Locally finite groups in which every non-cyclic subgroup is self-centralizing

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Abstract

Locally finite groups having the property that every non-cyclic subgroup contains its centralizer are completely classified.

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1. Introduction

A subgroup H of a group G is *self-centralizing* if the centralizer $C_G(H)$ is contained in H. In [1] it has been remarked that a locally graded group in which all non-trivial subgroups are self-centralizing has to be finite; therefore it has to be either cyclic of prime order or non-abelian of order being the product of two different primes.

In this article, we consider the more extensive class $\mathfrak{X}$ of all groups in which every non-cyclic subgroup is self-centralizing. In what follows we use the term $\mathfrak{X}$-groups in order to denote groups in the class $\mathfrak{X}$. The study of properties of $\mathfrak{X}$-groups was initiated in [1]. In particular, the first four authors determined the structure of finite $\mathfrak{X}$-groups which are either nilpotent, supersoluble or simple.

In this paper, Theorem 2.1 gives a complete classification of finite $\mathfrak{X}$-groups. We remark that this result does not depend on classification of the finite simple groups rather only on the classification of groups with dihedral or semidihedral Sylow 2-subgroups. We also determine the infinite soluble $\mathfrak{X}$-groups, and the infinite locally finite $\mathfrak{X}$-groups, the results being presented in Theorems 3.6 and 3.7. It turns out that these latter groups are suitable finite extensions either of the infinite cyclic group or of a Prüfer p-group, $\mathbb{Z}_{p^\infty}$, for some prime

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p. Theorem 3.7 together with Theorem 2.1 provides a complete classification of locally finite $\mathfrak{X}$-groups.

We follow [2] for basic group theoretical notation. In particular, we note that $F^*(G)$ denotes the generalized Fitting subgroup of G, that is the subgroup of G generated by all subnormal nilpotent or quasisimple subgroups of G. The latter subgroups are the components of G. We see from [2, Section 31] that distinct components commute. The fundamental property of the generalized Fitting subgroup that we shall use is that it contains its centralizer in G [2, (31.13)]. We denote the alternating group and symmetric group of degree n by $\mathrm{Alt}(n)$ and $\mathrm{Sym}(n)$ respectively. We use standard notation for the classical groups. The notation $\mathrm{Dih}(n)$ denotes the dihedral group of order n and Q_8 is the quaternion group of order 8. The term quaternion group will cover groups which are often called generalized quaternion groups. The cyclic group of order n is represented simply by n, so for example $\mathrm{Dih}(12) \cong 2 \times \mathrm{Dih}(6) \cong 2 \times \mathrm{Sym}(3)$. Finally $\mathrm{Mat}(10)$ denotes the Mathieu group of degree 10. The Atlas [3] conventions are used for group extensions. Thus, for example, $p^2{:}\mathrm{SL}_2(p)$ denotes the split extension of an elementary abelian group of order p^2 by $\mathrm{SL}_2(p)$.

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2. Finite $\mathfrak{X}$-groups

In this section we determine all the finite groups belonging to the class $\mathfrak{X}$. The main result is the following.

Theorem 2.1. *Let G be a finite $\mathfrak{X}$-group. Then one of the following holds:*

(1) *If G is nilpotent, then either*
- (1.1) *G is cyclic;*
- (1.2) *G is elementary abelian of order p^2 for some prime p;*
- (1.3) *G is an extraspecial p-group of order p^3 for some odd prime p; or*
- (1.4) *G is a dihedral, semidihedral or quaternion 2-group.*

(2) *If G is supersoluble but not nilpotent, then, letting p denote the largest prime divisor of $|G|$ and $P \in \mathrm{Syl}_p(G)$, we have that P is a normal subgroup of G and one of the following holds:*
- (2.1) *P is cyclic and either*
  - (2.1.1) *$G \cong D \ltimes C$, where C is cyclic, D is cyclic and every non-trivial element of D acts fixed point freely on C (so G is a Frobenius group);*
  - (2.1.2) *$G = D \ltimes C$, where C is a cyclic group of odd order, D is a quaternion group, and $C_G(C) = C \times D_0$ where D_0 is a cyclic subgroup of index 2 in D with G/D_0 a dihedral group; or*

(2.1.3) *$G = D \ltimes C$, where D is a cyclic q-group, C is a cyclic q'-group (here q denotes the smallest prime dividing the order of G), $1 < Z(G) < D$ and $G/Z(G)$ is a Frobenius group;*

(2.2) *P is extraspecial and G is a Frobenius group with cyclic Frobenius complement of odd order dividing $p - 1$.*

(3) *If G is not supersoluble and $F^*(G)$ is nilpotent, then either* (3.1) *or* (3.2) *below holds.*

(3.1) *$F^*(G)$ is elementary abelian of order p^2, $F^*(G)$ is a minimal normal subgroup of G and one of the following holds:*

(3.1.1) *$p = 2$ and $G \cong \mathrm{Sym}(4)$ or $G \cong \mathrm{Alt}(4)$; or*

(3.1.2) *p is odd and $G = G_0 \ltimes N$ is a Frobenius group with Frobenius kernel N and Frobenius complement G_0 which is itself an $\mathfrak{X}$-group. Furthermore, either*

(3.1.2.1) *G_0 is cyclic of order dividing $p^2 - 1$ but not dividing $p - 1$;*

(3.1.2.2) *G_0 is quaternion;*

(3.1.2.3) *G_0 is supersoluble as in* (2.1.2) *with $|C|$ dividing $p - \epsilon$ where $p \equiv \epsilon \pmod 4$;*

(3.1.2.4) *G_0 is supersoluble as in* (2.1.3) *with D a 2-group, $C_D(C)$ a non-trivial maximal subgroup of D and $|C|$ odd dividing $p - 1$ or $p + 1$;*

(3.1.2.5) *$G_0 \cong \mathrm{SL}_2(3)$;*

(3.1.2.6) *$G_0 \cong \mathrm{SL}_2(3){}^{\cdot}2$ and $p \equiv \pm 1 \pmod 8$; or*

(3.1.2.7) *$G_0 \cong \mathrm{SL}_2(5)$ and 60 divides $p^2 - 1$.*

(3.2) *$F^*(G)$ is extraspecial of order p^3 and one of the following holds:*

(3.2.1) *$G \cong \mathrm{SL}_2(3)$ or $G \cong \mathrm{SL}_2(3){}^{\cdot}2$ (with quaternion Sylow 2-subgroups of order 16); or*

(3.2.2) *$G = K \ltimes N$ where N is extraspecial of order p^3 and exponent p with p an odd prime, K centralizes $Z(N)$ and is cyclic of odd order dividing $p + 1$. Furthermore, $G/Z(N)$ is a Frobenius group.*

(4) *If $F^*(G)$ is not nilpotent, then either*

(4.1) *$F^*(G) \cong \mathrm{SL}_2(p)$ where p is a Fermat prime, $|G/F^*(G)| \leq 2$ and G has quaternion Sylow 2-subgroups; or*

(4.2) *$G \cong \mathrm{PSL}_2(9)$, $\mathrm{Mat}(10)$ or $\mathrm{PSL}_2(p)$ where p is a Fermat or Mersenne prime.*

Furthermore, all the groups listed above are $\mathfrak{X}$-groups.

We make a brief remark about the group $\mathrm{SL}_2(3){}^{\cdot}2$ and the groups appearing in part (4.1) of Theorem 2.1 in the case $G > F^*(G)$. To obtain such groups, take $F = \mathrm{SL}_2(p^2)$, then the groups in question are isomorphic to the normalizer in F of the subgroup isomorphic to $\mathrm{SL}_2(p)$. We denote these groups by $\mathrm{SL}_2(p){}^{\cdot}2$ to indicate that the extension is not split (there are no elements of order 2 in the outer half of the group).

We shall repeatedly use the fact that if L is a subgroup of an $\mathfrak{X}$-group X, then L is an $\mathfrak{X}$-group. Indeed, if $H \leq L$ is non-cyclic, then $C_L(H) \leq C_X(H) \leq H$.

The following elementary facts will facilitate our proof that the examples listed are indeed $\mathfrak{X}$-groups.

Lemma 2.2. *The finite group X is an $\mathfrak{X}$-group if and only if $C_X(x)$ is an $\mathfrak{X}$-group for all $x \in X$ of prime order.*

Proof. If X is an $\mathfrak{X}$-group, then, as $\mathfrak{X}$ is subgroup closed, $C_X(x)$ is an $\mathfrak{X}$-group for all $x \in X$ of prime order. Conversely, assume that $C_X(x)$ is an $\mathfrak{X}$-group for all $x \in X$ of prime order (and hence of any order). Let $H \leq X$ be non-cyclic. We shall show $C_X(H) \leq H$. If $C_X(H) = 1$, then $C_X(H) \leq H$ and we are done. So assume $x \in C_X(H)$ and $x \neq 1$. Then $H \leq C_X(x)$ which is an $\mathfrak{X}$-group. Hence $x \in C_{C_X(x)}(H) \leq H$. Therefore $C_X(H) \leq H$, and X is an $\mathfrak{X}$-group. $\square$

Lemma 2.3. *Suppose that X is a Frobenius group with kernel K and complement L. If K and L are $\mathfrak{X}$-groups, then X is an $\mathfrak{X}$-group.*

Proof. Let $x \in X$ have prime order. Then, as K and L have coprime orders, $x \in K$ or x is conjugate to an element of L. But then, since X is a Frobenius group, either $C_X(x) \leq K$ or $C_X(x)$ is conjugate to a subgroup of L. Since K and L are $\mathfrak{X}$-groups, $C_X(x)$ is an $\mathfrak{X}$-group. Hence X is an $\mathfrak{X}$-group by Lemma 2.2. $\square$

The rest of this section is dedicated to the proof of Theorem 2.1; therefore G always denotes a finite $\mathfrak{X}$-group. Parts (1) and (2) of Theorem 2.1 are already proved in [1, Theorems 2.2, 2.4, 3.2 and 3.4]. However, our statement in (2.1.3) adds further detail which we now explain. So, for a moment, assume that G is supersoluble, q is the smallest prime dividing $|G|$, D is a cyclic q-group and C is a cyclic q'-group. In addition, $1 \neq Z(G) = C_D(C)$. Assume that $d \in D \setminus Z(G)$. Then, as $d \notin Z(G)$, C is not centralized by d. By coprime action, $C = [C,d] \times C_C(d)$ and so $Y = [C,d]\langle d\rangle$ is centralized by $C_C(d)$. As Y is non-abelian and $C_C(d) \cap Y = 1$, we deduce that $C_C(d) = 1$. Hence $G/Z(G)$ is a Frobenius group. This means that we can assume that (1) and (2) hold and, in particular, we assume that G is not supersoluble.

The following lemma provides the basic case subdivision of our proof.

Lemma 2.4. *One of the following holds:*

(i) *$F^*(G)$ is elementary abelian of order p^2 for some prime p.*
(ii) *$F^*(G)$ is extraspecial of order p^3 for some prime p.*
(iii) *$F^*(G)$ is quasisimple.*

Proof. Suppose first that $F^*(G)$ is nilpotent. Then its structure is given in part (1) of Theorem 2.1. Suppose that $F^*(G)$ is cyclic. Since $C_G(F^*(G)) = F^*(G)$, we have $G/F^*(G)$ is isomorphic to a subgroup of $\mathrm{Aut}(F^*(G))$. Because the automorphism group of a cyclic group is abelian, we have that G is supersoluble. Therefore, by our assumption concerning G, $F^*(G)$ is not cyclic. Hence $F^*(G)$ is either elementary abelian of order p^2 for some prime p, is extraspecial of order p^3 for some odd prime p or $F^*(G)$ is a dihedral, semidihedral or quaternion 2-group. Since the automorphism groups of dihedral, semidihedral and quaternion groups of order at least 16 are 2-groups, we deduce that when $p = 2$ and $F^*(G)$ is non-abelian, $F^*(G)$ is extraspecial. This proves the lemma when $F^*(G)$ is nilpotent.

If $F^*(G)$ is not nilpotent, then there exists a component $K \le F^*(G)$. As $F^*(G) = C_{F^*(G)}(K)K$ and K is non-abelian, we have $F^*(G) = K$ and this is case (iii). □

Lemma 2.5. *Suppose that p is a prime and $F^*(G)$ is extraspecial of order p^3. Then one of the following holds:*

(i) $G \cong \mathrm{SL}_2(3)$, $G \cong \mathrm{SL}_2(3){\cdot}2$ *(with quaternion Sylow 2-subgroups of order 16); or*

(ii) $G = NK$ *where N is extraspecial of order p^3 of exponent p with p an odd prime, K centralizes $Z(N)$ and is cyclic of odd order dividing $p+1$. Furthermore, $G/Z(N)$ is a Frobenius group.*

Proof. Let $N = F^*(G)$. We have that N is extraspecial of order p^3 by assumption. Suppose first that $p = 2$, then we have $N \cong \mathrm{Q}_8$ as the dihedral group of order 8 has no odd order automorphisms and G is not a 2-group. Since $\mathrm{Aut}(\mathrm{Q}_8) \cong \mathrm{Sym}(4)$, $G/Z(N)$ is isomorphic to a subgroup of $\mathrm{Sym}(4)$ containing $\mathrm{Alt}(4)$. If $G/Z(N) \cong \mathrm{Alt}(4)$, then $G = NT \cong \mathrm{SL}_2(3)$ where T is a cyclic subgroup of order 3. When $G/Z(N) \cong \mathrm{Sym}(4)$, taking $T \in \mathrm{Syl}_3(G)$, we have $NT \cong \mathrm{SL}_2(3)$, $N_G(T)$ has order 12 and $N_G(T)/Z(N) \cong \mathrm{Sym}(3)$. Since $N_G(T)$ is an $\mathfrak{X}$-group and $N_G(T)$ is supersoluble, we see that $N_G(T)$ is a product DT where D is cyclic of order 4 by (2.1.3). Because the Sylow 2-subgroups of G are either dihedral, semidihedral or quaternion and $D \not\le N$, we see that ND is quaternion. Thus $G \cong \mathrm{SL}_2(3){\cdot}2$ as claimed in (i).

Assume that p is odd. We know that the outer automorphism group of N is isomorphic to a subgroup of $\mathrm{GL}_2(p)$ and $C_{\mathrm{Aut}(N)}(Z(N))/\mathrm{Inn}(N)$ is isomorphic to a subgroup of $\mathrm{SL}_2(p)$. Since p is odd and the Sylow p-subgroups of G are $\mathfrak{X}$-groups, we have $N \in \mathrm{Syl}_p(G)$ and G/N is a p'-group by part (1) of Theorem 2.1. Set $Z = Z(N)$. Since G/N and N have coprime orders, the Schur Zassenhaus Theorem says that G contains a complement K to N. Set $K_1 = C_K(Z)$. Then K_1 commutes with Z and so K_1 is cyclic. If $K_1 = 1$, then $|K|$ divides $p-1$ and we find that G is supersoluble, which is a contradiction. Hence $K_1 \ne 1$. Let $x \in K_1$. Then $[N, x]$ and $C_N(x)$ commute by the Three Subgroups Lemma. Hence $C_N(x)$ centralizes $[N, x]\langle x\rangle$ which is non-abelian. It follows that $[N, x] = N$ and $C_N(x) = Z$. If $\langle x\rangle$ does not act irreducibly on N/Z, then there exists $Z < N_1 < N$ which is $\langle x\rangle$-invariant. If N_1 is cyclic, then, as $\langle x\rangle$ centralizes $\Omega_1(N_1) = Z$, $\langle x\rangle$ centralizes $N_1 > Z$, a contradiction. If N_1 is elementary abelian, then, as $\langle x\rangle$ centralizes Z, $[N_1, \langle x\rangle]$ has order at most p by Maschke's Theorem. If $[N_1, \langle x\rangle] \ne 1$, then $[N_1, \langle x\rangle]\langle x\rangle$ is non-abelian and Z centralizes $[N_1, \langle x\rangle]\langle x\rangle$, a contradiction. Hence $\langle x\rangle$ centralizes N_1 contrary to $C_N(\langle x\rangle) = Z$. We conclude that every element of K_1 acts irreducibly on $N/Z(N)$. In particular, since K_1 is isomorphic to a subgroup of $\mathrm{SL}_2(p)$, we have that K_1 is cyclic of odd order dividing $p+1$. Furthermore, as K_1 acts irreducibly on $N/Z(N)$, N has exponent p.

By the definition of K_1, $|K/K_1|$ divides $|\mathrm{Aut}(Z)| = p-1$. Assume that $K \ne K_1$ and let $y \in K \setminus K_1$ have prime order r. Then r does not divide $|K_1|$ and $Z\langle y\rangle$ is non-abelian. Since K_1 centralizes Z, we have $C_{K_1}(y) = 1$. Let

$w \in K_1$ have prime order q. Then $\langle y\rangle\langle w\rangle$ is non-abelian and acts faithfully on $V = N/Z$. Therefore [2, 27.18] implies that $C_N(y) \neq 1$. As $C_N(y) \cap Z = 1$ and $C_N(y)$ centralizes $Z\langle y\rangle$, we have a contradiction. Hence $K = K_1$. Finally, we note that $NK/Z(N)$ is a Frobenius group.

It remains to show that the groups listed are $\mathfrak{X}$-groups. We consider the groups listed in (ii) and leave the groups in (i) to the reader. Assume that $H \leq G$ is non-cyclic. We shall show that $C_G(H) \leq H$. If $H \geq N$, then $C_G(H) \leq C_G(N) \leq N \leq H$ and we are done. Suppose that $H < N$. Then, as N is extraspecial of exponent p, H is elementary abelian of order p^2 and $C_N(H) = H$. Since G/N is cyclic of odd order dividing $p+1$, we see that $N_G(H) = N$ and so $C_G(H) = C_N(H) = H$ and we are done in this case. Suppose that $H \not\leq N$ and $N \not\leq H$. Let $h \in H \setminus N$. Then, as $|G/N|$ divides $p+1$ and is odd, we either have $H \cap N = N$ or $H \cap N = Z$. So we must have $H \cap N = Z = Z(G)$. Now $H/Z \cong G/N$ is cyclic of order dividing $p+1$ and so we get that H is cyclic, a contradiction. Thus G is an $\mathfrak{X}$-group. □

Lemma 2.6. *Suppose that $N = F^*(G)$ is elementary abelian of order p^2. Then one of the following holds:*

(i) $p = 2$, $G \cong \mathrm{Sym}(4)$ *or* $\mathrm{Alt}(4)$*; or*

(ii) *p is odd and $G = NG_0$ is a Frobenius group with Frobenius kernel N and Frobenius complement G_0 which is itself an $\mathfrak{X}$-group. Furthermore, either*
   - (a) *G_0 is cyclic of order dividing $p^2 - 1$ but not dividing $p - 1$;*
   - (b) *G_0 is quaternion;*
   - (c) *G_0 is supersoluble as in part* (2.1.2) *of Theorem* 2.1 *with $|C|$ dividing $p - \epsilon$ where $p \equiv \epsilon \pmod 4$;*
   - (d) *G_0 is supersoluble as in part* (2.1.3) *of Theorem* 2.1 *with D a 2-group, $C_D(C)$ a non-trivial maximal subgroup of D and $|C|$ odd dividing $p - 1$ or $p + 1$;*
   - (e) $G_0 \cong \mathrm{SL}_2(3)$*;*
   - (f) $\mathrm{SL}_2(3){\cdot}2$ *and* $p \equiv \pm 1 \pmod 8$*; or*
   - (g) $G_0 \cong \mathrm{SL}_2(5)$ *and* 60 *divides* $p^2 - 1$.

Furthermore, all the groups listed are $\mathfrak{X}$-groups.

Proof. We have N has order p^2, is elementary abelian and G/N is isomorphic to a subgroup of $\mathrm{GL}_2(p)$. If $p = 2$, then we quickly obtain part (i). So assume that p is odd.

Suppose that p divides the order of G/N. Let $P \in \mathrm{Syl}_p(G)$. Then P is extraspecial of order p^3 and P is not normal in G. Hence by [4, Theorem 2.8.4] there exists $g \in G$ such that $G \geq K = \langle P, P^g\rangle \cong p^2{:}\mathrm{SL}_2(p)$. Let $Z = Z(P)$, t be an involution in K, $K_0 = C_K(t)$ and $P_0 = P \cap K_0$. Then, as t inverts N, $K_0 \cong \mathrm{SL}_2(p)$, P_0 has order p and centralizes $Z\langle t\rangle$, which is a contradiction as $Z\langle t\rangle \cong \mathrm{Dih}(2p)$. Hence G/N is a p'-group.

Suppose that $x \in G \setminus N$. If $C_N(x) \neq 1$, then $C_N(x)$ centralizes $[N, x]\langle x\rangle$ which is non-abelian, a contradiction. Thus $C_N(x) = 1$ for all $x \in G \setminus N$. It follows that G is a Frobenius group with Frobenius kernel N. Let G_0 be a

Frobenius complement to N. As $G_0 \leq G$, G_0 is an $\mathfrak{X}$-group. Recall that the Sylow 2-subgroups of G_0 are either cyclic or quaternion and that the odd order Sylow subgroups of G_0 are all cyclic [5, V.8.7].

Assume that N is not a minimal normal subgroup of G. Then G/N is conjugate in $\mathrm{GL}_2(p)$ to a subgroup of the diagonal subgroup. Therefore G is supersoluble, which is a contradiction. Hence N is a minimal normal subgroup of G and G_0 is isomorphic to an irreducible subgroup of $\mathrm{GL}_2(p)$. This completes the general description of the structure of G. It remains to determine the structure of G_0.

If G_0 is nilpotent, then Theorem 2.1 (1) applies to give G_0 is either quaternion or cyclic. In the latter case, as G_0 acts irreducibly on N it is isomorphic to a subgroup of the multiplicative group of $\mathrm{GF}(p^2)$ and is not of order dividing $p-1$. This gives the structures in (ii) (a) and (b).

If G_0 is supersoluble, then the structure of G_0 is described in part (2.1) of Theorem 2.1, as $\mathrm{GL}_2(p)$ contains no extraspecial subgroups of odd order. We adopt the notation from (2.1). By [5, V.8.18 c)], $Z(G_0) \neq 1$. Hence (2.1.1) cannot occur. Case (2.1.2) can occur and, as C commutes with a non-central cyclic subgroup of order at least 4 and G_0 is isomorphic to a subgroup of $\mathrm{GL}_2(p)$, $|C|$ divides $p-1$ if $p \equiv 1 \pmod 4$ and $|C|$ divides $p+1$ if $p \equiv 3 \pmod 4$. In the situation described in part (2.1.3) of Theorem 2.1, the groups have no 2-dimensional faithful representations unless $q = 2$ and $C_D(C)$ has index 2. In this case $|C|$ is an odd divisor of $p-1$ or $p+1$.

Suppose that G_0 is not supersoluble. Refereing to Lemma 2.4 and using the fact that the Sylow subgroups of G_0 are either cyclic or quaternion, we have that $F^*(G_0)$ is either quaternion of order 8 or $F^*(G_0)$ is quasisimple. In the first case we obtain the structures described in parts (b), (e) and (f) from Lemma 2.5 where for part (f) we note that we require $\mathrm{SL}_2(p)$ to have order divisible by 16.

If $F^*(G_0)$ is quasisimple, then Zassenhaus's Theorem [6, Theorem 18.6, p. 204] gives $G_0 = WM$ where $W \cong \mathrm{SL}_2(5)$ and M is metacyclic. Since G_0 is an $\mathfrak{X}$-group, this means that $M \leq W$ and $G_0 \cong \mathrm{SL}_2(5)$. Since $\mathrm{SL}_2(5)$ is isomorphic to a subgroup of $\mathrm{GL}_2(p)$ only when $p = 5$ or 60 divides $p^2 - 1$ and $p \neq 5$ part (g) holds.

That Sym(4) and Alt(4) are $\mathfrak{X}$-groups is easy to check. The groups listed in (ii) are $\mathfrak{X}$-groups by Lemma 2.3. □

The finite simple $\mathfrak{X}$-groups are determined in [1]. We have to extend the arguments to the cases where $F^*(G)$ is simple or quasisimple. This is relatively elementary.

Lemma 2.7. *Suppose that $F^*(G)$ is simple. Then $G \cong \mathrm{SL}_2(4)$, $\mathrm{PSL}_2(9)$, $\mathrm{Mat}(10)$ or $\mathrm{PSL}_2(p)$ where p is a Fermat or Mersenne prime.*

Proof. Set $H = F^*(G)$. As $\mathfrak{X}$ is subgroup closed, H is an $\mathfrak{X}$-group and so H is one of the groups listed in the statement by Theorem 3.7 of [1]. Hence we obtain $H \cong \mathrm{SL}_2(4)$, $\mathrm{PSL}_2(9)$ or $\mathrm{PSL}_2(p)$ for p a Fermat or Mersenne prime.

Suppose that $G > H$. If $H \cong \mathrm{SL}_2(4)$, then $G \cong \mathrm{Sym}(5)$ and the subgroup $2 \times \mathrm{Sym}(3)$ witnesses the fact that Sym(5) is not an $\mathfrak{X}$-group. Suppose $H \cong$

$\mathrm{PSL}_2(9) \cong \mathrm{Alt}(6)$. If $G \geq K \cong \mathrm{Sym}(6)$, then G contains $\mathrm{Sym}(5)$ which is impossible. Therefore $G \cong \mathrm{PGL}_2(9)$ or $G \cong \mathrm{Mat}(10)$. In the first case, G contains a subgroup $\mathrm{Dih}(20) \cong 2 \times \mathrm{Dih}(10)$ which is impossible. Thus $G \cong \mathrm{Mat}(10)$ and this group is easily shown to satisfy the hypothesis as all the centralizer of elements of prime order are $\mathfrak{X}$-groups.

If $H \cong \mathrm{PSL}_2(p)$, p a Fermat or Mersenne prime, then $G \cong \mathrm{PGL}_2(p)$ and contains a dihedral group of order $2(p+1)$ and one of order $2(p-1)$. One of these is not a 2-group and this contradicts G being an $\mathfrak{X}$-group. $\square$

Lemma 2.8. *Suppose that $F^*(G)$ is quasisimple but not simple. Then $F^*(G) \cong \mathrm{SL}_2(p)$ where p is a Fermat prime, $|G/H| \leq 2$ and G has quaternion Sylow 2-subgroups.*

Proof. Let $H = F^*(G)$ and $Z = Z(H)$. Since H centralizes Z, we have Z is cyclic. Let $S \in \mathrm{Syl}_2(H)$. If $Z \not\leq S$, then S must be cyclic. Since groups with a cyclic Sylow 2-subgroup have a normal 2-complement [2, 39.2], this is impossible. Hence $Z \leq S$. In particular, $Z(G) \neq 1$ as the central involution of H is central in G. It follows also that all the odd order Sylow subgroups of G are cyclic. By part (1) of Theorem 2.1, S is either abelian, dihedral, semidihedral or quaternion. If S is abelian, then S/Z is cyclic and again we have a contradiction. So S is non-abelian. Thus S/Z is dihedral (including elementary abelian of order 4). Hence $H/Z \cong \mathrm{Alt}(7)$ or $\mathrm{PSL}_2(q)$ for some odd prime power q [4, Theorem 16.3]. Since the odd order Sylow subgroups of G are cyclic, we deduce that $H \cong \mathrm{SL}_2(p)$ for some odd prime p. If $p-1$ is not a power of 2, then H has a non-abelian subgroup of order pr where r is an odd prime divisor of $p-1$ which is centralized by Z. Hence p is a Fermat prime.

Suppose that $G > H$ with $H \cong \mathrm{SL}_2(p)$, p a Fermat prime. Note G/H has order 2. Let $S \in \mathrm{Syl}_2(G)$. Then $S \cap H$ is a quaternion group. Suppose that S is not quaternion Then there is an involution $t \in S \setminus H$. By the Baer-Suzuki Theorem, there exists a dihedral group D of order $2r$ for some odd prime r which contains t. Since D and Z commute, this is impossible. Hence S is quaternion. This gives the structure described in the lemma.

It remains to demonstrate that the groups $\mathrm{SL}_2(p)$ and $\mathrm{SL}_2(p)^{\cdot}2$ with p a Fermat prime are indeed $\mathfrak{X}$-groups. Let G denote one of these group, $H = F^*(G) \cong \mathrm{SL}_2(p)$. Recall from the comments just after the statement of Theorem 2.1 that G is isomorphic to a subgroup of $X = \mathrm{SL}_2(p^2)$. Let V be the natural $\mathrm{GF}(p^2)$ representation of X and thereby a representation of G. Assume that $L \leq G$ is non-cyclic. Since H has no abelian subgroups which are not cyclic, L is non-abelian and L acts irreducibly on V. Schur's Lemma implies that $C_X(L)$ consists of scalar matrices and so has order at most 2. If L has even order, then as G has quaternion Sylow 2-subgroups, $L \geq C_G(L)$. So suppose that L has odd order. Then using Dickson's Theorem [7, 260, page 285], as p is a Fermat prime, we find that L is cyclic, a contradiction. Thus G is an $\mathfrak{X}$-group. $\square$

Proof of Theorem 2.1. This follows from the combination of the lemmas in this section. $\square$

3. Infinite locally finite $\mathfrak{X}$-groups

It has been proved in [1, Theorem 2.2] that an infinite abelian group is in the class $\mathfrak{X}$ if and only if it is either cyclic or isomorphic to $\mathbb{Z}_{p^\infty}$ (the Prüfer p-group) for some prime p. Moreover, Theorem 2.3 and Theorem 2.5 of [1] imply that every infinite nilpotent $\mathfrak{X}$-group is abelian. We start this section by showing that some extensions of infinite abelian $\mathfrak{X}$-groups provide further examples of infinite $\mathfrak{X}$-groups.

Lemma 3.1. *The infinite dihedral group belongs to the class $\mathfrak{X}$.*

Proof. Write $G = \langle a, y \,|\, y^2 = 1, a^y = a^{-1}\rangle$. Then for every non-cyclic subgroup H of G there exist non-zero integers n and m such that $a^n, a^m y \in H$. It easily follows that $C_G(H) = 1$. $\square$

Lemma 3.2. *Let $G = A\langle y\rangle$ where $A \cong \mathbb{Z}_{2^\infty}$ and $\langle y\rangle$ has order 2 or 4, with $y^2 \in A$ and $a^y = a^{-1}$, for all $a \in A$. Then G belongs to the class $\mathfrak{X}$.*

Proof. It is clear that G/A has order 2, and A is the Fitting subgroup of G. Also $C_G(A) = A$ and $Z(G)$ is the subgroup of order 2 of A. Let H be a non-cyclic subgroup of G with $H \neq A$. Then $H \not\leq A$ as every proper subgroup of A is cyclic. Pick any element $h \in H \setminus A$. Then $G = A\langle h\rangle$ since $|G : A| = 2$. Therefore by the Dedekind modular law we get $H = C\langle h\rangle$, where $C = A \cap H > 1$ is finite.

Since $h = bv$ with $b \in A$ and $v \in \langle y\rangle \setminus A$, we get $a^h = a^{-1}$ for all $a \in A$. In particular, $C_A(h)$ has order 2 and $C_G(h)$ has order 4. Since C has a unique involution and $h \in C_G(H)$, we conclude that $C_G(H) \leq H$ and so G is an $\mathfrak{X}$-group. $\square$

When $\langle y\rangle$ has order 2, the group $G = A \rtimes \langle y\rangle$ of Lemma 3.2 is a generalized dihedral group.

Let p denote any odd prime. Then, by Hensel's Theorem (see for instance [8, Theorem 127.5]), the group $\mathbb{Z}_{p^\infty}$ has an automorphism of order $p-1$, say ϕ.

Lemma 3.3. *The groups $G = \mathbb{Z}_{p^\infty} \rtimes \langle \phi^j\rangle$ for $1 \leq j \leq p-1$ are $\mathfrak{X}$-groups.*

Proof. As $\mathfrak{X}$ is subgroup closed, it suffices to show that $G = \mathbb{Z}_{p^\infty} \rtimes \langle\phi\rangle$ is an $\mathfrak{X}$-group. Write the elements of G in the form ay with $a \in A \cong \mathbb{Z}_{p^\infty}$ and $y \in \langle\phi\rangle$. Suppose there exist non-trivial elements $a \in A$ and $y \in \langle\phi\rangle$ such that $a^y = a$. For a suitable non-negative integer n, the element a^{p^n} has order p and it is fixed by y. Then y centralizes all elements of order p in A, and therefore $y = 1$ by a result due to Baer (see, for instance, [9, Lemma 3.28]). This contradiction shows that $\langle\phi\rangle$ acts fixed point freely on A.

Let H be any non-cyclic subgroup of G. Then, as G/A is cyclic, $A \cap H \neq 1$. If $H = A$ then of course $C_G(H) = H$. Thus we can assume that there exist non-trivial elements $a, b \in A$ and $y \in \langle\phi\rangle$ such that $a, by \in H$. Let $g \in C_G(H)$. If $g \in A$ then $1 = [g, by] = [g, y]$, so $g = 1$. Now let $g = cz$ with $c \in A$ and $1 \neq z \in \langle\phi\rangle$. Thus $1 = [cz, a] = [z, a]$, and $a = 1$, a contradiction. Therefore $C_G(H) \leq H$ for all non-cyclic subgroups H of G, so G is an $\mathfrak{X}$-group. $\square$

Lemma 3.4. *An infinite polycyclic group belongs to the class $\mathfrak{X}$ if and only if it is either cyclic or dihedral.*

Proof. Arguing as in the proof of Theorem 3.1 of [1], one can easily prove that every infinite polycyclic $\mathfrak{X}$-group is either cyclic or dihedral. On the other hand, the infinite dihedral group belongs to the class $\mathfrak{X}$ by Lemma 3.1. □

Proposition 3.5. *A torsion-free soluble group belongs to the class $\mathfrak{X}$ if and only if it is cyclic.*

Proof. Let G be a torsion-free soluble $\mathfrak{X}$-group. Then every abelian subgroup of G is cyclic, so G satisfies the maximal condition on subgroups by a result due to Mal'cev (see, for instance, [10, 15.2.1]). Thus G is polycyclic by [10, 5.4.12]. Therefore G has to be cyclic. □

In next theorem we determine all infinite soluble $\mathfrak{X}$-groups.

Theorem 3.6. *Let G be an infinite soluble group. Then G is an $\mathfrak{X}$-group if and only if one of the following holds:*

(i) *G is cyclic;*
(ii) *$G \cong \mathbb{Z}_{p^\infty}$ for some prime p;*
(iii) *G is dihedral;*
(iv) *$G = A\langle y\rangle$ where $A \cong \mathbb{Z}_{2^\infty}$ and $\langle y\rangle$ has order 2 or 4, with $y^2 \in A$ and $a^y = a^{-1}$, for all $a \in A$;*
(v) *$G \cong A \rtimes D$, where $A \cong \mathbb{Z}_{p^\infty}$ and $1 \neq D \leq C_{p-1}$ for some odd prime p.*

Proof. First let G be an $\mathfrak{X}$-group. If G is abelian then (i) or (ii) holds by [1, Theorem 2.2]. Assume G is non-abelian, and let A be the Fitting subgroup of G. Then $A \neq 1$ and $C_G(A) \leq A$ as G is soluble. Let N be a nilpotent normal subgroup of G. Then N is finite, as, otherwise, using N is self-centralizing and $G/Z(N)$ is a subgroup of $\mathrm{Aut}(N)$, we obtain G is finite, which is a contradiction. Thus [1, Theorems 2.3 and 2.5] imply that N is abelian. In particular, as the product of any two normal nilpotent subgroups of G is again a normal nilpotent subgroup by Fitting's Theorem, we see that the generators of A commute. Hence A is abelian. As A is infinite and abelian, $A = C_G(A)$ is either infinite cyclic or isomorphic to $\mathbb{Z}_{p^\infty}$ for some prime p. In the former case clearly $G' \leq A$. In the latter case, let C be any proper subgroup of A. Thus C is finite cyclic. Moreover C is characteristic in A, so it is normal in G, and $G/C_G(C)$ is abelian since it is isomorphic to a subgroup of $\mathrm{Aut}(C)$. It follows that $G' \leq C_G(C)$, and again $G' \leq C_G(A) = A$. Therefore G/A is abelian.

If A is infinite cyclic, then the argument in the proof of Theorem 3.1 of [1] shows that G is dihedral. Thus (iii) holds.

Let $A \cong \mathbb{Z}_{p^\infty}$ for some prime p, and suppose there exists an element $x \in G$ of infinite order. Then $x \in G \setminus A$, and so there exists an element $y \in A$ such that $[x, y] \neq 1$. Then $\langle y\rangle$ is a finite normal subgroup of G, so conjugation by x induces a non-trivial automorphism of $\langle y\rangle$. Since $\mathrm{Aut}(\langle y\rangle)$ is finite, it follows that there is an integer n such that $[x^n, y] = 1$. Now y is a torsion element and

x^n has infinite order and so $\langle x^n, y\rangle$ is neither periodic nor torsion-free and this contradicts [1, Theorems 2.2]. Therefore G is periodic, and G/A is isomorphic to a periodic subgroup of automorphisms of $\mathbb{Z}_{p^\infty}$.

It is well-known that $\mathrm{Aut}(\mathbb{Z}_{p^\infty})$ is isomorphic to the multiplicative group of all p-adic units. It follows that periodic automorphisms of $\mathbb{Z}_{p^\infty}$ form a cyclic group having order 2 if $p = 2$, and order $p-1$ if p is odd (see, for instance, [11] for details). In the latter case (v) holds. Finally, let $p = 2$. Then $G/A = \langle yA\rangle$ has order 2, and $G = A\langle y\rangle$ with $y \notin A$ and $y^2 \in A$. Moreover $a^y = a^{-1}$, for all $a \in A$. If y has order 2 then $G = A \rtimes \langle y\rangle$. Otherwise from $y^2 \in A$ it follows $y^2 = (y^2)^y = y^{-2}$, so y has order 4. Therefore G has the structure described in (iv).

On the other hand, Lemmas 3.1 – 3.3 show that the groups listed in (i) – (v) are $\mathfrak{X}$-groups. $\square$

Finally, we determine all infinite locally finite $\mathfrak{X}$-groups.

Theorem 3.7. *Let G be an infinite locally finite group. Then G is an $\mathfrak{X}$-group if and only if one of the following holds:*

(i) *$G \cong \mathbb{Z}_{p^\infty}$ for some prime p;*
(ii) *$G = A\langle y\rangle$ where $A \cong \mathbb{Z}_{2^\infty}$ and $\langle y\rangle$ has order 2 or 4, with $y^2 \in A$ and $a^y = a^{-1}$, for all $a \in A$;*
(iii) *$G \cong A \rtimes D$, where $A \cong \mathbb{Z}_{p^\infty}$ and $1 \neq D \leq C_{p-1}$ for some odd prime p.*

Proof. Any abelian subgroup of G is either finite or isomorphic to $\mathbb{Z}_{p^\infty}$ for some prime p, so it satisfies the minimal condition on subgroups. Thus G is a Černikov group by a result due to Šunkov (see, for instance [10, page 436, I]). Hence there exists an abelian normal subgroup A of G such that $A \cong \mathbb{Z}_{p^\infty}$ for some prime p, and G/A is finite. It follows that G is metabelian, arguing as in the proof of Theorem 3.6. Therefore the result follows from Theorem 3.6. $\square$

Clearly Theorem 2.1 and Theorem 3.7 give a complete classification of locally finite $\mathfrak{X}$-groups.

Corollary 3.8. *Let G be an infinite locally nilpotent group. Then G is an $\mathfrak{X}$-group if and only if one of the following holds:*

(i) *G is cyclic;*
(ii) *$G \cong \mathbb{Z}_{p^\infty}$ for some prime p;*
(iii) *$G = A\langle y\rangle$ where $A \cong \mathbb{Z}_{2^\infty}$ and $\langle y\rangle$ has order 2 or 4, with $y^2 \in A$ and $a^y = a^{-1}$, for all $a \in A$.*

Proof. Suppose G is not abelian. Every finitely generated subgroup of G is nilpotent, so it is either abelian or finite. It easily follows that all torsion-free elements of G are central. Thus G is periodic (see [12, Proposition 1]). Therefore G is direct product of p-groups (see, for instance, [10, Proposition 12.1.1]). Actually only one prime can occur since G is an $\mathfrak{X}$-group, so G is a locally finite p-group. Thus the result follows by Theorem 3.7. $\square$

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