Adding One Edge to Planar Graphs Makes Crossing Number Hard*

Sergio Cabello
Department of Mathematics
IMFM and FMF, University of Ljubljana
Slovenia
sergio.cabello@fmf.uni-lj.si

Bojan Mohar†
Department of Mathematics
Simon Fraser University
Burnaby, B.C. V5A 1S6
mohar@sfu.ca

ABSTRACT

A graph is *near-planar* if it can be obtained from a planar graph by adding an edge. We show that it is NP-hard to compute the crossing number of near-planar graphs. The main idea in the reduction is to consider the problem of simultaneously drawing two planar graphs inside a disk, with some of its vertices fixed at the boundary of the disk. This approach can be used to prove hardness of some other geometric problems. As an interesting consequence we obtain a new, geometric proof of NP-completeness of the crossing number problem, even when restricted to cubic graphs. This resolves a question of Hliněný.

Categories and Subject Descriptors

F.2.2 [**Analysis of Algorithms and Problem Complexity**]: Nonnumerical algorithms and problems—*Geometric problems and computations; Computations on discrete structures*; G.2.2 [**Discrete Mathematics**]: Graph theory—*Graph algorithms*

General Terms

Algorithms, Theory

Keywords

Topological graph theory, planar graphs, crossing number, NP-hard, graph drawing, graph embedding

1. INTRODUCTION

A *drawing* of a graph G in the plane is a representation of G where vertices are represented by distinct points of $\mathbb{R}^2$, edges are represented by simple polygonal arcs in $\mathbb{R}^2$ joining points that correspond to their endvertices, and the interior of every arc representing an edge contains no points representing the vertices of G. A *crossing* of a drawing $\mathcal{D}$ is a pair $(\{e, e'\}, p)$, where e and e' are distinct edges and $p \in \mathbb{R}^2$ is a point that belongs to the interiors of both arcs representing e and e' in the drawing $\mathcal{D}$. The number of crossings of a drawing $\mathcal{D}$ is denoted by $\texttt{cr}(\mathcal{D})$ and is called the *crossing number* of the drawing. The *crossing number* $\texttt{cr}(G)$ of a graph G is the minimum $\texttt{cr}(\mathcal{D})$ taken over all drawings $\mathcal{D}$ of G. A drawing $\mathcal{D}$ with $\texttt{cr}(\mathcal{D}) = 0$ is called an *embedding* of G.

A graph is *near-planar* if it can be obtained from a planar graph G by adding an extra edge xy between vertices x and y of G. We denote such near-planar graph by $G + xy$. (The term *almost planar* has also been used for the same concept [10, 13].) Near-planarity is a very weak relaxation of planarity, and hence it is natural to study the crossing number of near-planar graphs. Graphs embeddable in the torus and apex graphs are a superfamily of near-planar graphs. We show that it is NP-hard to compute the crossing number of near-planar graphs. This result is not only surprising but also fundamental. It provides evidence that computing crossing numbers is an extremely challenging task, even for the simplest families of non-planar graphs.

Our reduction is based on considering the following optimization problem: draw two planar graphs inside a disk with some of its vertices at prescribed positions of the boundary, so as to minimize the number of crossings in the drawing. We show that this problem is NP-hard using a reduction from satisfiability (SAT). The reduction is inspired by the work of Werner [16], although the details in our proof are essentially different. We can then use a technique from [13] to provide NP-hardness of computing the crossing number of near-planar graphs.

Our approach can be used to prove hardness of some other geometric problems. As an interesting consequence we obtain a new, geometric proof of NP-completeness of the crossing number problem, even when restricted to cubic graphs. Hardness of the crossing number problem for cubic graphs was established by Hliněný [9], who asked if one can prove this result by a reduction from an NP-complete geometric problem instead of the LINEAR ARRANGEMENT problem used in his proof.

Related work. It has been known for quite some time that it is NP-hard to compute crossing numbers of graphs. Pre-

*Research partially supported by the Slovenian Research Agency, program P1-0297, by an NSERC Discovery Grant, and by the Canada Research Chair Program.

†On leave from IMFM & FMF, Department of Mathematics, University of Ljubljana, 1000 Ljubljana, Slovenia.

vious proofs involved reductions from the problem Linear Arrangement [6, 9, 14]. The spirit of our reduction is completely different from previous proofs and hence of interest in its own right. In particular, we provide an alternative proof that computing crossing numbers is NP-hard (even when restricted to cubic graphs). Our NP-hardness proof is more complicated, but it provides the additional bonus of having control over the structure of the graph and henceforth working for near-planar graphs.

The study of crossing numbers for near-planar graphs was initiated by Riskin [15], who showed that if G is a planar 3-connected cubic graph, then the crossing number of $G + xy$ is equal to the length of a shortest path in the geometric dual graph of the planar subgraph $G-x-y$. A consequence of his result it that the crossing number of a 3-connected cubic near-planar graph can be computed in polynomial time. Riskin asked if a similar result holds in more general situations. This was disproved by Mohar [13] and Gutwenger, Mutzel, and Weiskircher [8]. In fact, the result cannot be extended even assuming 5-connectivity.

For near-planar graphs of maximum degree Δ, Hliněný and Salazar [10] provided a Δ-approximation algorithm for the crossing number. Later, we [2] improved the approximation factor of this algorithm to $\lfloor \Delta/2 \rfloor$ using combinatorial bounds that relate the crossing number of $G + xy$ to the number of vertex-disjoint and edge-disjoint cycles in G that separate x and y. This separation has to be defined in a certain strong sense over all planar embeddings of G. Approximation algorithms for the crossing number have been provided for some superfamilies of near-planar graphs [3, 4, 11]. However, it should be noted that it was not known if computing the crossing number in any of those families is NP-hard. Combinatorial bounds have also been studied in [1, 5].

Our previous paper [2] contained a closely related result: namely, we showed that computing the crossing number of near-planar graphs is NP-hard for *weighted* graphs. Unfortunately, our reduction was from Partition, and hence required weights that are not polynomially bounded in the size of the graph. Moreover, the planarizing edge xy needed large weight, so $G + xy$ could not be transformed into an unweighted near-planar graph. See the discussion below. In this paper we use completely different techniques.

Kawarabayashi and Reed [12], improving upon a result of Grohe [7], have shown that for each constant k_0 there is a linear-time algorithm that decides if the crossing number of an input graph is at most k_0. Hence, it is clear that in our reduction the crossing number has to be an increasing function in the number of vertices.

Weighted vs. unweighted edges. Our discussion will be simplified by using weighted edges. When each edge e of G has a weight $w_e \in \mathbb{N}$, the crossing number of a drawing $\mathcal{D}$ is defined as $\sum w_e \cdot w_{e'}$, the sum taken over all crossings $(\{e, e'\}, p)$ in $\mathcal{D}$. The crossing number of G is then defined again as the minimum $\texttt{cr}(\mathcal{D})$ taken over all drawings $\mathcal{D}$ of G.

Let G be a weighted graph. Consider the unweighted graph H_G with $V(H_G) = V(G)$, in which there are w_{uv} subdivided "parallel" edges between u and v in H_G, for each edge $uv \in E(G)$. It is easy to see that $\texttt{cr}(H_G) = \texttt{cr}(G)$. If the weights of G are polynomially bounded in $|V(G)|$, then H_G can be constructed in polynomial time. Hence, in our reduction it will be enough to describe a weighted planar graph G whose weights are polynomially bounded, and then describe which extra edge xy we add. The resulting unweighted near-planar graph is $H_G + xy$. The additional edge xy that we add must have unit weight, as otherwise the resulting graph H_{G+xy} would not be near-planar.

Anchored graphs. The main idea in our proof is considering a concept of anchored graphs, and studying the corresponding crossing numbers. We are not aware of any previous work considering this concept.

An *anchored graph* is a triple (G, A_G, π_G), where G is a graph, A_G is a subset of vertices of G, and π_G is a circular ordering of A_G. For reasons that will become evident soon, we call the vertices A_G *anchors*. With a slight abuse of notation, we will sometimes use G to denote an anchored graph when the anchor set A_G and the ordering π_G are implicit.

Let $\Omega \subseteq \mathbb{R}^2$ be a topological disk whose boundary is a closed polygonal line. An *anchored drawing* of an anchored graph (G, A_G, π_G) is a drawing of G in Ω such that the vertices of A_G are represented by points on the boundary of the disk Ω, and the ordering of the anchors A_G along the boundary of Ω is π_G. An anchored drawing without crossings is an *anchored embedding*. An *anchored planar graph* is an anchored graph that has an anchored embedding. The *anchored crossing number* $\texttt{acr}(G, A_G, \pi_G)$, or simply $\texttt{acr}(G)$, of an anchored graph G is the minimum number of crossings over all anchored drawings of G.

Let (G, A_G, π_G) be an anchored graph. Any subgraph H of G naturally defines the *anchored subgraph* (H, A_H, π_H), where $A_H = A_G \cap V(H)$ and π_H is the restriction of π_G to the vertices in A_H. We say that an anchored graph (G, A_G, π_G) can be *decomposed into two anchored graphs* if there are anchored subgraphs (R, A_R, π_R) and (B, A_B, π_B) such that $R \cup B = G$ and $E(R) \cap E(B) = \emptyset$. The decomposition is *vertex-disjoint* when $V(R) \cap V(B) = \emptyset$. For helping the exposition we will refer to R as "red" graph and to B as "blue" graph.

2. ANCHORED CROSSING NUMBER IN A DISK

The problem of minimizing the number of crossings in drawings of anchored graphs is of independent interest. In this section we show that computing the crossing number of anchored graphs is hard even in a very special case when the anchored graph is decomposed into two vertex-disjoint planar anchored subgraphs.

Theorem 1. *Computing the anchored crossing number of anchored graphs is NP-hard, even if the input graph is decomposed into two vertex-disjoint planar anchored subgraphs (and the decomposition is part of the input).*

The rest of this section is devoted the proof of Theorem 1. We will use the notation $[m] = \{0, 1, \ldots, m\}$. The reduction will be from the decision problem of satisfiability:

> SAT.
> *Input:* A set of n variables $x_1, \ldots, x_n$ and a set of m disjunctive clauses $C_1, \ldots, C_m$. *Output:* Can we assign boolean values T/F to the variables such that the formula $C_1 \wedge \cdots \wedge C_m$ is satisfied?

Consider an instance I to SAT. Henceforth, we will use n to denote the number of variables and use m to denote the number of clauses. Let $w = 30nm$. It is convenient to think of w as a sufficiently large weight to make the reduction work. Let $k = (6nm + 6n + 2m + 1)w^3 - m(w^2 + w - 1)$. The upper bound $k < (6nm+6n+2m+1)w^3 \leq 15nmw^3 < (w^2-1)^2$ will be useful in our discussion.

We next provide an overview of our reduction. Our aim at this point is to provide intuition. An example showing how the whole reduction works is given in Figure 1. It may help getting the global picture through the discussion. We will describe an anchored *blue* planar graph $B = B(I)$ and an anchored *red* planar graph $R = R(I)$. The graphs B and R will be vertex-disjoint. We will then construct an anchored graph $G = G(I)$ that has a decomposition into R and B; the graph G is determined by R, B and by specifying the circular ordering of the anchors $A_B \cup A_R$. It will turn out from the construction that the anchored crossing number of G is at least k, and that it is equal to k if and only if the instance I can be satisfied.

The blue graph B has a grid-like structure. In an optimal drawing there are no blue-blue crossings. The weights of the blue edges are used to encode the clauses of the instance I. The red graph has the following structure. For each variable x_i there is a pair of 'vertical paths' in the red graph; they connect anchor $r(x_i)$ to anchor $r'(x_i)$ in Figure 1. The construction will enforce that in an optimal drawing such a pair will be drawn either to the left or to the right of the middle line, i.e. either in the gray or in the black region as shown in the left part of Figure 3. Each such option correspond to an assignment of the variable as T or F. For each clause C_j there is a 'horizontal path'; it connects anchor $r_{(0,j)}$ to anchor $r_{(2n+1,j)}$ in Figure 1. Such path must cross a 'horizontal line' of the blue grid once. The number of crossings with such horizontal path depends on where it crosses the 'blue line', and tells if the clause is satisfied with the assignment of the variables or not.

We next proceed with the formal proof. The blue graph $B = B(I)$ is constructed as follows; see Figure 2:

(i) Take a grid-like graph with vertices $b_{(\alpha,\beta)}$, $(\alpha,\beta) \in [2n+2] \times [2m+3]$, and an edge between vertices $b_{(\alpha,\beta)}$ and $b_{(\alpha',\beta')}$ if and only if $|\alpha - \alpha'| + |\beta - \beta'| = 1$.

(ii) Remove the vertices $b_{(2i,0)}$ and $b_{(2i,2m+3)}$ for each $i \in [n+1]$.

(iii) Define as anchors the vertices $b_{(2i+1,0)}$ and $b_{(2i+1,2m+3)}$ for each $i \in [n]$, and the vertices $b_{(0,\beta)}$ and $b_{(2n+2,\beta)}$ for each $\beta \in [2m+2] \setminus \{0\}$.

(iv) Remove the edges between any two anchors.

(v) The weights of the edges are defined as follows:

- each edge adjacent to an anchor has weight w^4;
- if the literal x_i appears in clause C_j, then the edge $b_{(2i-1,2j)}b_{(2i,2j)}$ has weight $w^2 - 1$;
- if the literal $\neg x_i$ appears in clause C_j, then the edge $b_{(2i,2j)}b_{(2i+1,2j)}$ has weight $w^2 - 1$;
- the edge $b_{(2i-1,2m+2)}b_{(2i,2m+2)}$ has weight $w^2 + |\{j \mid \text{literal } x_i \text{ appears in } C_j\}|$;
- the edge $b_{(2i,2m+2)}b_{(2i+1,2m+2)}$ has weight $w^2 + |\{j \mid \text{literal } \neg x_i \text{ appears in } C_j\}|$;
- all other edges have weight w^2.

Note that each edge in the blue graph B has weight at least $w^2 - 1$. Hence, independently of the red graph R to be defined below, a drawing of B with crossing number at most $k < (w^2-1)^2$ has to be an embedding of B. Henceforth, we will assume that B is anchored embedded. Note that the graph B has a unique combinatorial embedding with anchors because of 3-connectivity. For each variable x_i, we define two *columns*; see Figure 3. The column C_i^T is the region of the disk enclosed between the curves

$$b_{(2i-1,0)}b_{(2i-1,1)}\dots b_{(2i-1,2m+3)} \quad \text{and}$$
$$b_{(2i+1,0)}b_{(2i+1,1)}b_{(2i,1)}b_{(2i,2)}\dots$$
$$\dots b_{(2i,2m+2)}b_{(2i+1,2m+2)}b_{(2i+1,2m+3)},$$

and the column C_i^F is the region of the disk enclosed between the curves

$$b_{(2i+1,0)}b_{(2i+1,1)}\dots b_{(2i+1,2m+3)} \quad \text{and}$$
$$b_{(2i-1,0)}b_{(2i-1,1)}b_{(2i,1)}b_{(2i,2)}\dots$$
$$\dots b_{(2i,2m+2)}b_{(2i-1,2m+2)}b_{(2i-1,2m+3)}.$$

The blue edges of the form $b_{(\alpha,\beta)}b_{(\alpha+1,\beta)}$ are called *horizontal*. The blue edges of the form $b_{(\alpha,\beta)}b_{(\alpha,\beta+1)}$ are called *vertical*. The weights of the horizontal edges $b_{(2i-1,2j)}b_{(2i,2j)}$ and $b_{(2i-1,2m+2)}b_{(2i,2m+2)}$ contained in the column C_i^T have been chosen so that they add up to $2(m+1)w^2$: each time we have a -1 in the weight of $b_{(2i-1,2j)}b_{(2i,2j)}$ we have a $+1$ in the weight of $b_{(2i-1,2m+2)}b_{(2i,2m+2)}$. The same holds for the column C_i^F.

For each clause C_j, we define two *rows*; see Figure 3. The *upper row* U_j is the region of the disk enclosed between the curves

$$b_{(0,2j)}b_{(1,2j)}\dots b_{(2n+2,2j)} \quad \text{and}$$
$$b_{(0,2j+1)}b_{(1,2j+1)}\dots b_{(2n+2,2j+1)},$$

and the *lower row* L_j is the region of the disk enclosed between the curves

$$b_{(0,2j-1)}b_{(1,2j-1)}\dots b_{(2n+2,2j-1)} \quad \text{and}$$
$$b_{(0,2j)}b_{(1,2j)}\dots b_{(2n+2,2j)}.$$

There is an additional row, called *enforcing row* and denoted by R_{enf}, which is the region of the disk enclosed between the curves

$$b_{(0,2m+1)}b_{(1,2m+1)}\dots b_{(2n+2,2m+1)} \quad \text{and}$$
$$b_{(0,2m+2)}b_{(1,2m+2)}\dots b_{(2n+2,2m+2)}.$$

The role of the enforcing row R_{enf} is to reduce the number of possible drawings to a well-structured subset of drawings. We will use the columns and the rows as some sort of coordinate system to tell where some red vertices go. For example, we may refer to the face $C_i^T \cap L_j$.

The red graph $R = R(I)$ is constructed as follows; see Figure 4:

(i) Take a grid-like graph with vertices $r_{(\alpha,\beta)}$, $(\alpha,\beta) \in [2n+1] \times [m+2]$, and an edge between vertices $r_{(\alpha,\beta)}$ and $r_{(\alpha',\beta')}$ if and only if $|\alpha - \alpha'| + |\beta - \beta'| = 1$.

(ii) Remove the four vertices $r_{(0,0)}$, $r_{(0,m+2)}$, $r_{(2n+1,0)}$, and $r_{(2n+1,m+2)}$.

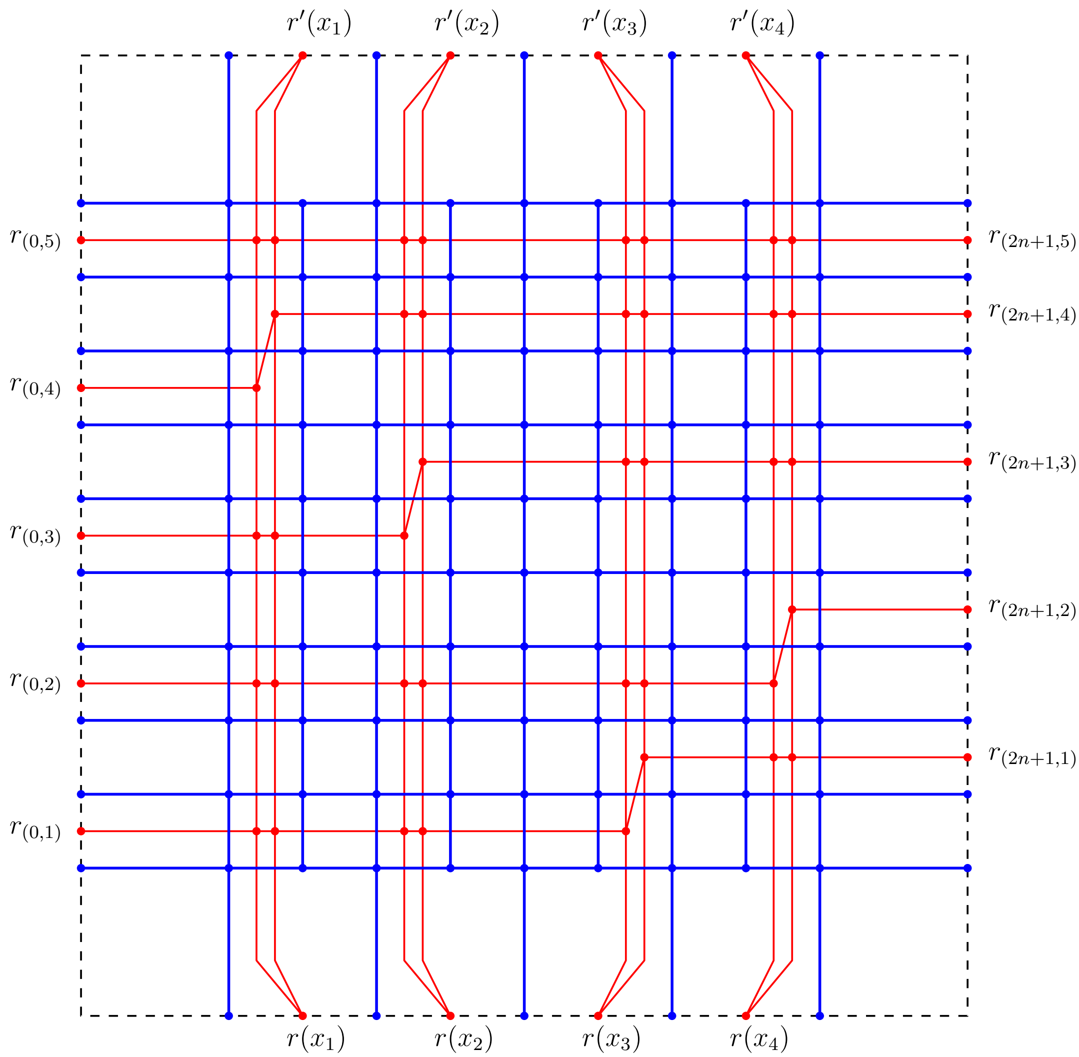

Figure 1: Example of the resulting reduction for the formula on 4 variables x_1, x_2, x_3, x_4 and clauses $\neg x_1 \vee \neg x_3 \vee x_4$, $\neg x_2 \vee \neg x_4$, $x_2 \vee \neg x_3$, $x_1 \vee x_2$. The optimal drawing in the figure corresponds to the boolean assignment $x_1 = x_2 = T$ and $x_3 = x_4 = F$.

(iii) For each variable x_i, identify the vertices $r_{(2i-1,0)}$ and $r_{(2i,0)}$ into a new vertex called $r(x_i)$, and identify the vertices $r_{(2i-1,m+2)}$ and $r_{(2i,m+2)}$ into a new vertex called $r'(x_i)$. For each variable x_i, the vertices $r(x_i)$ and $r'(x_i)$ are anchors for R. The vertices $r_{(0,j)}$ and $r_{(2n+1,j)}$ are also anchors for R, for every $j \in [m+1] \setminus \{0\}$.

(iv) Remove the edges between any two anchors.

(v) The weights of the edges are defined as follows:

- for each variable x_i the edge $r_{(2i-1,m+1)}r_{(2i,m+1)}$ has weight w^4;
- for each variable x_i and each clause C_j the edge $r_{(2i-1,j)}r_{(2i,j)}$ has weight $w-1$;
- all other edges have weight w.

For each variable x_i the following two *vertical paths* are important:

$$r(x_i)r_{(2i-1,1)}r_{(2i-1,2)}\dots r_{(2i-1,m+1)}r'(x_i) \quad \text{and}$$
$$r(x_i)r_{(2i,1)}r_{(2i,2)}\dots r_{(2i,m+1)}r'(x_i).$$

We will use V_i to denote their union. For each clause C_j, we will consider the *horizontal path* H_j defined by

$$r_{(0,j)}r_{(1,j)}\dots r_{(2n+1,j)}.$$

We also define the *horizontal enforcing path* H_{enf} as

$$r_{(0,m+1)}r_{(1,m+1)}\dots r_{(2n+1,m+1)}.$$

The role of the horizontal enforcing path H_{enf} will be to reduce the number of possible drawings to a well-structured subset of drawings.

It is important to note that the paths

$$V_1, V_2, \dots, V_n, H_1, H_2 \dots, H_m, H_{\text{enf}}$$

form a partition of the edge set of the red graph R. Hence, we can add the number of crossings that each of them con-

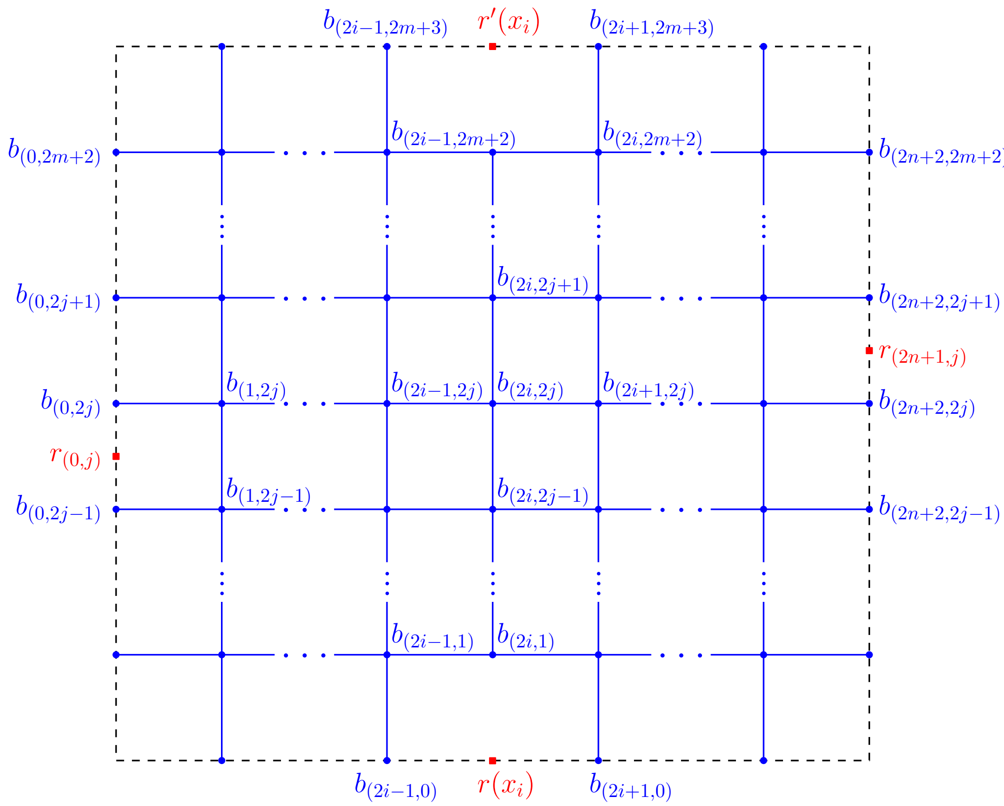

Figure 2: The graph $B = B(I)$. Anchors of the red graph are included to show the cyclic ordering of $A_B \cup A_R$.

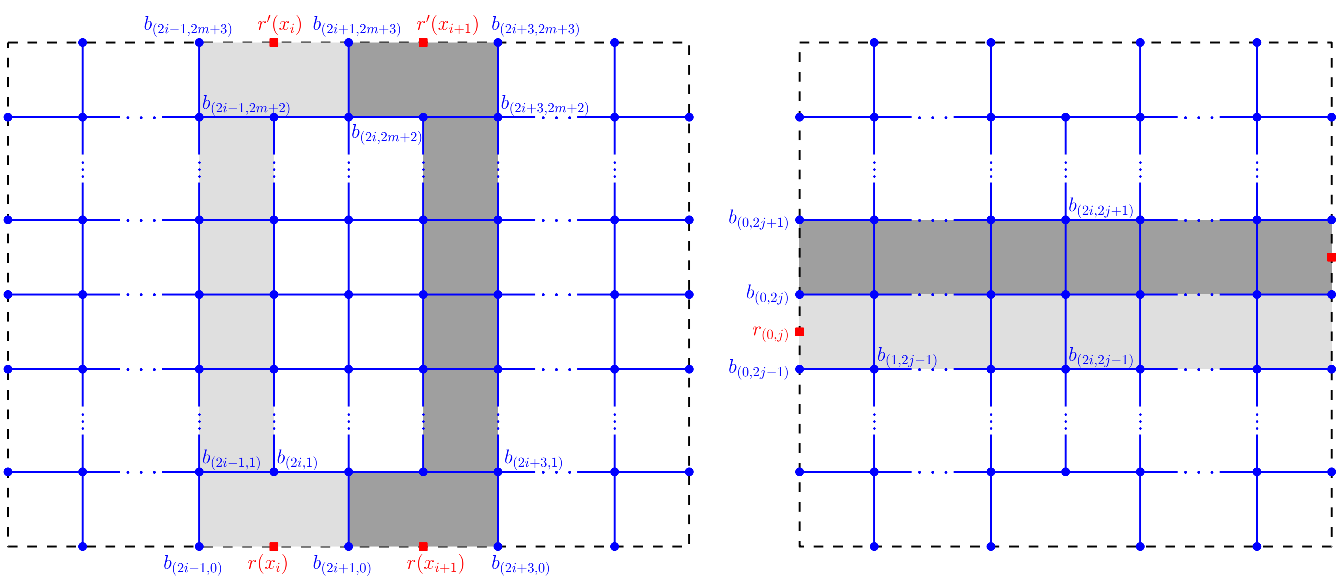

Figure 3: Left: columns C_i^T (lighter shading) and C_{i+1}^F (darker shading). Right: the upper row U_j (darker shading) and the lower row L_j (lighter shading).

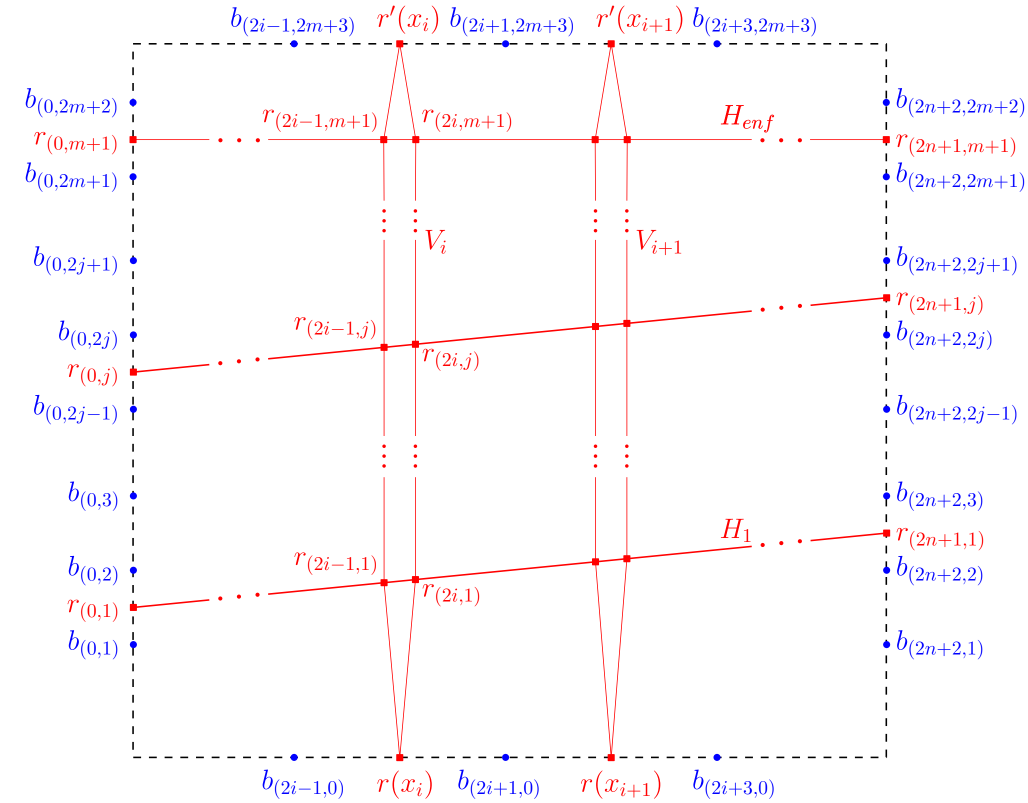

Figure 4: The graph $R = R(I)$. Anchors of the blue graph are also included to show the cyclic order of $A_B \cup A_R$.

tributes separately to obtain the crossing number of a drawing. Note also, that the intersection of a vertical pair of paths V_i and a horizontal path H_j (or H_{enf}) always consists of two vertices.

Let $G = G(I)$ be the anchored graph obtained by joining the red graph R and the blue graph B. The (clockwise) circular ordering of the anchors along the boundary of the disk as follows:

- For each clause C_j we have the sequence of anchors
$$b_{(0,2j-1)},\ r_{(0,j)},\ b_{(0,2j)},\ b_{(0,2j+1)},$$
and the sequence
$$b_{(2n+2,2j+2)},\ b_{(2n+2,2j+1)},\ r_{(2n+1,j)},\ b_{(2n+2,2j)}.$$
Hence, the anchor $r_{(0,j)}$ is in the lower row L_j and anchor $r_{(2n+1,j)}$ is in the upper row U_j.
- For each variable x_i, the anchor $r(x_i)$ is between anchors $b_{(2i-1,0)}$ and $b_{(2i+1,0)}$, and the anchor $r'(x_i)$ is between $b_{(2i-1,2m+3)}$ and $b_{(2i+1,2m+3)}$. Hence, $r(x_i)$ and $r'(x_i)$ are in both columns C_i^T and C_i^F.
- Anchor $r_{(0,m+1)}$ is between $b_{(0,2m+1)}$ and $b_{(0,2m+2)}$, anchor $r_{(2n+1,m+1)}$ is between anchors $b_{(2n+2,2m+2)}$ and $b_{(2n+2,2m+1)}$. Hence, anchors $r_{(0,m+1)}$ and $r_{(2n+1,m+1)}$ are in the enforcing row R_{enf}.
- Anchor $b_{(0,1)}$ comes immediately after $b_{(1,0)}$, anchor $b_{(1,2m+3)}$ comes immediately after $b_{(0,2m+2)}$, anchor $b_{(2n+2,2m+2)}$ comes immediately after $b_{(2n+1,2m+3)}$, anchor $b_{(2n+1,0)}$ comes immediately after $b_{(2n+2,1)}$.

This finishes the description of the graph G. We first show the easy direction of the proof, which will also give an idea of how the reduction works. Recall that we have defined $k = (6nm + 6n + 2m + 1)w^3 - m(w^2 + w - 1)$.

Lemma 1. *If the instance I is satisfiable, then there is an anchored drawing of G with k crossings.*

Proof. We draw the blue graph B without crossings. The corresponding embedding is unique. The red graph R is also going to be drawn without crossings. Hence, it is enough to describe in which face of B is each red vertex and (when not obvious) where the red edges cross the blue edges. See Figure 1 for a particular example. Let $b_i \in \{T, F\}$ be an assignment for each variable x_i of I that satisfies all clauses. We draw the two red vertical paths of V_i inside the column $C_i^{b_i}$. For each clause C_j we proceed as follows. Let x_t, where $t = t(j)$, be a variable whose value b_t makes the clause C_j true. We then draw the horizontal path H_j as follows: the subpath of H_j between $r_{(0,j)}$ and $r_{(2t-1,j)}$ is drawn in the lower row L_j, the edge $r_{(2t-1,j)}r_{(2t,j)}$ crosses from L_j to U_j through the blue edge in $L_j \cap U_j \cap C_t^{b_t}$, and the subpath of H_j between $r_{(2t,j)}$ and $r_{(2n+1,j)}$ is drawn in the upper row U_j. The path H_{enf} is drawn inside the row R_{enf}. Note that this description implicitly assigns to each non-anchor vertex of R a face of B. The drawing can be extended to a planar embedding of R in such a way that no red edge crosses twice any blue edge.

Let us now compute the crossing number of the drawing we have described. There are no monochromatic crossings in the construction; hence we only need to count the red-blue crossings. Each of the two paths in V_i contributes $2(m +$

$1)w^3$ to the crossing number of the drawing: edges in V_i have weight w, and each path in V_i crosses all the horizontal blue edges contained in $C_i^{b_i}$, whose weights add to $2(m+1)w^2$. Each horizontal path H_j contributes $(2n+1)w^3+(w-1)(w^2-1)$ to the crossing number of the drawing: the edges on the horizontal path H_j connecting V_i to V_{i+1} have weight w and cross $2n+1$ blue vertical edges whose weight is w^2; there is only one red edge in H_j, namely $r_{(2t(j)-1,j)}r_{(2t(j),j)}$ with weight $w-1$, that crosses the boundary between rows L_i and U_i, namely at the edge of $L_j \cap U_j \cap C_t^{b_t}$ with weight w^2-1 because the corresponding literal x_i or $\neg x_i$ makes C_j satisfied. The horizontal path H_{enf} contributes $(2n+1)w^3$ to the crossing number of the drawing: the edges of H_j connecting V_i to V_{i+1} have weight w and cross $2n+1$ blue edges whose weight is w^2.

The crossing number of the drawing is thus

$$n \cdot 2 \cdot 2(m+1)w^3 + m \cdot \big((2n+1)w^3 + (w-1)(w^2-1)\big) + (2n+1)w^3$$

which is

$$(6nm+6n+2m+1)w^3 - m(w^2+w-1) \;=\; k.$$

□

We next have to show the reverse implication: if the anchored crossing number of G is at most k, then the formula I is satisfiable. Henceforth, let us assume for the rest of this section that $\texttt{acr}(G) \le k$, and let us fix an anchored drawing $\mathcal{D}$ of G with at most k crossings. As mentioned before, $\mathcal{D}$ cannot have any blue-blue crossing because otherwise $\texttt{cr}(\mathcal{D}) > k$. In principle, $\mathcal{D}$ could contain red-red crossings; we will show below that in fact this is not possible, and hence all crossings are red-blue. It will be convenient to look at the number of red-blue crossings without taking into account the weights. We refer to such crossings as *unweighted crossings*. Simple arithmetic shows the following two properties.

Lemma 2. *The drawing $\mathcal{D}$ has at most $6nm+6n+2m+1$ unweighted red-blue crossings.*

Lemma 3. (i) *For each clause C_j, the horizontal path H_j is inside the rows $U_j \cup L_j$ and crosses precisely $2n+2$ blue edges.*

(ii) *The horizontal path H_{enf} is drawn inside the row R_{enf}.*

(iii) *For each variable x_i, both vertical paths of V_i are inside the column C_i^T or both are inside the column C_i^F.*

Lemma 4. *If $\texttt{cr}(\mathcal{D}) \le k$, then the instance I is satisfiable. Moreover, the restriction of the drawing $\mathcal{D}$ to R or to B is an embedding.*

Proof. By Lemma 3, in the drawing $\mathcal{D}$ both paths in V_i are contained either in C_i^T or C_i^F. Consider the assignment where variable x_i gets value $b_i = T$ if the two paths of V_i are contained in C_i^T, and $b_i = F$ otherwise. We will show that this assignment satisfies the formula $C_1 \wedge \cdots \wedge C_m$ of the instance I.

We will use in our analysis the properties of $\mathcal{D}$ obtained in Lemma 3. For a red subgraph X, let $\texttt{cr}(X)$ denote the crossing number of the subdrawing of $\mathcal{D}$ induced by X and the blue graph B.

All the edges in V_i have weight w. The weights of the horizontal blue edges contained in $C_i^{b_i}$ add to $2(m+1)w^2$. Furthermore, note that each of those blue horizontal edges is crossed by each of the two paths in V_i. Therefore we have $\texttt{cr}(V_i) = 2\cdot(2m+2)w^3$, and thus $\texttt{cr}(\bigcup_i V_i) = (4nm+4n)w^3$. We next look at H_{enf}. All the edges from the path H_{enf} have weight w or w^4, and hence only edges from H_{enf} with weight w may cross the vertical blue edges contained in the row R_{enf}. Inside the row R_{enf} there are $2n+1$ vertical blue edges of weight w^2, and thus $\texttt{cr}(H_{\text{enf}}) = (2n+1)w^3$. Since $cr(\mathcal{D}) \le k$ and the paths $V_1, \dots, V_n, H_1, \dots H_m, H_{\text{enf}}$ form an edge-disjoint partition of the edges of R, we have $\texttt{cr}(\bigcup_j H_j) + \texttt{cr}(\bigcup_i V_i) + \texttt{cr}(H_{\text{enf}}) \;\le\; k$, and therefore

$$\texttt{cr}(\bigcup_j H_j) \;\le\; k - (4nm+2n)w^3 - (2n+1)w^3 \tag{1}$$

$$= m\big((2n+2)w^3 - (w^2+w-1)\big). \tag{2}$$

The edges of H_j have weight w, if they connect a vertex in $V_i \cap H_j$ to a vertex in $V_{i+1} \cap H_j$ for some i, or weight $w-1$ if they connect both vertices of $V_i \cap H_j$ for some j. Because of Lemma 3, the edges with weight $w-1$ are always within the column $C_i^{b_i}$ for some i. This means that the boundary any column C_i^T or C_i^F, which has weight w^2, is always crossed by a red edge of H_j with weight w. Let ∂_j denote the boundary between the lower row L_j and the upper row U_j. This boundary ∂_j must also be crossed by an edge of H_j, and that crossing contributes weight at least $(w-1)(w^2-1)$ to $\texttt{cr}(H_j)$. We conclude that

$$\begin{aligned} \texttt{cr}(H_j) \;&\ge\; (2n+1)\cdot w \cdot w^2 + (w-1)(w^2-1) \\ &=\; (2n+2)w^3 - (w^2+w-1), \end{aligned}$$

with equality if and only if the crossing between H_j and ∂_j contributes exactly $(w-1)(w^2-1)$ to the crossing number. Combining with equations (1)–(2), we see that

$$\texttt{cr}(H_j) = (2n+2)w^3 - (w^2+w-1) \tag{3}$$

for each clause C_j. Therefore, the boundary ∂_j must be crossed at a blue edge $b_{(t,j)}b_{(t+1,j)}$ of weight w^2-1 by a red edge $r_{(2i-1,j)}r_{(2i,j)}$ of weight $w-1$. It may be that $t = 2i-1$ or $t = 2i$. Consider first the case when $t = 2i-1$. By the construction of the blue graph B, the edge $b_{(t,j)}b_{(t+1,j)}$ has weight w^2-1 because the literal x_i appears in the clause C_j of I. Moreover, the endpoints of $r_{(2i-1,j)}r_{(2i,j)}$ must be in the column C_i^T since they are part of the vertical paths V_i. Hence, the clause C_j is satisfied by the assignment $x_i = b_i = T$ we defined at the beginning of the proof. The case when $t = 2i$ is alike, but in this case the literal $\neg x_i$ appears in the clause C_j of I, and we took the assignment $x_i = b_i = F$.

Finally, note that our analysis shows that in $\mathcal{D}$ there are exactly k red-blue crossings, and therefore there cannot be any red-red crossings. Hence the restriction of the drawing $\mathcal{D}$ to the red graph is an embedding. □

Note that the graph G can be constructed in polynomial time and that it has weights that are polynomially bounded in n and m. Thus, in polynomial time we can replace in G each edge e by w_e parallel edges that are subdivided once, and the resulting graph has the same anchored crossing number as G; see the discussion in the introduction. Then Theorem 1 follows from Lemmas 1 and 4.

3. HARDNESS OF CROSSING NUMBER FOR NEAR-PLANAR GRAPHS

Theorem 2. *Computing the crossing number of near–planar graphs is NP-hard.*

Proof. Consider an (unweighted) anchored graph G that is decomposed into two planar anchored graphs R and B that are vertex-disjoint. Let m denote the number of edges of G.

Consider the weighted graph G', without anchors, obtained from G as follows: we start with G and assign weight $2m$ to each of its edges. For every two consecutive anchors a and a' of G in the cyclic ordering π_G, we introduce in G' an edge aa' with weight $5m^4$. The set of added edges defines a cycle, which we denote by C. Finally, we choose an arbitrary vertex r of R that is not an anchor, an arbitrary vertex b of B that is not an anchor, and add the edge rb to G' with unit weight. (If all vertices of R are anchors we just subdivide an edge and take r as the new vertex. A similar procedure can be done with B.) This completes the description of G'.

Firstly, we show that G' is a near-planar graph. The graph $G'-rb$ consists of the cycle C connecting consecutive anchors of G, and two planar anchored graphs R and B. We can thus embed $G' - xy$ taking an embedding of C in $\mathbb{R}^2$, embedding R in the disk bounded by C in $\mathbb{R}^2$, and embedding B in the exterior of the disk. Note that the embeddings of R and B exist because they are planar anchored graphs. Since the weight of the edge rb of G' is one, it follows that G' is a near-planar graph, even when replacing each edge distinct from rb by the corresponding number of subdivided parallel edges.

We claim that $\texttt{acr}(G) = \lfloor \texttt{cr}(G')/(4m^2)\rfloor$. Consider an optimal anchored drawing $\mathcal{D}$ of G in a topological disk Ω with $\texttt{cr}(\mathcal{D}) = \texttt{acr}(G)$. We then extend the drawing $\mathcal{D}$ to obtain a drawing $\mathcal{D}'$ of G' as follows. Pushing the interior of the edges inside Ω, we may assume that $\mathcal{D}$ touches the boundary $\partial\Omega$ of Ω only at the anchors. We then draw the edges aa' of G' between consecutive anchors along the boundary of the disk Ω. Finally, we draw the edge rb so as to minimize the number of crossings it contributes. Each crossing of $\mathcal{D}$ contributes $(2m)^2$ crossings to $\mathcal{D}'$. The edge rb can cross each edge of G' at most once in the drawing $\mathcal{D}'$ because of optimality of $\mathcal{D}'$ and the drawing of rb. Therefore $\texttt{cr}(\mathcal{D}') \leq \texttt{cr}(\mathcal{D}) \cdot 4m^2 + 2m^2$, and thus

$$\begin{aligned} \lfloor \texttt{cr}(G')/4m^2 \rfloor &\leq \lfloor \texttt{cr}(\mathcal{D}')/4m^2 \rfloor \\ &\leq \lfloor \texttt{acr}(G) + 1/2m^2 \rfloor \\ &= \texttt{acr}(G). \end{aligned}$$

Let $\mathcal{D}'$ be a drawing of G' in the plane such that no edge crosses itself and no two edges cross more than once. If the cycle C is embedded and no edge of C is involved in a crossing, then each pair of edges not in C can cross at most once, and hence $\texttt{cr}(D')$ is upper bounded by $\binom{m}{2}2m \cdot 2m + 1 \cdot m \cdot 2m < 5m^4$. If the restriction of $\mathcal{D}'$ to C is not an embedding or some edge of C participates in a crossing, then $\texttt{cr}(D') \geq 5m^4$. It follows that in every optimal drawing $\mathcal{D}'$ of G', the cycle C is embedded and no edge crosses C.

Consider now an optimal drawing $\mathcal{D}'$ of G' with $\texttt{cr}(\mathcal{D}') = \texttt{cr}(G')$. Let Ω be the disk bounded by the image of C in $\mathcal{D}'$. Since no edge of G' can cross an edge of C, the drawing $\mathcal{D}'$ is contained in the closure of Ω. If in $\mathcal{D}'$ we remove the image of C and the image of the edge rb, then we obtain an anchored drawing of G, which we shall denote by $\mathcal{D}$. Note that $\texttt{cr}(\mathcal{D})$ is equal to $\texttt{cr}(\mathcal{D}')$ minus the number of crossings contributed by rb and scaled down by $(2m)^2$ because of the weights introduced in G'. We thus have

$$\texttt{acr}(G) \leq \texttt{acr}(\mathcal{D}) \leq \texttt{cr}(\mathcal{D}')/(4m^2) = \texttt{cr}(G')/(4m^2),$$

and we may take the floor on the right-hand side because the value is an integer. This finishes the proof of the claim $\texttt{acr}(G) = \lfloor \texttt{cr}(G')/4m^2 \rfloor$.

The graph G' can be constructed from G in polynomial time. Moreover, since the weights of G' are polynomially bounded, we can also replace each edge by parallel subdivided edges to obtain an unweighted graph $H_{G'}$, as described in the introduction, that satisfies $\texttt{cr}(G') = \texttt{cr}(H_{G'})$. Since the graph $H_{G'}$ is near-planar and $\texttt{acr}(G) = \lfloor \texttt{cr}(H_{G'})/4m^2 \rfloor$, the result follows. □

Extensions.

Our NP-hardness proof in Section 2 has the following additional property: we know the local rotation of the edges around each vertex in any optimal anchored drawing. This essentially follows from Lemma 4 because R and B have a unique anchored embedding. Hence, it is NP-hard to compute the *anchored* crossing number restricted to drawings that have a prescribed local rotation system in each vertex. We can also make the anchored crossing number with rotation systems to be the usual crossing number with rotation systems by taking large enough weight on the edges of the cycle C connecting the anchors, as we did in the proof of Theorem 2. Hence, it follows that computing the crossing number for graphs with prescribed rotation systems is NP-hard. This has been shown recently in [14] using a reduction from Linear Arrangement. It is then possible to replace each vertex by a 3-regular grid where the edges are attached preserving the local rotation; the techniques is detailed in [14]. Hence, we provide an alternative, geometric proof that crossing number of cubic graphs is NP-hard. This settles a question of Hliněný [9], who asked if one can prove this result by a reduction from an NP-complete problem with geometric flavor, instead of reducing from Linear Arrangement.

Acknowledgments

We thank the anonymous reviewers for several helpful suggestions.

4. REFERENCES

[1] K. J. Böröczky, J. Pach, and G. Tóth. Planar crossing numbers of graphs embeddable in another surface. *Int. J. Found. Comput. Sci.*, 17(5):1005–1016, 2006.

[2] S. Cabello and B. Mohar. Crossing and weighted crossing number of near-planar graphs. *Algorithmica*, accepted. Available at http://dx.doi.org/10.1007/s00453-009-9357-5. Preliminary version in Graph Drawing 2008.

[3] M. Chimani and P. Hliněný. Approximating the crossing number of graphs embeddable in any orientable surface. In *Proc. SODA 2010*, to appear.

[4] M. Chimani, P. Hliněný, and P. Mutzel. Vertex insertion approximates the crossing number for apex graphs. Manuscript available at http:

//www.fi.muni.cz/~hlineny/Research/papers/. Preliminary version in Graph Drawing 2008.

[5] H. Djidjev and I. Vrto. Planar crossing numbers of genus g graphs. In *ICALP 2006*, volume 4051 of *LNCS*, pages 419–430, 2006.

[6] M. R. Garey and D. S. Johnson. Crossing number is NP-complete. *SIAM J. Alg. Discr. Meth.*, 4:312–316, 1983.

[7] M. Grohe. Computing crossing numbers in quadratic time. In *Proc. STOC 2001*, pages 231–236, 2001.

[8] C. Gutwenger, P. Mutzel, and R. Weiskircher. Inserting an edge into a planar graph. *Algorithmica*, 41:289–308, 2005.

[9] P. Hlinený. Crossing number is hard for cubic graphs. *J. Comb. Theory, Ser. B*, 96(4):455–471, 2006.

[10] P. Hlinený and G. Salazar. On the crossing number of almost planar graphs. In M. Kaufmann and D. Wagner, editors, *GD 2006*, volume 4372 of *LNCS*, pages 162–173. Springer, 2007.

[11] P. Hliněný and G. Salazar. Approximating the crossing number of toroidal graphs. In T. Tokuyama, editor, *ISAAC 2007*, volume 4835 of *LNCS*, pages 148–159, 2007.

[12] K.-I. Kawarabayashi and B. Reed. Computing crossing number in linear time. In *Proc. STOC 2007*, pages 382–390, 2007.

[13] B. Mohar. On the crossing number of almost planar graphs. *Informatica*, 30:301–303, 2006.

[14] M. J. Pelsmajer, M. Schaefer, and D. Štefankovic. Crossing number of graphs with rotation systems. *Algorithmica*, accepted. Available at http://dx.doi.org/10.1007/s00453-009-9343-y.

[15] A. Riskin. The crossing number of a cubic plane polyhedral map plus an edge. *Studia Sci. Math. Hungar.*, 31:405–413, 1996.

[16] S. Werner. On the complexity of the planar edge-disjoint paths problem with terminals on the outer boundary. *Combinatorica*, 29(1):121–126, 2009.