ON NONABELIAN TENSOR ANALOGUES OF 2-ENGEL CONDITIONS

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Abstract. Tensor analogues of right 2-Engel elements in groups were introduced by D. P. Biddle and L.-C. Kappe. We investigate the properties of right 2-Engel tensor elements and introduce the concept of $2_\otimes$-Engel margin. With the help of these results we describe the structure of $2_\otimes$-Engel groups. In particular, we prove a tensor version of Levi's theorem for 2-Engel groups and determine tensor squares of two-generator $2_\otimes$-Engel p-groups.

1. Introduction

For any group G, *the nonabelian tensor square* $G \otimes G$ is a group generated by the symbols $g \otimes h$, subject to the relations

$$gg' \otimes h = (g^{g'} \otimes h^{g'})(g' \otimes h) \quad \text{and} \quad g \otimes hh' = (g \otimes h')(g^{h'} \otimes h^{h'}),$$

where $g, g', h, h' \in G$ and $g^h = h^{-1}gh$. The more general concept of nonabelian tensor product of groups acting on each other in certain compatible way was introduced by R. Brown and J.-L. Loday in [5], following the ideas of R. K. Dennis [6]. This construction has its origins in algebraic K-theory as well as in homotopy theory, yet it has become interesting from a purely group-theoretical point of view since the paper of R. Brown, D. L. Johnson and E. F. Robertson [4]. Since then, many authors have been concerned with explicit computations of nonabelian tensor squares; see the paper of L.-C. Kappe [9] for a comprehensive survey of these results.

The main topic of [3] is consideration of tensor analogues of the center and centralizers in groups. More precisely, for a given group G the subgroup $Z^\otimes(G)$ consisting of all $a \in G$ with $a \otimes x = 1_\otimes$ for every $x \in G$ is called *the tensor center*. This concept was introduced by G. J. Ellis [7]. Moreover, for a group G and a non-empty subset X, the subgroup $C_G^\otimes(X) = \{a \in G : a \otimes x = 1_\otimes \text{ for all } x \in X\}$ is said to be *the tensor annihilator* of X in G. Also, tensor analogues of right n-Engel elements have been defined. Recall that *the set of right n-Engel elements* of a group G is defined by $R_n(G) = \{a \in G : [a, {}_n x] = 1 \text{ for all } x \in G\}$. Here $[a, {}_n x]$ stands for the commutator $[\cdots[[a, x], x], \cdots]$ with n copies of x. It is well-known that $R_1(G) = Z(G)$ and that $R_2(G)$ is a subgroup of G [13]. In contrast

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with this, it was shown that for $n \geq 3$ the set $R_n(G)$ is not necessarily a subgroup [14]. *The set of right $n_\otimes$-Engel elements* of a group G is then defined as

$$R_n^\otimes(G) = \{a \in G \,:\, [a,_{n-1}x] \otimes x = 1_\otimes \text{ for all } x \in G\}.$$

One of the results of [3] shows that $R_2^\otimes(G)$ is always a characteristic subgroup of G containing $Z(G)$ and contained in $R_2(G)$. It is also shown by an example that these inclusions may be proper.

The purpose of this paper is to further investigate tensor analogues of 2-Engel structure in groups. In the first part of the paper we determine some further information about $R_2^\otimes(G)$ and provide some new characterizations of this subgroup. In particular, we define the tensor analogue of 2-Engel margin and show that there is a striking resemblance between the results about 2-Engel margin and the results about its tensor analogue. We use these results to obtain the structure of $2_\otimes$-Engel groups. Here the group G is said to be *$n_\otimes$-Engel* when $[x,_{n-1}y] \otimes y = 1_\otimes$ for any $x, y \in G$. It is straightforward to see that every $2_\otimes$-Engel group is also 2-Engel. A well-known result of F. W. Levi (see [15, pp. 45–46]) states that every 2-Engel group G is metabelian and nilpotent of class ≤ 3 and the exponent of $\gamma_3(G)$ divides 3. Therefore it is hardly surprising that the following result is obtained: If G is a $2_\otimes$-Engel group, then $G \otimes G$ is abelian, $\gamma_3(G) \leq Z^\otimes(G)$ and $([x, y] \otimes z)^3 = 1_\otimes$ for every $x, y, z \in G$. As a consequence, we obtain several characterizations of $2_\otimes$-Engel groups, once again indicating the strong correspondence between 2-Engel groups and $2_\otimes$-Engel groups.

Let $\mathfrak{G}$ be a group-theoretic property. A group G is said to have *a finite covering by $\mathfrak{G}$-subgroups* if G equals, as a set, to the union of finite family of $\mathfrak{G}$-subgroups. The finite coverings of groups by their 2-Engel subgroups were studied by L.-C. Kappe [10]. It is proved in that paper that a group G has a finite covering by 2-Engel subgroups if and only if $|G : R_2(G)| < \infty$. The situation is similar in the context of $2_\otimes$-Engel groups. We prove that a group G can be covered by a finite family of $2_\otimes$-Engel subgroups if and only if $|G : R_2^\otimes(G)| < \infty$. Another result of [10] in this direction is that G has a finite covering by 2-Engel normal subgroups if and only if G is 3-Engel and $|G : R_2(G)| < \infty$. It is to be expected that there is a tensor analogue of this result, but we leave it for future consideration. It is not difficult to see that if G has a finite covering by $2_\otimes$-Engel normal subgroups, then G is $3_\otimes$-Engel and $|G : R_2^\otimes(G)| < \infty$. For the reverse conclusion one would probably need the characterization of $3_\otimes$-Engel groups by their normal closures analogous to [12].

Since every $2_\otimes$-Engel group has an abelian tensor square, there is a good chance to compute tensor squares of $2_\otimes$-Engel groups explicitly. We reduce these computations to consideration of tensor squares of groups of class ≤ 2.

With the help of this we compute tensor squares of two-generator $2_{\otimes}$-Engel p-groups, using the results of [1] and [11]. It is worth mentioning that there is a minor error in the classification of two-generator p-groups of class 2 given by [1], so we give the correct result here. We also compute the kernel of the commutator map $\kappa : G \otimes G \to G'$ given by $g \otimes h \mapsto [g, h]$ for any nonabelian two-generator $2_{\otimes}$-Engel p-group G. The group $\ker \kappa$ is of interest as it is isomorphic to the third homotopy group of the space $SK(G, 1)$ [5]. Beside that, we compute the Schur multiplier of G.

2. Preliminary results

In this section we summarize without proofs some basic results regarding computations in tensor squares and the results concerning 2-Engel groups which will be used throughout the paper without any further reference. The first lemma gives the right action version of [5, Proposition 3].

Lemma 1 ([5])**.** *Let $g, g', h, h' \in G$. The following relations hold in $G \otimes G$:*

(a) $(g^{-1} \otimes h)^g = (g \otimes h)^{-1} = (g \otimes h^{-1})^h$.
(b) $(g' \otimes h')^{g \otimes h} = (g' \otimes h')^{[g,h]}$.
(c) $[g, h] \otimes g' = (g \otimes h)^{-1}(g \otimes h)^{g'}$.
(d) $g' \otimes [g, h] = (g \otimes h)^{-g'}(g \otimes h)$.
(e) $[g, h] \otimes [g', h'] = [g \otimes h, g' \otimes h']$.

Note here that G acts on $G \otimes G$ by $(g \otimes h)^{g'} = g^{g'} \otimes h^{g'}$. The next result is crucial in studying the analogy between commutators and tensors.

Proposition 1 ([4])**.** *For a given group G there exists a homomorphism $\kappa : G \otimes G \to G'$ such that $\kappa : g \otimes h \mapsto [g, h]$. Moreover,* $\ker \kappa \leq Z(G \otimes G)$ *and G acts trivially on* $\ker \kappa$.

An element a of a group G is called *a right 2-Engel element* of G if $[a, x, x] = 1$ for each $x \in G$. In a similar fashion, an element a is said to be *a left 2-Engel element* of G if $[x, a, a] = 1$ for each $x \in G$. The sets of right 2-Engel elements and left 2-Engel elements of G are denoted by $R_2(G)$ and $L_2(G)$, respectively. For the properties of right 2-Engel elements we refer to [15, Theorem 7.13] and [16, Lemma 2.2, Theorem 2.3]. We list here some of them, especially those which turn out to have tensor analogues.

Proposition 2 ([15], [16])**.** *Let G be a group, $a \in R_2(G)$ and $x, y, z \in G$.*

(a) *a is also a left 2-Engel element and a^G is abelian.*
(b) $[a, x]^{rs} = [a^r, x^s]$ *for all* $r, s \in \mathbb{Z}$.
(c) $[a, x, y] = [a, y, x]^{-1}$.
(d) $[a, [x, y]] = [a, x, y]^2$.
(e) $a^2 \in Z_3(G)$.
(f) $[a, [x, y], z] = 1$.

Here a^G denotes the normal closure of a in G. This result is the main ingredient of the proof of Levi's theorem [15, pp. 45–46] that every 2-Engel group G is nilpotent of class ≤ 3 and the exponent of $\gamma_3(G)$ divides 3. We also list some characterizations of 2-Engel groups which will serve as a model for $2_\otimes$-Engel groups.

Proposition 3 ([15])**.** *For a group G the following assertions are equivalent:*

(a) *G is a 2-Engel group.*
(b) *$C_G(x)$ is a normal subgroup of G for every $x \in G$.*
(c) *$[x,[y,z]] = [x,y,z]^2$ for any $x,y,z \in G$.*
(d) *$[x,z,y]^{-1} = [x,y,z]$ for any $x,y,z \in G$.*
(e) *x^G is abelian for every $x \in G$.*

3. Right $2_\otimes$-Engel elements of groups

The main object of this section is the study of tensor analogues of right (left) 2-Engel elements of a given group. More precisely, for an arbitrary group G we define *the sets of right (left) $2_\otimes$-Engel elements* of G by $R_2^\otimes(G) = \{a \in G \,:\, [a,x] \otimes x = 1_\otimes \text{ for all } x \in G\}$ and $L_2^\otimes(G) = \{a \in G \,:\, [x,a] \otimes a = 1_\otimes \text{ for all } x \in G\}$, respectively. At the beginning we formulate some elementary properties of these two sets.

Lemma 2. *Let G be any group. We have:*

(a) *$R_2^\otimes(G) \subseteq R_2(G)$, $L_2^\otimes(G) \subseteq L_2(G)$.*
(b) *Every right $2_\otimes$-Engel element of G also belongs to $L_2^\otimes(G)$.*
(c) *$L_2^\otimes(G) = \{a \in G \,:\, a^x \otimes a^y = a \otimes a \text{ for all } x,y \in G\}$.*

Proof. Let $\kappa : G \otimes G \to G'$ be the commutator map. Let $a \in R_2^\otimes(G)$ and $x \in G$. Then we get $1 = \kappa([a,x] \otimes x) = [a,x,x]$, hence $a \in R_2(G)$. The inclusion $L_2^\otimes(G) \subseteq L_2(G)$ is proved in a similar way, therefore (a) is proved. To prove (b), pick $a \in R_2^\otimes(G)$ and $x \in G$. Then we have $1_\otimes = [a,ax] \otimes ax = [a,x] \otimes ax = ([a,x] \otimes a)^x = ([x,a] \otimes a)^{-[a,x]x}$, hence $[x,a] \otimes a = 1_\otimes$ and therefore $a \in L_2^\otimes(G)$. So we are left with the proof of (c). Let $S = \{a \in G \,:\, a^x \otimes a^y = a \otimes a \text{ for all } x,y \in G\}$. For $a \in S$ and $x \in G$ we have $[a,x] \otimes a = a^{-1}a^x \otimes a = (a^{-1} \otimes a)^{a^x}(a^x \otimes a) = 1_\otimes$, hence $a \in L_2^\otimes(G)$. Conversely, let $a \in L_2^\otimes(G)$ and $x,y \in G$. Then we obtain $a^x \otimes a^y = (a^{xy^{-1}} \otimes a)^y = (a[a,xy^{-1}] \otimes a)^y = (a \otimes a)^{[a,xy^{-1}]y}([a,xy^{-1}] \otimes a)^y$. Since G acts trivially on $\ker \kappa$, we have $(a \otimes a)^{[a,xy^{-1}]y} = a \otimes a$, whereas $[a,xy^{-1}] \otimes a = 1_\otimes$ by (b). This proves the assertion. □

The following theorem is already proved in [3]:

Theorem 1 ([3])**.** *For any group G, the set of all right $2_\otimes$-Engel elements of G is a characteristic subgroup of G.*

The computations with tensors involving right $2_\otimes$-Engel elements are facilitated by the following result which has roots in corresponding rules for computation with 2-Engel elements [15, Theorem 7.13]. Before formulating the result, note that

$$Z_n^\otimes(G) = \{a \in G \,:\, [a, x_1, \ldots, x_{n-1}] \otimes x_n = 1_\otimes \text{ for all } x_1, \ldots, x_n \in G\}$$

is a characteristic subgroup of G contained in the n-th center $Z_n(G)$. This subgroup is called *the n-th tensor center* of G [3].

Proposition 4. *Let G be a group, $x, y, z \in G$ and $a \in R_2^\otimes(G)$.*

(a) $[a, x] \otimes y = ([a, y] \otimes x)^{-1}$.
(b) $[a, x] \in C_G^\otimes(x^G)$.
(c) $[a, x]^n \otimes y = ([a, x] \otimes y)^n$ *for any* $n \in \mathbb{Z}$.
(d) $a \otimes x^n = (a \otimes x)^n$ *for any* $n \in \mathbb{Z}$.
(e) $[a, x] \otimes [y, z] = 1_\otimes$.
(f) $[x, y] \otimes a = ([x, a] \otimes y)^2$ *and* $a \otimes [x, y] = ([a, x] \otimes y)^2$.
(g) $a^2 \in Z_3^\otimes(G)$.

Proof. The identities (a) and (b) are already proved in [3, Lemma 5.1 and Lemma 5.2]. To prove (c), it suffices to assume that $n > 0$. Now observe that $[a, x]^n \otimes y = ([a, x] \otimes y)([a, x]^{n-1} \otimes y)$, hence (c) follows by an induction on n.

Before we proceed, note first that (a) implies that the elements of the form $b \otimes z$, where $b \in a^G$ and $z \in G$, commute with each other. Expanding $a \otimes xy$ and $xy \otimes a$ using the tensor product rules, we have

$$a \otimes xy = (a \otimes x)(a \otimes y)([a, x] \otimes y) \tag{1}$$

and

$$xy \otimes a = (x \otimes a)(y \otimes a)([x, a] \otimes y). \tag{2}$$

The first equation yields

$$a \otimes [x, y] = a \otimes (yx)^{-1}(xy) = (a \otimes xy)(a \otimes yx)^{-1}([a, (yx)^{-1}] \otimes xy)$$

by [3, Lemma 5.1]. Since xy is a conjugate of yx, we have $[a, (yx)^{-1}] \otimes xy = 1_\otimes$ by (b), hence $a \otimes [x, y] = ([a, x] \otimes y)^2$. Similarly we prove $a \otimes [x, y] = ([a, x] \otimes y)^2$. It is also clear that the equation (1) also implies (d).

It remains to prove that $[a, x] \otimes [y, z] = 1_\otimes$ and $a^2 \in Z_3^\otimes(G)$. Expanding the identity $[a, x] \otimes yz = ([a, yz] \otimes x)^{-1}$, we obtain $([a, x] \otimes z)([a, x] \otimes y)^z = ([a, z] \otimes x)^{-[a,y]^z}([a, y] \otimes [z^{-1}, x^{-1}]x)^{-z}$. Since $[a, z, x] \otimes [a^z, y^z] = 1_\otimes$, it follows that $[a, y]^z$ acts trivially on $[a, z] \otimes x$. Thus we obtain, after cancellation and relabeling, $1_\otimes = [a, y] \otimes [x, z] = ([a, [x, z]] \otimes y)^{-1} = ([a, x, z]^2 \otimes y)^{-1}$, hence $[a^2, x, y] \otimes z = 1_\otimes$. □

The immediate consequence of Proposition 4 is the following characterization of $R_2^\otimes(G)$.

Corollary 1. *For any group G we have $R_2^{\otimes}(G) = \{a \in G : [a,x] \in C_G^{\otimes}(x^G)$ for all $x \in G\}$.*

It is known that $a \in R_2(G)$ implies that a^G is abelian. The following corollary gives the corresponding result for right $2_{\otimes}$-Engel elements.

Corollary 2. *Let $a \in R_2^{\otimes}(G)$. Then the normal closure $(a \otimes x)^{G \otimes G}$ is an abelian group for any $x \in G$.*

Proof. Let $a \in R_2^{\otimes}(G)$ and $\tau \in G \otimes G$. As usual, denote with κ the commutator map $G \otimes G \to G'$. Then we have $[(a \otimes x), (a \otimes x)^{\tau}] = [a \otimes x, (a \otimes x)^{\kappa(\tau)}] = [a,x] \otimes [a^{\kappa(\tau)}, x^{\kappa(\tau)}] = 1_{\otimes}$ by Proposition 4. It follows by conjugation that every two elements of $(a \otimes x)^{G \otimes G}$ commute, as required. □

Let $\phi(x_1, \ldots, x_n)$ be any word in the variables $x_1, \ldots, x_n$. For a group G *the associated marginal subgroup* $\phi^*(G)$ (also called *the ϕ-margin* of G) consists of all $a \in G$ such that $\phi(g_1, \ldots, ag_i, \ldots, g_n) = \phi(g_1, \ldots, g_i, \ldots, g_n)$ for every $g_i \in G$ and $1 \leq i \leq n$. It is clear that $\phi^*(G)$ is always a characteristic subgroup of G. Margins were first introduced by P. Hall [8]. In particular, marginal subgroups for the 2-Engel word $\phi(x,y) = [x,y,y]$ were studied by T. K. Teague [16]. Let $E_1(G) = \{a \in G : [ax,y,y] = [x,y,y]$ for all $x, y \in G\} = R_2(G)$ and $E_2(G) = \{a \in G : [x, ay, ay] = [x,y,y]$ for all $x, y \in G\}$. Then *the 2-Engel margin* of G is $E(G) = E_1(G) \cap E_2(G)$. Now, the tensor analogues of these subgroups can be defined as

$$E_1^{\otimes}(G) = \{a \in G : [ax,y] \otimes y = [x,y] \otimes y \text{ for all } x, y \in G\},$$
$$E_2^{\otimes}(G) = \{a \in G : [x,ay] \otimes ay = [x,y] \otimes y \text{ for all } x, y \in G\},$$

and let $E^{\otimes}(G) = E_1^{\otimes}(G) \cap E_2^{\otimes}(G)$. It is not difficult to see that these sets are characteristic subgroups of G. Using Proposition 4, we also conclude that $E_1^{\otimes}(G) = R_2^{\otimes}(G)$.

In [16, Theorem 2.4] it is proved that $E(G) = \{a \in G : [x,a,y][x,y,a] = 1$ for all $x, y \in G\}$. The following result is therefore hardly surprising:

Theorem 2. *For any group G we have*

$$E^{\otimes}(G) = \{a \in G : ([x,a] \otimes y)([x,y] \otimes a) = 1_{\otimes} \text{ for all } x, y \in G\}.$$

Proof. Let $S = \{a \in G : ([x,a] \otimes y)([x,y] \otimes a) = 1_{\otimes}$ for all $x, y \in G\}$, let $a \in S$ and $x, y \in G$. It is clear that $a \in R_2^{\otimes}(G) = E_1^{\otimes}(G)$. Using Proposition 4, we have $[x,ay] \otimes ay = [x,y][x,a]^y \otimes ay = ([x,y][x,a]^y \otimes y)([x,y][x,a]^y \otimes a)^y = ([x,y] \otimes y)^{[x,a]^y}([x,a] \otimes y)^y([x,y] \otimes a)^{[x,a]^y y}([x,a]^y \otimes a)^y = ([x,y] \otimes y)^{[x,a]^y}([x,a]^y \otimes a)^y$. Observe that $([x,a]^y \otimes a)^y = (a^{-xy}a^y \otimes a)^y = (a \otimes a)^{-1}(a \otimes a) = 1_{\otimes}$ by Lemma 2, hence we only have to prove that $[x,a]^y$ acts trivially on $[x,y] \otimes y$. To see this, we first note that $y^{[x,a]^y} = [y,[x,a]]y$, hence $([x,y] \otimes y)^{[x,a]^y} = [x,y] \otimes [y,[x,a]]y$. As $[x,a] \in R_2^{\otimes}(G)$, we get $[[x,a],y] \otimes [x,y] = ([[x,a],[x,y]] \otimes y)^{-1} = 1_{\otimes}$ by Proposition 4, thus the inclusion $S \subseteq E^{\otimes}(G)$ is proved. Conversely, every $a \in E^{\otimes}(G)$ also belongs to $R_2^{\otimes}(G)$. Reversing the above arguments, we obtain $a \in S$, as required. □

Let us mention an important consequence of this theorem.

Corollary 3. *Let G be a group, $x, y \in G$ and $a \in E^{\otimes}(G)$. Then $([a,x] \otimes y)^3 = [a^3, x] \otimes y = 1_{\otimes}$.*

Proof. For $a \in E^{\otimes}(G)$ we get $1_{\otimes} = ([x,y] \otimes a)([x,a] \otimes y) = ([x,a] \otimes y)^3$ by Proposition 4, hence also $[a^3, x] \otimes y = 1_{\otimes}$. □

It is proved in [16] that $Z_2(G) \leq E(G) \leq Z_3(G)$ for any group G. Similar arguments show the following.

Proposition 5. *For any group G we have $Z_2^{\otimes}(G) \leq E^{\otimes}(G) \leq Z_3^{\otimes}(G)$.*

Proof. It is clear that $Z_2^{\otimes}(G) \leq E^{\otimes}(G)$. Now, if $a \in E^{\otimes}(G)$, then $a^3 \in Z_2^{\otimes}(G) \leq Z_3^{\otimes}(G)$. On the other hand, we have $a^2 \in Z_3^{\otimes}(G)$ by Proposition 4, hence $a \in Z_3^{\otimes}(G)$. □

4. $2_{\otimes}$-Engel groups

A group G is said to be $2_{\otimes}$-*Engel* when $[x,y] \otimes y = 1_{\otimes}$ for any $x, y \in G$. It is worth noting that G is $2_{\otimes}$-Engel precisely when $R_2^{\otimes}(G) = G$, which is equivalent to $L_2^{\otimes}(G) = G$ and is also equivalent to $E^{\otimes}(G) = G$. Using the commutator map argument, it becomes clear that every $2_{\otimes}$-Engel group is also 2-Engel. The structure of $2_{\otimes}$-Engel groups is described in the next result which corresponds to the well-known Levi's theorem about 2-Engel groups [15, pp. 45–46]:

Theorem 3. *Let G be a $2_{\otimes}$-Engel group. Then we have:*

(a) *$G \otimes G$ is abelian group.*
(b) *$\gamma_3(G) \leq Z^{\otimes}(G)$.*
(c) *$([x,y] \otimes z)^3 = 1_{\otimes}$ for any $x, y, z \in G$.*

Proof. It follows directly from Proposition 4 that $G \otimes G$ is abelian. From the same proposition we obtain $([x,y,z] \otimes v)^2 = [x,y,z]^2 \otimes v = [x,[y,z]] \otimes v = ([x,v] \otimes [y,z])^{-1} = 1_{\otimes}$. Furthermore, since $E^{\otimes}(G) = G$, we get (b) and (c) by Corollary 3. □

In contrast with this result, there exists a 2-Engel group G such that $\mathrm{cl}\,(G \otimes G) = 2$ [2]. The following is a tensor analogue of Proposition 3:

Corollary 4. *The following statements for a group G are equivalent:*

(a) *G is $2_{\otimes}$-Engel.*
(b) *$[x,y] \otimes z = ([x,z] \otimes y)^{-1}$ for any $x, y, z \in G$.*
(c) *$x \otimes [y,z] = ([x,y] \otimes z)^2$ for any $x, y, z \in G$.*
(d) *$x^y \otimes x^z = x \otimes x$ for any $x, y, z \in G$.*

Additionally, if G is a $2_\otimes$-Engel group, then $C_G^\otimes(g) \triangleleft G$ for any $g \in G$.

Proof. By Proposition 4, (a), (b) and (c) are equivalent. The equivalence between (a) and (d) is established in Lemma 2, (c). Now let G be a $2_\otimes$-Engel group, let $g, y \in G$ and let $x \in C_G^\otimes(g) \le C_G(g)$. Then we have $x^y \otimes g = x[x,y] \otimes g = [x,y] \otimes g = ([x,g] \otimes y)^{-1} = 1_\otimes$, thus $x^y \in C_G^\otimes(g)$. This proves the corollary. □

It is evident that the condition "$C_G^\otimes(g) \triangleleft G$ for any $g \in G$" may fail to imply that G is $2_\otimes$-Engel, as $C_G^\otimes(g)$ does not necessarily contain g.

Turning our attention to finite coverings by $2_\otimes$-Engel subgroups, we mention here a related result of L.-C. Kappe [10] which states that a group G has a finite covering by 2-Engel subgroups if and only if $|G : R_2(G)| < \infty$. Our proof of the tensor analogue follows the lines of Kappe's proof.

Theorem 4. *A group G has a finite covering by $2_\otimes$-Engel subgroups if and only if $|G : R_2^\otimes(G)| < \infty$.*

Proof. Suppose that $G = \bigcup_{i=1}^{n} H_i$, where H_i are $2_\otimes$-Engel subgroups of G. The standard reduction step, due to B. H. Neumann (see [10]), shows that we may assume that $|G : H_i| < \infty$ for every i. Hence the subgroup $D = \bigcap_{i=1}^n H_i$ has a finite index in G. It is clear that $D \le R_2^\otimes(G)$, hence $|G : R_2^\otimes(G)| < \infty$.

Assume now $|G : R_2^\otimes(G)| < \infty$. Let $\{g_1, \ldots, g_n\}$ be a transversal of $R_2^\otimes(G)$ in G and let $H_i = \langle g_i \rangle R_2^\otimes(G)$. We have $G = \bigcup_{i=1}^{n} H_i$, hence it suffices to prove that each H_i is $2_\otimes$-Engel. Let $y = g^i a$ and $x = g^j b$ be arbitrary elements of $\langle g \rangle R_2^\otimes(G)$, where $i, j \in \mathbb{Z}$ and $a, b \in R_2^\otimes(G)$. Since $R_2^\otimes(G) = E_1^\otimes(G)$, we obtain, using Proposition 4, $[x,y] \otimes y = [g^j, g^i a] \otimes g^i a = [g^j, a] \otimes g^i a = ([g^j, a] \otimes a)([g^j, a] \otimes g^i)^a = (([g,a] \otimes g)^a)^{ij} = 1_\otimes$, as required. □

Remark. Suppose that a group G has a finite covering by $2_\otimes$-Engel normal subgroups $N_1, \ldots, N_n$. Again we may assume that $|G : N_i| < \infty$ and by Theorem 4 we also have $|G : R_2^\otimes(G)| < \infty$. Since for every $x \in G$ we have $x^G \le N_i$ for some i, we conclude that every normal closure of an element of G is $2_\otimes$-Engel. In particular, we have $1_\otimes = [x^{-y}, x] \otimes x = ([y,x,x] \otimes x)^{x^{-1}}$, hence G is $3_\otimes$-Engel. In view of [10] it is likely that a $3_\otimes$-Engel group G with $|G : R_2^\otimes(G)| < \infty$ has a finite normal covering by $2_\otimes$-Engel subgroups, but we have not been able to (dis)prove this, since there are no known tensor analogues of results regarding 3-Engel groups [12].

5. Tensor squares of $2_\otimes$-Engel groups

We have proved in the previous section that $2_\otimes$-Engel groups have abelian tensor squares. Moreover, if G is a $2_\otimes$-Engel group, then $\gamma_3(G) \le Z^\otimes(G)$ by Theorem 3. Using a result of G. J. Ellis [7], we see that $G \otimes G \cong G/\gamma_3(G) \otimes G/\gamma_3(G)$, hence the calculations of tensor squares reduce to the calculations of tensor squares of class 2 groups (of course, the situation becomes even better when G is abelian).

Let G be a nonabelian two-generator $2_\otimes$-Engel p-group. The group $G/\gamma_3(G)$ is a two-generator $2_\otimes$-Engel p-group of class 2. From [1] and [11] we obtain the complete classification of two-generator p-groups of class 2, hence we only have to check which of these groups are $2_\otimes$-Engel. The following lemma provides a useful criterion for this task:

Lemma 3. *Let G be a two-generator group of class two. Then G is $2_\otimes$-Engel if and only if $G \otimes G \cong G^{ab} \otimes G^{ab}$.*

Proof. Let $G = \langle a, b\rangle$ be a group of class two and let $x, y \in G$. Then $x = a^i b^j [a, b]^k$ and $y = a^{i'} b^{j'} [a, b]^{k'}$ for some $i, i', j, j', k, k' \in \mathbb{Z}$. By means of linear expansion we obtain $[x, y] = [a, b]^{ij'-i'j}$, hence $[x, y] \otimes y = (a \otimes [a, b])^{j'-ii'j'+i'^2 j}(b \otimes [a, b])^{-i'-ij'^2+i'jj'}$. Therefore G is $2_\otimes$-Engel if and only if $a \otimes [a, b] = b \otimes [a, b] = 1_\otimes$, which is equivalent to $x \otimes [y, z] = 1_\otimes$ for all $x, y, z \in G$. By [9, Theorem 3], G is $2_\otimes$-Engel if and only if $G \otimes G \cong G^{ab} \otimes G^{ab}$. □

The recipe for computing tensor squares of two-generator $2_\otimes$-Engel p-groups therefore consists of looking for those two-generator p-groups G of class two which satisfy the condition $G \otimes G \cong G^{ab} \otimes G^{ab}$. Note also that if $G^{ab} \cong \mathbb{Z}_{a_1} \times \cdots \times \mathbb{Z}_{a_r}$, then $G^{ab} \otimes G^{ab}$ is isomorphic to the direct product of all $\mathbb{Z}_{\gcd(a_i,a_j)}$, where $i, j = 1, \ldots, r$.

First assume p is odd. Then we have the following cases [1]:

(*Case 1.*) $G \cong (\langle c\rangle \times \langle a\rangle) \rtimes \langle b\rangle$, where $[a, b] = c$, $[a, c] = [b, c] = 1$, $|a| = p^\alpha$, $|b| = p^\beta$, $|c| = p^\gamma$ and $\alpha \ge \beta \ge \gamma \ge 1$. Here we have $G\otimes G \cong \mathbb{Z}_{p^\alpha} \times \mathbb{Z}^3_{p^\beta} \times \mathbb{Z}^2_{p^\gamma}$, hence $G \otimes G \ncong G^{ab} \otimes G^{ab}$.

(*Case 2.*) $G \cong \langle a\rangle \rtimes \langle b\rangle$, where $[a, b] = a^{p^{\alpha-\gamma}}$, $|a| = p^\alpha$, $|b| = p^\beta$, $|[a, b]| = p^\gamma$ and $\beta \ge \gamma \ge 1$, $\alpha \ge 2\gamma$; by a closer inspection of the proof of [1, Theorem 2.4] it becomes clear that the extra condition $\alpha \ge \beta$ given there is irrelevant. By [1, Theorem 4.2] we have $G\otimes G \cong \langle a\otimes a\rangle \times \langle b\otimes b\rangle \times \langle (b\otimes a)(a\otimes b)\rangle \times \langle b\otimes a\rangle$, where $|a \otimes a| = p^{\alpha-\gamma}$, $|b \otimes b| = p^\beta$, $|(b \otimes a)(a \otimes b)| = p^{\min\{\alpha-\gamma,\beta\}}$ and $|b\otimes a| = n$, where $n = \gcd(p^\alpha, \sum_{k=0}^{p^\beta-1}(p^\alpha - p^{\alpha-\gamma} + 1)^k)$. Applying [1, Lemma 4.1], we immediately obtain $n = p^{\min\{\alpha,\beta\}}$, hence $G \otimes G$ is isomorphic to $\mathbb{Z}_{p^\beta} \times \mathbb{Z}_{p^{\alpha-\gamma}} \times \mathbb{Z}_{p^{\min\{\alpha,\beta\}}} \times \mathbb{Z}_{p^{\min\{\alpha-\gamma,\beta\}}}$. Since $G^{ab} \cong \mathbb{Z}_{p^{\alpha-\gamma}} \times \mathbb{Z}_{p^\beta}$, we get $G^{ab} \otimes G^{ab} \cong \mathbb{Z}_{p^\beta} \times \mathbb{Z}_{p^{\alpha-\gamma}} \times \mathbb{Z}^2_{p^{\min\{\alpha-\gamma,\beta\}}}$. This yields that G is $2_\otimes$-Engel if and only if $\min\{\alpha - \gamma, \beta\} = \min\{\alpha, \beta\}$ which is equivalent to $\alpha \ge \beta + \gamma$.

(*Case 3.*) $G \cong (\langle c\rangle\times\langle a\rangle)\rtimes\langle b\rangle$, where $[a, b] = a^{p^{\alpha-\gamma}} c$, $[c, b] = a^{-p^{2(\alpha-\gamma)}} c^{-p^{\alpha-\gamma}}$, $|a| = p^\alpha$, $|b| = p^\beta$, $|[a, b]| = p^\gamma$, $|c| = p^\sigma$, $\alpha \ge \beta \ge \gamma > \sigma \ge 1$ and $\alpha + \sigma \ge 2\gamma$. Let $\delta = \min\{\alpha - \gamma, \beta\}$ and $\tau = \min\{\alpha - \gamma, \sigma\}$. Then we have $G \otimes G \cong \mathbb{Z}_{p^{\alpha-\gamma}} \times \mathbb{Z}^3_{p^\delta} \times \mathbb{Z}^2_{p^\tau}$, hence it is not isomorphic to $G^{ab} \otimes G^{ab}$.

For $p = 2$ the situation is more complicated [11]:

(*Case 4.*) $G \cong (\langle c\rangle \times \langle a\rangle) \rtimes \langle b\rangle$, where $[a, b] = c$, $[a, c] = [b, c] = 1$, $|a| = 2^\alpha$,

$|b| = 2^\beta$, $|c| = 2^\gamma$ and $\alpha \geq \beta \geq \gamma \geq 1$. Here we have

$$G \otimes G \cong \begin{cases} \mathbb{Z}_{2^\alpha} \times \mathbb{Z}_{2^\beta}^3 \times \mathbb{Z}_{2^\gamma}^2, & : \ \beta > \gamma, \\ \mathbb{Z}_{2^\alpha} \times \mathbb{Z}_{2^\gamma}^2 \times \mathbb{Z}_{2^{\gamma+1}} \times \mathbb{Z}_{2^{\gamma-1}} \times \mathbb{Z}_{2^{\min\{\alpha-1,\gamma\}}} & : \ \beta = \gamma. \end{cases}$$

It follows from here that $G \otimes G \not\cong G^{ab} \otimes G^{ab}$.
(*Case 5.*) $G \cong \langle a \rangle \rtimes \langle b \rangle$, where $[a,b] = a^{2^{\alpha-\gamma}}$ $|a| = 2^\alpha$, $|b| = 2^\beta$, $|[a,b]| = 2^\gamma$ and $\alpha, \beta, \gamma \in \mathbb{N}$, $\alpha \geq 2\gamma$, $\beta \geq \gamma$ and $\alpha + \beta > 3$. In this particular case, $G \otimes G$ is isomorphic to $\mathbb{Z}_{2^\beta} \times \mathbb{Z}_{2^{\alpha-\gamma+1}} \times \mathbb{Z}_{2^{\min\{\alpha-\gamma,\beta\}}} \times \mathbb{Z}_{2^{\min\{\alpha,\beta\}}}$. It is straightforward to verify that $G \otimes G \not\cong G^{ab} \otimes G^{ab}$.
(*Case 6.*) $G \cong (\langle c \rangle \times \langle a \rangle) \rtimes \langle b \rangle$, where $[a,b] = a^{2^{\alpha-\gamma}} c$, $[c,b] = a^{-2^{2(\alpha-\gamma)}} c^{-2^{\alpha-\gamma}}$, $|a| = 2^\alpha$, $|b| = 2^\beta$, $|[a,b]| = 2^\gamma$, $|c| = 2^\sigma$ with $\alpha, \beta, \gamma, \sigma \in \mathbb{N}$, $\alpha + \sigma \geq 2\gamma$ and $\beta \geq \gamma > \sigma$. Let $\rho = \min\{\alpha - \gamma + \sigma, \beta\}$. Then we have

$$G \otimes G \cong \begin{cases} \mathbb{Z}_{2^\gamma}^3 \times \mathbb{Z}_{2^{\gamma+1}} \times \mathbb{Z}_{2^{\gamma-1}}^2 & : \ \alpha = \gamma + 1\,,\ \beta = \gamma, \\ \mathbb{Z}_{2^{\alpha-\gamma+\sigma+1}} \times \mathbb{Z}_{2^\beta} \times \mathbb{Z}_{2^{\min\{\alpha,\beta\}}} \times \mathbb{Z}_{2^\rho} \times \mathbb{Z}_{2^\sigma}^2 & : \ \alpha \geq \gamma + 2 \text{ or } \beta \geq \gamma + 1. \end{cases}$$

It is clear that $G \otimes G$ is not isomorphic to $G^{ab} \otimes G^{ab}$.

We summarize our conclusions in the following theorem:

Theorem 5. *Let G be a nonabelian two-generator $2_\otimes$-Engel p-group. Then $p \neq 2$ and $G/\gamma_3(G) \cong \langle a \rangle \rtimes \langle b \rangle$, where $[a,b] = a^{p^{\alpha-\gamma}}$, $|a| = p^\alpha$, $|b| = p^\beta$, $|[a,b]| = p^\gamma$ with $\alpha \geq \beta \geq \gamma \geq 1$, $\alpha \geq 2\gamma$ and $\alpha \geq \beta + \gamma$. We have $G \otimes G \cong \langle a \otimes a \rangle \times \langle b \otimes b \rangle \times \langle (b \otimes a)(a \otimes b) \rangle \times \langle b \otimes a \rangle \cong \mathbb{Z}_{p^\beta}^3 \times \mathbb{Z}_{p^{\alpha-\gamma}}$.*

Our considerations also show the following.

Corollary 5. *Every $2_\otimes$-Engel 2-group is abelian.*

More generally, if G is a $2_\otimes$-Engel group without elements of order 3, then $G' \leq Z^\otimes(G)$ by Theorem 3. This, together with the result of Ellis [7], implies $G \otimes G \cong G^{ab} \otimes G^{ab}$.

Let G be a group. From a topological point of view, the third homotopy group $\pi_3 SK(G,1)$ of the suspension of $K(G,1)$ is of some interest. A combinatorial description of $\pi_n SK(G,1)$ has been given by J. Wu [17]. Observing the formula $\pi_3 SK(G,1) \cong \ker \kappa$ [5], one can use a different approach when $G \otimes G$ is explicitly computed. Applying Theorem 5, we describe $\pi_3 SK(G,1)$ for any nonabelian two-generator $2_\otimes$-Engel p-group G. We also determine the Schur multiplier $H_2(G)$ of G.

Corollary 6. *Let G be a nonabelian two-generator $2_\otimes$-Engel p-group, let $\kappa : G \otimes G \to G'$ be the commutator map and let a, b, α, β, γ be as in Theorem 5. Then $\pi_3 SK(G,1) \cong \ker \kappa \cong \langle a \otimes a \rangle \times \langle b \otimes b \rangle \times \langle (b \otimes a)(a \otimes b) \rangle \times \langle (b \otimes a)^{p^\gamma} \rangle \cong \mathbb{Z}_{p^\beta}^2 \times \mathbb{Z}_{p^{\alpha-\gamma}} \times \mathbb{Z}_{p^{\beta-\gamma}}$ and $H_2(G) \cong \mathbb{Z}_{p^{\beta-\gamma}}$.*

Proof. As $\kappa(a \otimes a) = \kappa(b \otimes b) = \kappa((b \otimes a)(a \otimes b)) = \kappa((b \otimes a)^{p^\gamma}) = 1$, Theorem 5 gives $\ker \kappa \cong \langle a \otimes a \rangle \times \langle b \otimes b \rangle \times \langle (b \otimes a)(a \otimes b) \rangle \times \langle (b \otimes a)^{p^\gamma} \rangle \cong \mathbb{Z}_{p^\beta}^2 \times \mathbb{Z}_{p^{\alpha-\gamma}} \times \mathbb{Z}_{p^{\beta-\gamma}}$, as required. To compute the Schur multiplier of G, note for instance that the exactness of rows and columns in commutative diagram (1) in [4] implies

$H_2(G) \cong \ker \kappa/\Delta(G)$, where $\Delta(G) = \langle x \otimes x : x \in G \rangle$. Now, every $x \in \langle a, b \rangle$ can be written in the form $x = a^m b^n [a, b]^k$, where $m, n, k \in \mathbb{Z}$. Expanding $x \otimes x$ linearly, we obtain $x \otimes x = (a \otimes a)^{m^2} (b \otimes b)^{n^2} ((b \otimes a)(a \otimes b))^{mn}$. This yields $\Delta(G) \cong \langle a \otimes a \rangle \times \langle b \otimes b \rangle \times \langle (b \otimes a)(a \otimes b) \rangle \cong \mathbb{Z}_{p^\beta}^2 \times \mathbb{Z}_{p^{\alpha-\gamma}}$, hence the result. □

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