r$, having constructed Q_{i-1}, we take $int(Q_{i-1})$ together with all facial cycles that intersect Q_{i-1} and denote this subgraph of G by $int(Q_i)$. The sum of all facial cycles forming $int(Q_i)$ is a surface-separating subgraph Q_i of G. (If G is a cubic graph, then Q_i is disjoint union of one or more cycles, but in general, it is just an Eulerian subgraph of G.)

The rest of the proof is the same as for Theorem 6. □

5. Concluding remarks

In view of the Cycle double cover conjecture and its embedding counterpart (Conjecture 2), it is of interest to show that (cubic) graphs of small genus admit strong embeddings in some surface. Some existing work in this area is [15,16]. Let us remark that for cubic graphs this follows from known results about possible counterexamples to the Cycle double cover conjecture. Goddyn [5] proved that a minimum counterexample has girth at least 10, and Huck [6] extended this further by proving that its girth is at least 12. The reductions used in those proofs can be made so that embeddability in a fixed surface is preserved. They can be summarized as follows.

Theorem 9. *Let $g \geq 0$ be an integer. If there is a 2-edge-connected graph whose genus (or non-orientable genus) is at most g and that does not have a cycle double cover, then there is such a graph G with the following properties:*

(a) *G is cubic and 3-connected.*
(b) *G has girth at least 12.*

This yields the following straightforward corollary.

Corollary 10. *If a 2-edge-connected graph G has genus at most 16 or has nonorientable genus at most 33, then it has a cycle double cover. In particular, if it is cubic, then it admits a strong embedding in some surface.*

Proof. By Theorem 9, we may assume that G is cubic and has girth at least 12. The girth condition implies that the order $n = |V(G)|$ of G satisfies

$$n \geq 1 + 3 + 6 + 12 + 24 + 48 + 32 = 126.$$

Since G is cubic, $|E(G)| = \frac{3}{2}n$. Since every facial walk has length at least 12, we have that $12f \leq 2|E(G)| = 3n$, where f denotes the number of facial walks. By Euler's formula, $f = c - n + |E(G)| \geq \frac{1}{2}n + c$, where c is the Euler characteristic of the surface in which G is embedded. This implies that $-c \geq \frac{1}{2}n - f \geq \frac{1}{4}n \geq \frac{63}{2}$. In particular, the genus (or the nonorientable genus) is at least 17 (respectively, 34). □

The proofs of Goddyn [5] and Huck [6] do not preserve orientability of embeddings, so it remains an open problem if the last conclusion of Corollary 10 can be strengthened by concluding that a strong embedding in some orientable surface exists. A recent paper by Ellingham and Zha [4] discusses this problem and provides a solution for projective-planar graphs.

The following computational problems are of interest.

Problem 11. What is the computational complexity of determining if a given (cubic) graph admits a strong embedding in some (orientable) surface?

If Conjecture 2 is true, then the answer to Problem 11 is trivial—such an embedding exists if and only if the input graph is 2-edge-connected. On the other hand, if Conjecture 2 fails, it is likely that the problem of existence of strong embeddings would be NP-hard.

In this note we have made a small step towards the study of the following problems.

Problem 12. For a given input graph G that has at least one strong embedding, find a strong embedding of minimum genus.

Problem 13. For a fixed integer $g \geq 0$ decide if a given input graph G, whose genus is at most g, admits a closed 2-cell embedding of genus at most g.

This problem is trivial for $g = 0$, and the methods of this paper may provide a way to a fast recognition for the case $g = 1$.

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