Locally Graded Bell Groups

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Abstract

For any integer $n \neq 0, 1$, a group is said to be n-Bell if it satisfies the law $[x^n, y] = [x, y^n]$. In this paper we prove that every finitely generated locally graded n-Bell group embeds into the direct product of a finite n-Bell group and a torsion-free nilpotent group of class ≤ 2. We prove that n-Bell groups which are not locally graded always have infinite simple sections of finite exponent. Additionally, we obtain similar results for varieties of n-Levi groups and n-abelian groups defined by the laws $[x^n, y] = [x, y]^n$ and $(xy)^n = x^n y^n$, respectively. We give characterizations of these groups in the locally graded case.

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1 Introduction

Let n be an integer. Following [9], we say that a group G is *n-Bell* if it satisfies the identity $[x^n, y] = [x, y^n]$ for all $x, y \in G$. H. E. Bell studied the identity $[x^n, y] = [x, y^n]$ in rings, see e.g. [3], [4] and [5]. The study of n-Bell groups has also been a subject of several papers, see for instance [6], [7] and [8].

For a group G we denote the set of all right 2-Engel elements of G by $R_2(G)$, i.e., $R_2(G) = \{a \in G : [a, x, x] = 1 \text{ for all } x \in G\}$. W. P. Kappe showed in [10] that this set is always a characteristic subgroup of G. We say that a group G is *n-Kappe* if the factor group $G/R_2(G)$ has finite exponent dividing n. It has been proved in [7, Theorem A] that every n-Bell group is $n(n-1)$-Kappe. Conversely, in [8] it is proved that every n-Kappe group is n^2-Bell. These two results are in a certain sense best possible. In [8] the authors construct an n-Bell group which is not k-Kappe for any $1 < k < n(n-1)$. On the other hand, given a prime p, the group $G = F/F''\gamma_3(F)^p\gamma_{p+2}(F)$, where F is a noncyclic free group, is p-Kappe, yet it is not k-Bell for any $1 < k < p^2$; see [12].

Following [8], we say that a group G is a *Bell group* if it is n-Bell for some integer $n \neq 0, 1$. Similarly, a group G is said to be a *Kappe group* if it is n-Kappe for some nonzero integer n. By the above mentioned results these two classes of groups coincide.

This paper is a further investigation of Bell groups and certain subclasses of groups. From Adjan's examples of torsion-free center-by-finite-exponent groups [1] it becomes apparent that locally graded groups seem to provide the appropriate class in which the Bell property should be discussed. Here a group G is said to be *locally graded* if every nontrivial finitely generated subgroup of G has a nontrivial finite image. The class of locally graded groups includes all locally finite groups, as well as all locally soluble and all residually finite groups. Several papers have been concerned with locally graded groups; see, for instance, [11] and [14]. In particular, H. Smith proved in [14] that if G is a locally graded group then $G/Z_i(G)$ is locally graded, for all $i \geq 0$. Besides, well known results due to Zel'manov [16, 17] show that a locally graded group of finite exponent is locally finite. These results are important ingredients of the characterization of locally graded Bell groups as exactly those groups G for which the factor group $G/Z_2(G)$ is locally finite of finite exponent [8]. We go a step further and prove that every finitely generated locally graded n-Bell group embeds into a direct product of a finite n-Bell group and a torsion-free nilpotent group of class two. This, together with a classification of finite n-Bell groups in [7, Theorem D], gives the complete picture of these groups.

We also study some subclasses of Bell groups. A group G is said to be n-*Levi* if $[x^n, y] = [x, y]^n$ for all $x, y \in G$. The class of n-Levi groups has been introduced by L.-C. Kappe in [9]. Clearly, every n-Levi group is n-Bell. A group G is said to be a *Levi group* if it is n-Levi for some $n \neq 0, 1$. We prove that in the locally graded case the classes of Bell groups and Levi groups coincide. This is no longer true in general, as it is shown by examples due to Adjan [1].

A group G is n-*abelian* if it satisfies the identity $(xy)^n = x^n y^n$. Equivalently, G is n-abelian if the map $x \mapsto x^n$ is an endomorphism of G. If G is n-abelian, then the equality $[x, y]^n = (x^{-1}x^y)^n = x^{-n}(x^n)^y = [x^n, y]$ implies that G is also n-Levi. In [2], n-abelian groups were classified in terms of direct products. In the same manner as above we define a group to be an *Alperin group* if it is n-abelian for some $n \neq 0, 1$. We prove that a locally graded group G is an Alperin group if and only if $G/Z(G)$ has finite exponent.

When groups in question are not locally graded, one can obtain some nasty examples, such as Tarski monsters or the above mentioned groups of Adjan. We show that every Bell group which is not locally graded contains an infinite simple section of finite exponent. A slightly stronger result is obtained for Levi groups. We also prove that every Alperin group which is not locally graded contains an infinite finitely generated subgroup of finite exponent.

2 Results

Our first result shows that, when studying locally graded n-Bell groups, only finite n-Bell groups are interesting.

Theorem 1. *Let G be a finitely generated locally graded n-Bell group. Then G can be embedded into the direct product of a finite n-Bell group and a finitely generated torsion-free nilpotent group of class ≤ 2.*

Proof. Suppose G is a finitely generated locally graded n-Bell group. Since the group $G/Z_2(G)$ is locally graded and periodic by [8, Theorem 3.9], it is finite by [8, Theorem 4.1]. This shows that G is polycyclic-by-finite. From [8, Corollary 4.2] it follows that the elements in G of finite order form a locally finite subgroup T. Furthermore, we may assume G is infinite, hence $Z_2(G)$ is also infinite. By Hirsch's Theorem [13, Part 2, p. 139], $Z_2(G)$ contains a torsion-free characteristic subgroup N of finite index. From this we conclude that N is a normal torsion-free subgroup of finite index in G. As $N \cap T = 1$, we have the natural embedding $g \mapsto (gN, gT)$ of G into $G/N \times G/T$. The group G/N is finite n-Bell, whereas G/T is torsion-free nilpotent of class ≤ 2 by [8, Corollary 4.6]. □

Recall that a group G is said to be *n-Engel* if $[x,_n y] = 1$ for all x and y in G.

Corollary 1. *Let G be a finitely generated locally graded n-Bell group. Let π_1, π_2 be the sets of primes dividing n and $n-1$, respectively. Then G is isomorphic to a subgroup of the direct product $A \times B \times C \times D$, where A is a finite n-Kappe π_1-group, B is a finite $(n-1)$-Kappe π_2-group, C is a finite 2-Engel group of order coprime to $n(n-1)$ and D is a torsion-free nilpotent group of class ≤ 2.*

Proof. This follows at once from Theorem 1 and [7, Theorem D]. □

As the classes of Bell groups and Kappe groups coincide, we can use the same argument as in the proof of Theorem 1 to improve Corollary 2 in [12].

Proposition 1. *Let G be a finitely generated, locally graded n-Kappe group. Then G can be embedded into the direct product of a finite n-Kappe group and a torsion-free nilpotent group of class ≤ 2.*

In particular, we have:

Corollary 2. *Let p be a prime and let G be a finitely generated, locally graded p^n-Kappe group. Then G is isomorphic to a subgroup of the direct product of a finite p^n-Kappe p-group and a 2-Engel group.*

Proof. Suppose G is a finitely generated, locally graded p^n-Kappe group. By Proposition 1, G can be embedded into $A \times B$, where A is a finite p^n-Kappe group and B is torsion-free nilpotent of class ≤ 2. Since $A/R_2(A)$ is of exponent dividing p^n, it is nilpotent. Let c be its nilpotency class. Then A is a $(c+2)$-Engel group. Since A is finite it follows that A is nilpotent by a result of Zorn (see e.g. [13, Part 2, p. 52]). Thus A is the direct product of its Sylow subgroups. But if p' is a prime different from p and P is the p'-Sylow subgroup of A, then $P = R_2(P)$, hence P is 2-Engel. This concludes the proof. □

We turn our attention to Levi groups. They form a subclass of the class of Bell groups. Yet it turns out that these two classes coincide if we put some additional finiteness conditions on the groups. More precisely, we have:

Proposition 2. *The following conditions for a group G are equivalent:*
(a) *G is a locally graded Bell group;*
(b) *G is a locally graded Levi group;*
(c) *$G/Z_2(G)$ is locally finite of finite exponent.*
Moreover, every locally graded n-Bell group is also m-Levi for some integer $m \neq 0, 1$ depending only on n.

Before proving this result, we state here two lemmas that are immediate consequences of well-known results. The first lemma follows from the proof of [13, Part 1, Lemma 4.22].

Lemma 1. *Let G be a group, and let H, K, M, N be normal subgroups of G, with $M \leq H$, $N \leq K$, $[H, N] = 1 = [K, M]$ and $[H, H \cap K] = 1 = [K, H \cap K]$. Then $[H, K]$ is a homomorphic image of $H/H'M \otimes K/K'N$.*

Lemma 2. *Let G be a group with $|G : Z_2(G)|$ finite of order at most n. Then $\gamma_3(G)$ has finite order depending only on n.*

Proof. J. Wiegold [15] proved that if $|G : Z(G)| = n$, then $|G'| \leq n^{(\log_p n - 1)/2}$, where p is the least prime dividing n. This implies that if $|G : Z_2(G)|$ is finite of order at most n then $\gamma_2(G/Z(G)) = G'Z(G)/Z(G)$ is finite of order bounded by a function $f(n)$ which depends only on n. Put $H = G'Z(G)$, $K = G$, $M = Z(G)$ and $N = Z_2(G)$. Since $[H, N] = 1 = [K, M]$, Lemma 1 yields that the group $[H, K]/[H, H \cap K][K, H \cap K] = \gamma_3(G)/G''$ is isomorphic to a factor group of the tensor product $G'Z(G)/G''Z(G) \otimes G/G'Z_2(G)$. Clearly $|G'Z(G)/G''Z(G)| \leq f(n)$ and $|G/G'Z_2(G)| \leq n$. It follows that there exists a function $g(n)$, which depends only on n, such that $|\gamma_3(G)/G''| \leq g(n)$. Thus it suffices to show that the order of G'' is bounded by a function of n. Let $\bar{H} = H/Z(H)$ and let $S = \bar{H} \ltimes H$, where the action of $\bar{H}$ on H is induced by conjugation. The group $H/(N \cap H)$ is isomorphic to a subgroup of $K/N = G/Z_2(G)$, hence its order is at most n. As $M \leq Z(H)$, we conclude that $\bar{H}$ is a homomorphic image of $H/M = G'Z(G)/Z(G) = \gamma_2(G/Z(G))$. This yields that $|\bar{H}| \leq f(n)$, whence the order of $S/(N \cap H)$ is bounded by some function $h(n)$ depending only on n. Note that $N \cap H$ is contained in the center of S, hence $|S/Z(S)| \leq h(n)$. By the above mentioned result of Wiegold, $|S'|$ is bounded by $k(n) = h(n)^{(\log_p h(n) - 1)/2}$, where p is the least prime dividing $h(n)$. Let $H^\star$ be the preimage of $\bar{H}$ in G. Then we have $G'' = [H, H] = [H^\star, H] \leq S'$, whence $|G''| \leq k(n)$, as required. □

Proof of Proposition 2. By Theorem 3.9 and Corollary 4.5. in [8], (a) implies (c). Suppose G is such that $G/Z_2(G)$ is locally finite of exponent e. In particular, G is e-Kappe, hence G is also e^2-Bell by [8, Theorem 2.1]. Therefore (c) implies (a). Thus it remains to show that (a) implies (b).

Let G be a locally graded n-Bell group, and let H be any two-generator subgroup of G. Then $H/Z_2(H)$ is locally graded (by [14]) of exponent dividing $e = 12n^5(n-1)^5$ (by [8, Theorem 3.9]), thus it is finite. Now the solution of the Restricted Burnside Problem [16, 17] implies that the order of $H/Z_2(H)$ is bounded by $|R(2,e)|$, where $R(2,e)$ is the largest two-generator group of exponent e. Moreover, $|R(2,e)|$ depends only on e. ¿From here we conclude that the order of $H/Z_2(H)$ is bounded by some function of n. By Lemma 2, it follows that the order of $\gamma_3(H)$ is bounded by some function of n. Let m be the exponent of $\gamma_3(H)$. Since H is also k-Kappe for $k = n(n-1)$, we have $[x^k, y] \in Z(\langle x, y\rangle)$. Furthermore, we have $[x,y]^k \equiv [x^k, y] \mod \gamma_3(\langle x,y\rangle)$, hence $[x,y]^k = [x^k,y]c$ for some $c \in \gamma_3(\langle x,y\rangle)$. This implies $[x,y]^{km} = [x^k,y]^m c^m = [x^k,y]^m = [x^{km},y]$, hence H is (km)-Levi. As km depends only on n, this concludes the proof. □

Note however that there exist n-Bell groups which are not k-Levi for any $k \neq 0,1$. Let $r > 1$ be an integer and let n be an odd integer, $n \geq 665$. S. I. Adjan [1] has constructed a finitely generated torsion-free group $G = A(r,n)$ such that $G/Z(G)$ is isomorphic to the free r-generator Burnside group $B(r,n)$ of exponent n. It is clear that G is n-central, i.e., $G/Z(G)$ is of exponent dividing n, hence G is also n-Bell. Suppose G is k-Levi for some $k \neq 0,1$. It is not difficult to observe that G is also $k(1-k)$-Levi (see [9]). But G is also a $k(1-k)$-Kappe group by [7, Theorem A]. This implies $[x^{k(1-k)}, y] \in Z(\langle x,y\rangle)$, hence $[x^{k(1-k)t}, y] = [x^{k(1-k)}, y]^t = [x,y]^{k(1-k)t}$ for any integer t. In particular, G is $k(1-k)n$-Levi and $k(1-k)n$-central, hence $[x,y]^{k(1-k)n} = 1$. But G is a torsion-free nonabelian group, hence we have a contradiction. Thus G is not k-Levi for any $k \neq 0,1$.

It is also easy to see that there exist locally graded n-Bell groups which are not k-abelian for any $k \neq 0,1$. For instance, if G is a torsion-free nilpotent group of class two, then G is n-Bell for all integers n. On the other hand, if G were k-abelian for some $k \neq 0,1$, then the identity $(xy)^k = x^k y^k [y,x]^{k(k-1)/2}$ would imply $[x,y] = 1$ for all $x,y \in G$, which is impossible.

In [8, Theorem 4.7] it is proved that if G is a Bell group which is not locally graded, then it contains a finitely generated subgroup H such that $H/Z(H)$ is infinite of finite exponent. Additionally, we have the following.

Theorem 2. *Let G be a Bell group. Then either $G/Z_2(G)$ is a locally finite group or G has an infinite simple section of finite exponent.*

Proof. Let G be n-Bell. If G is locally graded, we are done by Theorem 4.7 of [8]. So we can assume G is not locally graded. By definition there exists a finitely generated subgroup H in G with no nontrivial finite quotients. We have that H is perfect and it follows from [8, Theorem 4.7] that $H/Z(H)$ is infinite of finite exponent. In particular, H is not nilpotent. Let $\{N_i : i \in I\}$ be a chain of normal subgroups of H such that H/N_i is not nilpotent for each $i \in I$. Let N be the union of this chain. If H/N is nilpotent, then it is polycyclic. Hence H/N is finitely presented. By a result of P. Hall (see e.g. [13, Part 1, Lemma 1.43]), N is finitely generated as a normal subgroup of H, that is, $N = \langle a_1, \ldots, a_r \rangle^H$ for some $a_1, \ldots, a_r$ in H. This gives that $N = N_i$ for some $i \in I$, a contradiction. Hence H/N is not nilpotent and Zorn's Lemma implies that there exists a normal subgroup M of H such that H/M is infinite and nonnilpotent, but every proper quotient of H/M is nilpotent. As H is perfect, this implies that H/M is simple. On the other hand, H/M is also m-central for some m, thus it has finite exponent. □

Note that the Adjan groups $A(r, n)$ are torsion-free Bell groups, so the subgroup H in Theorem 4.7 of [8] can be torsion-free. Our next result shows that in Levi groups such a subgroup H has to be generated by elements of finite order.

Proposition 3. *Let G be a Levi group. Then either $G/Z_2(G)$ is locally finite or there exists an infinite subgroup H of G generated by finitely many periodic elements such that $H/Z(H)$ is of finite exponent.*

Proof. Let G be an n-Levi group. If G is locally graded, then the first claim follows from Theorem 2. In case G is not locally graded, we can find a perfect subgroup H of G such that $H/Z(H)$ is of finite exponent m. As $H = H'$, we can assume that H is generated by finitely many commutators. Since H is $n(1-n)$-Levi and $n(1-n)$-Kappe, we obtain that H is also $mn(1-n)$-Levi. Whence $[x, y]^{mn(1-n)} = [x^{mn(1-n)}, y] = 1$ for all $x, y \in H$. This concludes the proof. □

The results can be further improved when we restrict ourselves to Alperin groups. First we prove the following elementary fact.

Proposition 4. *Every n-abelian group is $n(n-1)$-central.*

Proof. Let G be an n-abelian group. This implies that G is $(1-n)$-abelian, hence it is also $n(1-n)$-abelian [9]. In particular, G is $n(1-n)$-Levi, hence we have $[x^{n(1-n)}, y] = [x, y]^{n(1-n)} = [x^n, y^{1-n}] = x^{-n}y^{-1}y^n x^n y^{-n} y = x^{-n}y^{-1}(yxy^{-1})^n y = x^{-n}y^{-1}(x^n)^{y^{-1}} y = 1$, as desired. □

On the other hand, there exist n-central groups which are not m-abelian for any $m \neq 0, 1$; consider, for instance, the Adjan groups $A(r, n)$.

Note that it is proved in [2] that every n-abelian group is isomorphic to a subgroup of the direct product of a group of exponent n, a group of exponent $n-1$ and an abelian group. Using Theorem 1 and Corollary 1 in [2], we can improve this result for finitely generated locally graded n-abelian groups.

Corollary 3. *Let G be a finitely generated, locally graded n-abelian group. Then G is isomorphic to a subgroup of the direct product of a finite group of exponent n, a finite group of exponent $n-1$, and an abelian group.*

It is also possible to improve Theorem 2 for Alperin groups. First we give a characterization of locally graded Alperin groups.

Theorem 3. *Let G be a locally graded group. Then G is an Alperin group if and only if $G/Z(G)$ has finite exponent.*

Proof. Let G be a locally graded n-abelian group. By Proposition 4, the group $G/Z(G)$ is of exponent dividing $n(n-1)$. Conversely, suppose G is a locally graded group such that $G/Z(G)$ is of finite exponent k. Then $G/Z(G)$ is also locally graded, hence it is locally finite. Let H be any two-generator subgroup of G. Then $H/Z(H)$ is finite of order n, and n depends only on k by the solution of the Restricted Burnside Problem [16, 17]. Hence H' is finite by Schur's theorem (see, for instance, [13, Theorem 4.12]), and we have $(H')^n = 1$. Let $x, y \in H$. Then $(xy)^k = x^k y^k c$ for some $c \in H'$, which implies $(xy)^{kn} = x^{kn} y^{kn} c^n = x^{kn} y^{kn}$. Therefore G is (kn)-abelian. □

Theorem 4. *Let G be an Alperin group and suppose G is not locally graded. Then G contains a finitely generated infinite subgroup of finite exponent.*

Proof. Let G be an n-abelian group. If G is not locally graded, there exists a finitely generated subgroup H of G with no nontrivial finite quotients. We have $H = H'$. Since the order of $[x, y]$ divides $n(n-1)$ for all $x, y \in G$ and G is $n(1-n)$-abelian, we conclude that H has finite exponent dividing $n(n-1)$. □

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