Bogomolov multipliers of groups of order 128

Urban Jezernik * Primož Moravec †

January 17, 2014

Abstract

This note describes an algorithm for computing Bogomolov multipliers of finite solvable groups. Compared to the existing ones, this algorithm has improved performance and is able to recognize the commutator relations of the group that constitute its Bogomolov multiplier. As a sample case we use the algorithm to effectively determine the multipliers of groups of order 128. The two serving purposes are a continuation of the results of Chu *et al.* on Bogomolov multipliers of groups of order 64, and to utilize one of the key steps of another paper by the authors dealing with probabilistic aspects of universal commutator relations.

1 Introduction

Let G be a group and let $G \curlywedge G$ be the group generated by the symbols $x \curlywedge y$ for all pairs $x, y \in G$, subject to the following relations:

$$xy \curlywedge z = (x^y \curlywedge z^y)(y \curlywedge z), \quad x \curlywedge yz = (x \curlywedge z)(x^z \curlywedge y^z), \quad a \curlywedge b = 1,$$

for all $x, y, z \in G$ and all $a, b \in G$ with $[a, b] = 1$. The group $G \curlywedge G$ was first studied in [Moravec 2012] and is called the *curly exterior square* of G. There is a canonical epimorphism $G \curlywedge G \to [G, G]$ whose kernel is denoted by $\mathrm{B}_0(G)$. Its significance was pointed out in [Moravec 2012] where it was shown that $\mathrm{Hom}(\mathrm{B}_0(G), \mathbb{Q}/\mathbb{Z})$ is naturally isomorphic to the unramifed Brauer group of a field extension $\mathbb{C}(V)^G/\mathbb{C}$ over $\mathbb{Q}/\mathbb{Z}$. The unramified Brauer group is a well known obstruction to Noether's problem [Noether 1916] asking whether or not $\mathbb{C}(V)^G$ is purely transcendental over $\mathbb{C}$. Following Kunyavskiĭ [Kunyavskiĭ 2008], we say that $\mathrm{B}_0(G)$ is the *Bogomolov multiplier* of G. Bogomolov multipliers can also be interpreted as measures of how the commutator relations in groups fail to follow from the so-called universal ones, see [JM 2013b] for further details.

Based on the above description of Bogomolov multipliers, an algorithm for computing $\mathrm{B}_0(G)$ and $G \curlywedge G$ when G is a polycyclic group was developed in [Moravec 2012]. Subsequently, Ellis developed a significantly more efficient algorithm for computing Bogomolov multipliers of arbitrary finite groups. It is now available as a part of a homological algebra library HAP,

*Institute of Mathematics, Physics, and Mechanics, Jadranska 19, 1000 Ljubljana, Slovenia.
E-mail address: urban.jezernik@imfm.si
†Department of Mathematics, University of Ljubljana, Jadranska 21, 1000 Ljubljana, Slovenia.
E-mail address: primoz.moravec@fmf.uni-lj.si

cf. [HAP] for further details. The purpose of this paper is to describe a new algorithm for computing Bogomolov multipliers and curly exterior squares of polycyclic groups. It is based on an algorithm for computing Schur multipliers that was developed by Eick and Nickel [EN 2008], and a Hopf-type formula for $B_0(G)$ that was found in [Moravec 2012]. An advantage of the new algorithm is that it enables a systematic trace of which elements of $B_0(G)$ are in fact non-trivial, thus providing an efficient tool of double-checking non-triviality of Bogomolov multipliers by hand. The algorithm has been implemented in GAP [GAP] and is available at the second author's website [GAP code].

Hand calculations of Bogomolov multipliers were done for groups of order 32 by Chu, Hu, Kang, and Prokhorov [CHKP 2008], and groups of order 64 by Chu, Hu, Kang, and Kunyavskiĭ [CHKK 2009]. In a similar way, Bogomolov multipliers of groups of order p^5 were determined in [HK 2011, HKK 2012], and for groups of order p^6 this was done recently by Chen and Ma [CM 2013]. We apply the above mentioned algorithm to determine Bogomolov multipliers of all groups of order 128. Our contribution is an explicit description of generators of Bogomolov multipliers of these groups. There are 2328 groups of order 128, and they were classified by James, Newman, and O'Brien [JNO 1990]. Instead of considering all of them, we use the fact [JNO 1990] that these groups belong to 115 isoclinism families according to Hall [Hall 1940], together with the fact that isoclinic groups have isomorphic Bogomolov multipliers [Moravec 2014]. It turns out that there are precisely eleven isoclinism families whose Bogomolov multipliers are non-trivial. For each of these families we explicitly determine $B_0(G)$ for a chosen representative G. An extended version of the paper where the calculations for all 115 isoclinism families are described is posted at `arXiv` [JM 2013a]. Finally we mention that the results of this paper form a basis for proving the main result of [JM 2013b].

The outline of the paper is as follows. In Section 2 we describe the new algorithm for computing Bogomolov multipliers and curly exterior squares of polycyclic groups. We then proceed to determine the multipliers of groups of order 128. A short summary of the results is provided in Section 3. Section 4 gives full details of the calculation for the isoclinism family Γ_{16}, and results for the remaining ten isoclinism families of groups of order 128 whose Bogomolov multipliers are not trivial.

2 The algorithm

Let G be a finite polycyclic group, presented by a power-commutator presentation with a polycyclic generating sequence g_i with $1 \leq i \leq n$ for some n subject to the relations

$$\begin{aligned} g_i^{e_i} &= \textstyle\prod_{k=i+1}^{n} g_k^{x_{i,k}} \qquad \text{for } 1 \leq i \leq n, \\ [g_i, g_j] &= \textstyle\prod_{k=i+1}^{n} g_k^{y_{i,j,k}} \qquad \text{for } 1 \leq j < i \leq n. \end{aligned}$$

Note that when printing such a presentation, we hold to standard practice and omit the trivial commutator relations, i.e. those for which $y_{i,j,k} = 0$ for all k. For every relation except the trivial commutator relations (the reason being these get factored out in the next step), introduce a new abstract generator, a so-called *tail*, append the tail to the relation, and make it central. In this way, we obtain a group generated by g_i with $1 \leq i \leq n$ and t_ℓ with

$1 \leq \ell \leq m$ for some m, subject to the relations

$$\begin{aligned} g_i^{e_i} &= \textstyle\prod_{k=i+1}^{n} g_k^{x_{i,k}} \cdot t_{\ell(i)} && \text{for } 1 \leq i \leq n, \\ [g_i, g_j] &= \textstyle\prod_{k=i+1}^{n} g_k^{y_{i,j,k}} \cdot t_{\ell(i,j)} && \text{for } 1 \leq j < i \leq n, \end{aligned}$$

with the tails t_ℓ being central. This presentation gives a central extension $G_\emptyset^*$ of $\langle t_\ell \mid 1 \leq \ell \leq m \rangle$ by G, but the given relations may not determine a consistent power-commutator presentation. Evaluating the consistency relations

$$\begin{aligned} g_k(g_j g_i) &= (g_k g_j) g_i && \text{for } k > j > i, \\ (g_j^{e_j}) g_i &= g_j^{e_j - 1}(g_i g_i) && \text{for } j > i, \\ g_j(g_i^{e_i}) &= (g_j g_i) g_i^{e_i - 1} && \text{for } j > i, \\ (g_i^{e_i}) g_i &= g_i(g_i^{e_i}) && \text{for all } i \end{aligned}$$

in the extension gives a system of relations between the tails. Having these in mind, the above presentation of $G_\emptyset^*$ amounts to a pc-presented quotient of the universal central extension G^* of the quotient system in the sense of [Nickel 1993], backed by the theory of the *tails routine* and *consistency checks*, see [Nickel 1993, Sims 1994, EN 2008]. Beside the consistency enforced relations, we evaluate the commutators $[g, h]$ in the extension with the elements g, h commuting in G, which potentially impose some new tail relations. In the language of exterior squares, this step amounts to determining the subgroup $\mathrm{M}_0(G)$ of the Schur multiplier, see [Moravec 2012]. This is computationally the most demanding part of the algorithm, since it does in general not suffice to inspect only commuting pairs made up of the polycyclic generators. The procedure may be simplified by noticing that the conjugacy class of a single commutator induces the same relation throughout. For this purpose, we work with a pc-presented version of the group in our algorithm, for which the implemented algorithm for determining conjugacy classes in GAP is much faster than the corresponding one for polycyclic groups. Let G_0^* be the group obtained by factoring $G_\emptyset^*$ by these additional relations. Computationally, we do this by applying Gaussian elimination over the integers to produce a generating set for all of the relations between the tails at once, and collect them in a matrix T. Applying a transition matrix Q^{-1} to obtain the Smith normal form of $T = PSQ$ gives a new basis for the tails, say t_ℓ^*. The abelian invariants of the group generated by the tails are recognised as the elementary divisors of T. Finally, the Bogomolov multiplier of G is identified as the torsion subgroup of $\langle t_\ell^* \mid 1 \leq \ell \leq m \rangle$ inside G_0^*, the theoretical background of this being the following proposition.

Proposition 2.1. *Let G be a finite group, presented by $G = F/R$ with F free of rank n. Denote by $\mathrm{K}(F)$ the set of commutators in F. Then $\mathrm{B}_0(G)$ is isomorphic to the torsion subgroup of $R/\langle \mathrm{K}(F) \cap R \rangle$, and the torsion-free factor $R/([F, F] \cap R)$ is free abelian of rank n. Moreover, every complement C to $\mathrm{B}_0(G)$ in $R/\langle \mathrm{K}(F) \cap R \rangle$ yields a commutativity preserving central extension of $\mathrm{B}_0(G)$ by G.*

Proof. Using the Hopf formula for the Bogomolov multiplier $\mathrm{B}_0(G) \cong ([F, F] \cap R)/\langle \mathrm{K}(F) \cap R \rangle$ from [Moravec 2012], the proposition follows from the arguments given in [Karpilovsky 1987, Corollary 2.4.7]. By construction and [EN 2008], we have $G_0^* \cong F/\langle \mathrm{K}(F) \cap R \rangle$, and the complement C gives the extension G_0^*/C. □

Taking the derived subgroup of the extension G_0^* and factoring it by a complement of the torsion part of the subgroup generated by the tails thus gives a consistent power-commutator presentation of the curly exterior square $G \curlywedge G$, see [EN 2008, Moravec 2012]. With each of the groups below, we also output the presentation of G_0^* factored by a complement of $\mathrm{B}_0(G)$ and expressed in the new tail basis t_i^* as to explicitly point to the nonuniversal commutator relations with respect to the commutator presentation of the original group.

Lastly, we compare our algorithm to the one given in [Moravec 2012] and existing algorithms based on other approaches [HAP]. The original algorithm from [Moravec 2012] was designed only to determine $\mathrm{B}_0(G)$; our approach furthermore explicitly constructs a central extension of the Bogomolov multiplier by the group G, which makes it possible to trace and in the end also recognize the commutator relations that constitute $\mathrm{B}_0(G)$. Moreover, our implementation adapts the algorithm [EN 2008] rather than directly extending it by not adding the tails that correspond to trivial commutators of the polycyclic generating sequence in the first place. With respect to more homological, cohomological and tensor implementations [HAP], our algorithm is specialized for polycyclic groups. As such, it is as a rule more efficient, particularly with groups of larger orders. This is of course also a limitation of our algorithm, but in fact not a big obstacle, since the p-part of $\mathrm{B}_0(G)$ embeds into $\mathrm{B}_0(S)$, where S is the Sylow p-subgroup of G, see [BMP 2004].

Time tests on different classes of groups are presented in Table 1, time is given in seconds. Our algorithm is implemented in the function `DetermineBog` [GAP code], the original algorithm from [Moravec 2012] in `Bog`, and HAP's standard version in `BogomolovMultiplier`. They have been run on a standard laptop computer. When using HAP's algorithm, all the groups have been transformed into pc-presented groups, as the algorithm works significantly slower for polycyclic groups.

Table 1: Time comparison with existing algorithms for determining the Bogomolov multiplier.

	`DetermineBog`	`Bog`	`BogomolovMultiplier`
`SmallGroup(128,100)`	0.08	0.20	0.09
`SmallGroup(128,1544)`	0.07	0.17	1.69
`AllSmallGroups(128)`	207.43	542.81	309.89
`DihedralGroup(2^14)`	25.52	184.82	64.62
`UnitriangularGroup(5,3)`	1.25	N/A	10.57

3 A summary of results

There are precisely 11 isoclinism families of groups of order 128 whose Bogomolov multipliers are nontrivial. These are the families Φ_i with $i \in \{16, 30, 31, 37, 39, 43, 58, 60, 80, 106, 114\}$ of [JNO 1990]. Their multipliers are all isomorphic to C_2, except those of the family Φ_{30} with which we get $C_2 \times C_2$. The exceptional groups belonging to the latter family have been, together with their odd prime counterpart, further investigated in [JM 2013b]. For each of

the families with nontrivial multipliers, we also give the identification number as implemented in GAP of a selected representative that was used for determining the family's multiplier. The results are collected in Table 3.

Table 3: Isoclinism families of groups of order 128 with nontrivial Bogomolov multipliers.

Family	GAP ID	B_0
16	227	C_2
30	1544	$C_2 \times C_2$
31	1345	C_2
37	242	C_2
39	36	C_2
43	1924	C_2
58	417	C_2
60	446	C_2
80	950	C_2
106	144	C_2
114	138	C_2

All-in-all, there are 230 groups of order 128 with nontrivial Bogomolov multipliers out of a total of 2328 groups of this order. For all these groups, Noether's rationality problem [Noether 1916] therefore has a negative solution.

4 The calculations

16. Let the group G be the representative of this family given by the presentation

$$\begin{aligned}\langle g_1, g_2, g_3, g_4, g_5, g_6, g_7 \mid\ & g_1^2 = g_5, \\ & g_2^2 = 1, \quad [g_2, g_1] = g_4, \\ & g_3^2 = 1, \quad [g_3, g_1] = g_7, \quad [g_3, g_2] = g_6 g_7, \\ & g_4^2 = g_6, \quad [g_4, g_1] = g_6, \quad [g_4, g_2] = g_6, \\ & g_5^2 = g_7, \\ & g_6^2 = 1, \\ & g_7^2 = 1\rangle.\end{aligned}$$

We add 12 tails to the presentation as to form a quotient of the universal central extension of the system: $g_1^2 = g_5 t_1$, $g_2^2 = t_2$, $[g_2, g_1] = g_4 t_3$, $g_3^2 = t_4$, $[g_3, g_1] = g_7 t_5$, $[g_3, g_2] = g_6 g_7 t_6$, $g_4^2 = g_6 t_7$, $[g_4, g_1] = g_6 t_8$, $[g_4, g_2] = g_6 t_9$, $g_5^2 = g_7 t_{10}$, $g_6^2 = t_{11}$, $g_7^2 = t_{12}$. Carrying out

consistency checks gives the following relations between the tails:

$$\begin{aligned}
g_4^2 g_2 = g_4(g_4 g_2) &\implies t_9^2 t_{11} = 1 \\
g_4^2 g_1 = g_4(g_4 g_1) &\implies t_8^2 t_{11} = 1 \\
g_3^2 g_2 = g_3(g_3 g_2) &\implies t_6^2 t_{11} t_{12} = 1 \\
g_3^2 g_1 = g_3(g_3 g_1) &\implies t_5^2 t_{12} = 1 \\
g_2^2 g_1 = g_2(g_2 g_1) &\implies t_3^2 t_7 t_9 t_{11} = 1 \\
g_2 g_1^2 = (g_2 g_1) g_1 &\implies t_3^2 t_7 t_8 t_{11} = 1
\end{aligned}$$

Scanning through the conjugacy class representatives of G and the generators of their centralizers, we see that no new relations are imposed. Collecting the coefficients of these relations into a matrix yields

$$T = \begin{array}{c} \begin{array}{cccccccccccc} t_1 & t_2 & t_3 & t_4 & t_5 & t_6 & t_7 & t_8 & t_9 & t_{10} & t_{11} & t_{12} \end{array} \\ \begin{pmatrix} & & 2 & & & & 1 & & 1 & & 1 & \\ & & & & 2 & & & & & & & 1 \\ & & & & & 2 & & & & & 1 & 1 \\ & & & & & & & 1 & 1 & & 1 & \\ & & & & & & & & 2 & & 1 & \end{pmatrix} \end{array}.$$

A change of basis according to the transition matrix (specifying expansions of t_i^* by t_j)

$$\begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \\ t_6 \\ t_7 \\ t_8 \\ t_9 \\ t_{10} \\ t_{11} \\ t_{12} \end{array} \begin{array}{c} \begin{array}{ccccccccccc} t_1^* & t_2^* & t_3^* & t_4^* & t_5^* & t_6^* & t_7^* & t_8^* & t_9^* & t_{10}^* & t_{11}^* & t_{12}^* \end{array} \\ \begin{pmatrix} & & & & & -1 & -1 & 1 & & & 1 & \\ & & & & & -1 & & -1 & & & -1 & -1 \\ -2 & & & & -2 & & & -1 & & & -1 & \\ & & & & & 4 & & -1 & & & & \\ 4 & & & & 3 & 1 & & & & & & \\ -16 & 2 & -2 & & -13 & -4 & & & & & & \\ -1 & & & & -1 & & 1 & & & & & \\ 16 & -2 & 2 & 1 & 13 & & & 1 & & & & \\ -27 & 4 & -2 & -3 & -21 & & & & 1 & & & \\ & & & & & & & & & 1 & & \\ -14 & 2 & -1 & -1 & -11 & & & & & & 1 & \\ -6 & 1 & -1 & & -5 & & & & & & & 1 \end{pmatrix} \end{array}$$

shows that the nontrivial elementary divisors of the Smith normal form of T are 1, 1, 1, 1, 2. The element corresponding to the divisor that is greater than 1 is t_5^*. This already gives $\mathrm{B}_0(G) \cong \langle t_5^* \mid {t_5^*}^2 \rangle$.

We now deal with explicitly identifying the nonuniversal commutator relation generating $\mathrm{B}_0(G)$. First, factor out by the tails t_i^* whose corresponding elementary divisors are either trivial or 1. Transforming the situation back to the original tails t_i, this amounts to the nontrivial expansion $t_6 = t_5^*$ and all the other tails t_i are trivial. We thus obtain a commutativity preserving central extension of the tails subgroup by G, generated by the sequence g_1, g_2, g_3, g_4, g_5, g_6, g_7, t_5^*,

subject to the following relations:

$$g_1^2 = g_5,\ g_2^2 = g_3^2 = 1,\ g_4^2 = g_6,\ g_5^2 = g_7,\ g_6^2 = g_7^2 = {t_5^*}^2 = 1,$$
$$[g_2, g_1] = g_4,\ [g_3, g_1] = g_7,\ [g_3, g_2] = g_6 g_7 t_5^*,\ [g_4, g_1] = g_6,\ [g_4, g_2] = g_6.$$

Its derived subgroup is isomorphic to the curly exterior square $G \curlywedge G$, whence the nonuniversal commutator relation of G is identified as $t_5^* = [g_3, g_1][g_3, g_2]^{-1}[g_4, g_2]$.

30. Choosing a representative group G of this family and applying the algorithm, we obtain the commutativity preserving central extension of the tails subgroup by G, generated by the sequence g_1, g_2, g_3, g_4, g_5, g_6, g_7, t_4^*, t_5^*, subject to the following relations:

$$g_1^2 = g_2^2 = 1,\ g_3^2 = t_4^*,\ g_4^2 = g_5^2 = g_6^2 = g_7^2 = {t_4^*}^2 = {t_5^*}^2 = 1,$$
$$[g_2, g_1] = g_5,\ [g_3, g_1] = g_6 t_4^*,\ [g_3, g_2] = g_7 t_5^*,\ [g_4, g_2] = g_5 g_6,\ [g_4, g_3] = g_5 t_5^*.$$

Its derived subgroup is isomorphic to the curly exterior square $G \curlywedge G$, whence the nonuniversal commutator relations of G are identified as $t_4^* = [g_2, g_1][g_3, g_1][g_4, g_2]^{-1}$ and $t_5^* = [g_2, g_1][g_4, g_3]^{-1}$, and we have $\mathrm{B}_0(G) \cong \langle t_4^*,\ t_5^* \mid {t_4^*}^2,\ {t_5^*}^2 \rangle$.

31. Choosing a representative group G of this family and applying the algorithm, we obtain the commutativity preserving central extension of the tails subgroup by G, generated by the sequence g_1, g_2, g_3, g_4, g_5, g_6, g_7, t_4^*, subject to the following relations:

$$g_1^2 = g_2^2 = g_3^2 = g_4^2 = g_5^2 = g_6^2 = g_7^2 = {t_4^*}^2 = 1,$$
$$[g_2, g_1] = g_5,\ [g_3, g_1] = g_6 t_4^*, [g_3, g_2] = g_7,\ [g_4, g_3] = g_5 t_4^*.$$

Its derived subgroup is isomorphic to the curly exterior square $G \curlywedge G$, whence the nonuniversal commutator relation of G is identified as $t_4^* = [g_2, g_1][g_4, g_3]^{-1}$, and we have $\mathrm{B}_0(G) \cong \langle t_4^* \mid {t_4^*}^2 \rangle$.

37. Choosing a representative group G of this family and applying the algorithm, we obtain the commutativity preserving central extension of the tails subgroup by G, generated by the sequence g_1, g_2, g_3, g_4, g_5, g_6, g_7, t_5^*, subject to the following relations:

$$g_1^2 = g_5 t_5^*,\ g_2^2 = g_3^2 = 1,\ g_4^2 = g_7,\ g_5^2 = g_6^2 = g_7^2 = {t_5^*}^2 = 1,$$
$$[g_2, g_1] = g_4 t_5^*,\ [g_3, g_1] = g_7 t_5^*,\ [g_4, g_1] = g_6,\ [g_4, g_2] = g_7,\ [g_5, g_2] = g_6 g_7.$$

Its derived subgroup is isomorphic to the curly exterior square $G \curlywedge G$, whence the nonuniversal commutator relation of G is identified as $t_5^* = [g_3, g_1][g_4, g_2]^{-1}$, and we have $\mathrm{B}_0(G) \cong \langle t_5^* \mid {t_5^*}^2 \rangle$.

39. Choosing a representative group G of this family and applying the algorithm, we obtain the commutativity preserving central extension of the tails subgroup by G, generated by the sequence g_1, g_2, g_3, g_4, g_5, g_6, g_7, t_5^*, subject to the following relations:

$$g_1^2 = g_4,\ g_2^2 = g_5,\ g_3^2 = t_5^*,\ g_4^2 = g_5^2 = g_6^2 = g_7^2 = {t_5^*}^2 = 1,$$
$$[g_2, g_1] = g_3,\ [g_3, g_1] = g_6 t_5^*,\ [g_3, g_2] = g_7 t_5^*,\ [g_4, g_2] = g_6,\ [g_5, g_1] = g_7.$$

Its derived subgroup is isomorphic to the curly exterior square $G \curlywedge G$, whence the nonuniversal commutator relation of G is identified as $t_5^* = [g_3, g_2][g_5, g_1]^{-1}$, and we have $\mathrm{B}_0(G) \cong \langle t_5^* \mid {t_5^*}^2 \rangle$.

43. Choosing a representative group G of this family and applying the algorithm, we obtain the commutativity preserving central extension of the tails subgroup by G, generated by the sequence g_1, g_2, g_3, g_4, g_5, g_6, g_7, t_6^*, subject to the following relations:

$$g_1^2 = t_6^*,\ g_2^2 = t_6^*,\ g_3^2 = 1,\ g_4^2 = t_6^*,\ g_5^2 = 1,\ g_6^2 = g_7,\ g_7^2 = {t_6^*}^2 = 1,$$
$$[g_2, g_1] = g_5,\ [g_3, g_1] = g_6 t_6^*,\ [g_3, g_2] = g_5 g_7 t_6^*,\ [g_4, g_1] = g_5,\ [g_6, g_1] = g_7,\ [g_6, g_3] = g_7.$$

Its derived subgroup is isomorphic to the curly exterior square $G \curlywedge G$, whence the nonuniversal commutator relation of G is identified as $t_6^* = [g_3, g_2][g_4, g_1]^{-1}[g_6, g_3]^{-1}$, and we have $\mathrm{B}_0(G) \cong \langle t_6^* \mid {t_6^*}^2 \rangle$.

58. Choosing a representative group G of this family and applying the algorithm, we obtain the commutativity preserving central extension of the tails subgroup by G, generated by the sequence g_1, g_2, g_3, g_4, g_5, g_6, g_7, t_6^*, subject to the following relations:

$$g_1^2 = 1,\ g_2^2 = g_4,\ g_3^2 = 1,\ g_4^2 = g_6,\ g_5^2 = g_7,\ g_6^2 = g_7^2 = {t_6^*}^2 = 1,$$
$$[g_2, g_1] = g_4,\ [g_3, g_1] = g_5,\ [g_3, g_2] = g_6 t_6^*,\ [g_4, g_1] = g_6,\ [g_5, g_1] = g_7,\ [g_5, g_3] = g_7.$$

Its derived subgroup is isomorphic to the curly exterior square $G \curlywedge G$, whence the nonuniversal commutator relation of G is identified as $t_6^* = [g_3, g_2][g_4, g_1]^{-1}$, and we have $\mathrm{B}_0(G) \cong \langle t_6^* \mid {t_6^*}^2 \rangle$.

60. Choosing a representative group G of this family and applying the algorithm, we obtain the commutativity preserving central extension of the tails subgroup by G, generated by the sequence g_1, g_2, g_3, g_4, g_5, g_6, g_7, t_5^*, subject to the following relations:

$$g_1^2 = t_5^*,\ g_2^2 = g_4,\ g_3^2 = g_5,\ g_4^2 = g_6,\ g_5^2 = g_7,\ g_6^2 = g_7^2 = {t_5^*}^2 = 1,$$
$$[g_2, g_1] = g_4 t_5^*,\ [g_3, g_1] = g_5 t_5^*,\ [g_3, g_2] = g_6 t_5^*,\ [g_4, g_1] = g_6,\ [g_5, g_1] = g_7.$$

Its derived subgroup is isomorphic to the curly exterior square $G \curlywedge G$, whence the nonuniversal commutator relation of G is identified as $t_5^* = [g_3, g_2][g_4, g_1]^{-1}$, and we have $\mathrm{B}_0(G) \cong \langle t_5^* \mid {t_5^*}^2 \rangle$.

80. Choosing a representative group G of this family and applying the algorithm, we obtain the commutativity preserving central extension of the tails subgroup by G, generated by the sequence g_1, g_2, g_3, g_4, g_5, g_6, g_7, t_5^*, subject to the following relations:

$$g_1^2 = t_5^*,\ g_2^2 = g_4 g_6,\ g_3^2 = 1,\ g_4^2 = g_6 g_7 t_5^*,\ g_5^2 = 1,\ g_6^2 = g_7,\ g_7^2 = {t_5^*}^2 = 1,$$
$$[g_2, g_1] = g_4 t_5^*,\ [g_3, g_1] = g_5 t_5^*,\ [g_3, g_2] = g_7 t_5^*,\ [g_4, g_1] = g_6 t_5^*,\ [g_6, g_1] = g_7.$$

Its derived subgroup is isomorphic to the curly exterior square $G \curlywedge G$, whence the nonuniversal commutator relation of G is identified as $t_5^* = [g_3, g_2][g_6, g_1]^{-1}$, and we have $\mathrm{B}_0(G) \cong \langle t_5^* \mid {t_5^*}^2 \rangle$.

106. Choosing a representative group G of this family and applying the algorithm, we obtain the commutativity preserving central extension of the tails subgroup by G, generated by the sequence g_1, g_2, g_3, g_4, g_5, g_6, g_7, t_9^*, subject to the following relations:

$$g_1^2 = g_4,\ g_2^2 = g_6 t_9^*,\ g_3^2 = g_6 g_7 t_9^*,\ g_4^2 = 1,\ g_5^2 = g_7,\ g_6^2 = g_7^2 = {t_9^*}^2 = 1,$$
$$[g_2, g_1] = g_3,\ [g_3, g_1] = g_5 t_9^*,\ [g_3, g_2] = g_6 t_9^*,\ [g_4, g_2] = g_5 g_6,\ [g_4, g_3] = g_6 g_7,$$
$$[g_5, g_1] = g_6,\ [g_5, g_2] = g_7,\ [g_5, g_4] = g_7,\ [g_6, g_1] = g_7.$$

Its derived subgroup is isomorphic to the curly exterior square $G \curlywedge G$, whence the nonuniversal commutator relation of G is identified as $t_9^* = [g_3, g_2][g_5, g_1]^{-1}$, and we have $\mathrm{B}_0(G) \cong \langle t_9^* \mid {t_9^*}^2 \rangle$.

114. Choosing a representative group G of this family and applying the algorithm, we obtain the commutativity preserving central extension of the tails subgroup by G, generated by the sequence g_1, g_2, g_3, g_4, g_5, g_6, g_7, t_9^*, subject to the following relations:

$$g_1^2 = g_4,\ g_2^2 = t_9^*,\ g_3^2 = g_6 t_9^*,\ g_4^2 = 1,\ g_5^2 = g_7,\ g_6^2 = g_7^2 = {t_9^*}^2 = 1,$$
$$[g_2, g_1] = g_3,\ [g_3, g_1] = g_5 t_9^*,\ [g_3, g_2] = g_6 t_9^*,\ [g_4, g_2] = g_5 g_6 g_7,\ [g_4, g_3] = g_6 g_7,$$
$$[g_5, g_1] = g_6,\ [g_5, g_2] = g_7,\ [g_5, g_4] = g_7,\ [g_6, g_1] = g_7.$$

Its derived subgroup is isomorphic to the curly exterior square $G \curlywedge G$, whence the nonuniversal commutator relation of G is identified as $t_9^* = [g_3, g_2][g_5, g_1]^{-1}$, and we have $\mathrm{B}_0(G) \cong \langle t_9^* \mid {t_9^*}^2 \rangle$.

References

[BMP 2004] F. A. Bogomolov, J. Maciel, and T. Petrov, *Unramified Brauer groups of finite simple groups of type A_ℓ*, Amer. J. Math. **126** (2004), 935–949.

[CM 2013] Y. Chen, and R. Ma, *Bogomolov multipliers of groups of order p^6*, arXiv:1302.0584 [math.AG] (2013).

[CHKK 2009] H. Chu, S. Hu., M. Kang, and B. E. Kunyavskiĭ, *Noether's problem and the unramified Brauer group for groups of order* 64, Int. Math. Res. Not. IMRN **12** (2010), 2329–2366.

[CHKP 2008] H. Chu, S. Hu., M. Kang, and Y. G. Prokhorov, *Noether's problem for groups of order* 32, J. Algebra **320** (2008), 3022–3035.

[EN 2008] B. Eick and W. Nickel, *Computing the Schur multiplicator and the nonabelian tensor square of a polycyclic group*, J. Algebra **320** (2008), no. 2, 927–944.

[HAP] G. Ellis, GAP *package HAP – Homological Algebra Programming, Version 1.10.15*; 2014, (`http://www.gap-system.org/Packages/hap.html`).

[GAP] The GAP Group, GAP – *Groups, Algorithms, and Programming, Version 4.7.2*; 2013, (`http://www.gap-system.org`).

[Hall 1940] P. Hall, *The classification of prime-power groups*, J. Reine Angew. Math. **182** (1940), 130–141.

[HK 2011] A. Hoshi, and M. Kang, *Unramified Brauer groups for groups of order p^5*, arXiv:1109.2966 [math.AC] (2011).

[HKK 2012] A. Hoshi, M. Kang, and B. E. Kunyavskiĭ, *Noether problem and unramified Brauer groups*, arXiv:1202.5812 [math.AG] (2012).

[JNO 1990] R. James, M. F. Newman, and E. A. O'Brien, *The groups of order* 128, J. Algebra **129** 1 (1990), 136–158.

[JM 2013a] U. Jezernik and P. Moravec, *Bogomolov multipliers of all groups of order* 128, arXiv:1312.4754 [math.GR] (2013).

[JM 2013b] U. Jezernik and P. Moravec, *Universal commutator relations, Bogomolov multipliers, and commuting probability*, arXiv:1307.6533 [math.GR] (2013).

[GAP code] U. Jezernik and P. Moravec, GAP *code for computing Bogomolov multipliers of finite solvable groups*, 2014,
(`http://www.fmf.uni-lj.si/~moravec/Papers/computing-bog.g`).

[Karpilovsky 1987] G. Karpilovsky, *The Schur multiplier*, London Mathematical Society Monographs. New Series, 2. The Clarendon Press, Oxford University Press, New York, 1987.

[Kunyavskiĭ 2008] B. E. Kunyavskiĭ, *The Bogomolov multiplier of finite simple groups*, Cohomological and geometric approaches to rationality problems, 209–217, Progr. Math., 282, Birkhäuser Boston, Inc., Boston, MA, 2010.

[Moravec 2012] P. Moravec, *Unramified groups of finite and infinite groups*, Amer. J. Math. **134** (2012), no. 6, 1679–1704.

[Moravec 2014] P. Moravec, *Unramified Brauer groups and isoclinism*, Ars Math. Contemp. **7** (2014), 337–340.

[Nickel 1993] W. Nickel, *Central extensions of polycyclic groups*, PhD thesis, Australian National University, Canberra, 1993.

[Noether 1916] E. Noether, *Gleichungen mit vorgeschriebener Gruppe*, Math. Ann. **78** (1916), 221–229.

[Sims 1994] C. C. Sims, *Computation with finitely presented groups*, Cambridge University Press, Cambridge, 1994.