Transitivity of properties of 2-generator subgroups

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In this paper we study infinite groups in which the polycyclicity is a transitive relation on the set of all 2-generator subgroups. We also present a short overview of known results about finite and infinite groups in which a given property is transitive on the set of all 2-generator subgroups.

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1. Introduction

Let $\mathfrak{X}$ be any group theoretical class. A group G is said to be $\mathfrak{X}$-*transitive* (or an $\mathfrak{X}$T-*group*) if for all $x, y, z \in G\backslash\{1\}$ the relations $\langle x, y\rangle \in \mathfrak{X}$ and $\langle y, z\rangle \in \mathfrak{X}$ imply $\langle x, z\rangle \in \mathfrak{X}$. In graph theoretical terms, let $\Gamma_{\mathfrak{X}}(G)$ be the simple graph whose vertices are all non-trivial elements of G, and vertices a and b are connected by an edge if and only if $\langle a, b\rangle \in \mathfrak{X}$. Then G is an $\mathfrak{X}$T-group precisely when every connected component of $\Gamma_{\mathfrak{X}}(G)$ is a complete graph. It is an obvious fact that the class of all $\mathfrak{X}$T-groups is closed under taking subgroups; but, in general, it is not closed under taking homomorphic images. Several authors have studied $\mathfrak{X}$T-groups for some special classes $\mathfrak{X}$. In Section 2 we give a short overview of known results about finite $\mathfrak{X}$T-groups, with emphasis on commutative-transitive groups (i.e., $\mathfrak{X}$T-groups where $\mathfrak{X}$ is the class of all abelian groups; these groups are briefly said CT-groups), soluble-transitive groups, supersoluble-transitive groups, and nilpotent-transitive groups. It is not surprising that the structure of infinite $\mathfrak{X}$T-groups is hard to describe. Results on this topic are relatively rare. Some known structural results are mentioned in Section 3, where we give a characterization of locally finite commutative-transitive groups together with further generalizations to nilpotent-transitive groups, and structural information on polycyclic commutative-transitive groups. Finally, in Section 4 we study infinite soluble-transitive and polycyclic-transitive groups. Here results of Lennox[7] suggest that hyper(abelian-by-finite) groups might be a good setting for studying such groups. We first show that if G is a finitely generated hyper(abelian-by-finite) solub! le-transitive group, then every 2-generator subgroup of G is soluble. From this we deduce that a finitely generated hyper(abelian-by-finite) polycyclic-transitive group is either polycyclic or polycyclic-semisimple. Moreover we show that both of these cases can actually occur. Our final result is about soluble groups of finite rank. We prove that if G is a finitely generated soluble group of finite rank which is polycyclic-transitive, then G is residually finite. Furthermore, if the Fitting subgroup of G is finitely generated then G is polycyclic.

2. Finite $\mathfrak{X}$T-groups

It is quite hard to describe the general structure of finite $\mathfrak{X}$T-groups. One of the obstacles is the fact that one often needs to deduce properties of the whole group from the structure of its 2-generator subgroups. In 1973, Lennox[7] introduced the following definition. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be group theoretical classes. Then the class $\mathfrak{X}$ is said to be *bigenetic* within the class $\mathfrak{Y}$ when a group from $\mathfrak{Y}$ belongs to $\mathfrak{X}$ provided that all of its 2-generator subgroups belong to $\mathfrak{X}$. Equipped with this notion, we say that a group theoretical class $\mathfrak{X}$ is a *good class* if it is subgroup closed, contains all finite abelian groups, and is bigenetic within the class of all finite groups. Examples of good classes are the class of all abelian groups, all nilpotent groups, all supersoluble groups and all soluble groups. It turns out that good classes of groups are suitable for studying the corresponding transitivity properties. Our first evidence is the following. Given a group theoretical class $\mathfrak{X}$, let $R_{\mathfrak{X}}(G)$ be the product of all normal $\mathfrak{X}$-subgroups of G (the $\mathfrak{X}$-*radical* of G). In general $R_{\mathfrak{X}}(G)$ does not belong to $\mathfrak{X}$. The crucial observation here is that this is however true within the class of all finite $\mathfrak{X}$-transitive groups when $\mathfrak{X}$ is a good class of groups. This fact is one of the key ingredients of our proof of the main structure result on finite $\mathfrak{X}$T-groups. Before formulating the result, we recall that, according Robinson[11], a group G is said to be $\mathfrak{X}$-*semisimple* if G has no normal $\mathfrak{X}$-subgroups.

Theorem 2.1 (Delizia, Moravec and Nicotera[3]). *Let $\mathfrak{X}$ be a good class of groups, and let G be a finite $\mathfrak{X}$T-group. Then one of the follow-*

ing holds:

(1) *G belongs to $\mathfrak{X}$;*
(2) *G is a Frobenius group with kernel and complement both in $\mathfrak{X}$;*
(3) *G is $\mathfrak{X}$-semisimple.*

When $\mathfrak{X}$ is the class of all abelian groups, then all three possibilities of Theorem 2.1 can occur. The following results, due to Weisner, Suzuki and Yu-Fen Wu, characterize the structure of finite CT-groups.

Theorem 2.2 (Weisner[15]). *A finite* CT*-group is soluble or simple.*

Theorem 2.3 (Suzuki[12]). *A finite non-abelian simple group is a* CT*-group if and only if it is isomorphic with some* $\mathrm{PSL}(2, 2^f)$, $f > 1$.

Theorem 2.4 (Yu-Fen Wu[16]). *If G is a finite soluble* CT*-group, then $G = F \rtimes \langle x\rangle$, where $F = \mathrm{Fit}\, G$ is abelian and $\langle x\rangle$ is a fixed-point-free group of automorphisms of F. Moreover, any two complements of F are conjugate in G. Conversely, if $G = F \rtimes \langle x\rangle$, where F is finite abelian and $\langle x\rangle$ is a fixed-point-free group of automorphisms of F, then G is a finite soluble* CT*-group with $F = \mathrm{Fit}\, G$.*

When $\mathfrak{X}$ is the class of all nilpotent groups, then again all three possibilities of Theorem 2.1 can occur.

Theorem 2.5 (Delizia, Moravec and Nicotera[3]). *Let G be a finite nilpotent-transitive group. Then one of the following holds:*

(1) *G is nilpotent;*
(2) *G is a Frobenius group with nilpotent complement;*
(3) *$G \cong \mathrm{PSL}(2, 2^f)$ for some $f > 1$, or $G \cong \mathrm{Sz}(q)$ with $q = 2^{2n+1} > 2$.*

Conversely, every finite group under (1) – (3) *is nilpotent-transitive.*

For certain good classes $\mathfrak{X}$, however, we are able to exclude the existence of $\mathfrak{X}$-semisimple $\mathfrak{X}$T-groups by using the well known Thompson's classification[13] of minimal simple groups, i.e., finite non-abelian simple groups all whose proper subgroups are soluble. We refer to the paper by Delizia, Moravec and Nicotera[3] for further details. As a consequence we get the following unexpected result.

Theorem 2.6 (Delizia, Moravec and Nicotera[3]). *A finite group is soluble-transitive if and only if it is soluble.*

An interesting open question, posed by Bechtell, is whether or not the class of all soluble groups contains a proper subformation $\mathfrak{X}$ that is also a good class of groups, such that $G \in \mathfrak{X}$ if and only if G is an $\mathfrak{X}$T-group.

Every finite supersoluble-transitive group is soluble. More precisely:

Theorem 2.7 (Delizia, Moravec and Nicotera[3]). *Let G be a finite supersoluble-transitive group. If G is not supersoluble, then G is a Frobenius group with supersoluble complement. In particular, G is always soluble.*

On the other hand, there exist finite supersoluble-transitive groups which are not supersoluble, as well as finite Frobenius groups having supersoluble complement but not being supersoluble-transitive (see Delizia, Moravec and Nicotera[3]).

Given a positive integer c, let $\mathfrak{N}_c$ denote the class of all nilpotent groups of class $\leqslant c$. Moreover, given positive integers $k > 1$ and c, let $\mathfrak{N}(k, c)$ denote the class of all groups in which every k-generator subgroup is nilpotent of class $\leqslant c$. It is easy to show that $\mathfrak{N}_c$T-groups form an ascending chain of classes of groups (see Delizia, Moravec and Nicotera[2]). When $c > 1$, the class $\mathfrak{N}_c$ is not a good class of groups. So Theorem 2.5 cannot be used to describe the structure of $\mathfrak{N}_c$T-groups. Nevertheless we get a complete classification of such groups.

Theorem 2.8 (Delizia, Moravec and Nicotera[2]). *Let G be a finite $\mathfrak{N}_c$T-group with $c > 1$. Then G is either soluble or simple. More precisely:*

(1) *G is soluble if and only if it is either an $\mathfrak{N}(2, c)$-group or a Frobenius group with the kernel which is an $\mathfrak{N}(2, c)$-group and complement which is nilpotent of class $\leqslant c$;*
(2) *G is a non-abelian simple group if and only if it is isomorphic either to $\mathrm{PSL}(2, 2^f)$, where $f > 1$, or to $\mathrm{Sz}(q)$, the Suzuki group with parameter $q = 2^{2n+1} > 2$.*

The existence of the second family of non-abelian simple $\mathfrak{N}_c$T-groups is probably the strongest evidence showing the gap between CT-groups and $\mathfrak{N}_c$T-groups with $c > 1$.

3. Infinite $\mathfrak{X}$T-groups

It would probably be too optimistic to expect that one could obtain a structural description of infinite $\mathfrak{X}$T-groups in general. An obvious evidence for this is the fact that every free group is a CT-group. Nevertheless, infinite CT-groups have played a major role in the model theory of groups.

Remeslennikov[9] and independently Gaglione and Fine[5] proved that a residually free group is a CT-group if and only it is fully residually free, and this is further equivalent to the fact that the group in question is universally free, i.e., it shares the same universal theory as the class of free groups.

The structure of locally finite CT-groups was investigated by Yu-Fen Wu[16]. She obtained results which are rather similar to the ones in the finite case.

Theorem 3.1 (Yu-Fen Wu[16]). *If G is a locally finite soluble* CT*-group, then $G = F \rtimes H$, where $F = \operatorname{Fit} G$ is abelian and H is a locally cyclic group of fixed-point-free automorphisms of F. Moreover, any two complements of F are conjugate in G. Conversely, if F is a locally finite abelian group and H a locally cyclic group of fixed-point-free automorphisms of F, then $G = F \rtimes H$ is a locally finite soluble* CT*-group.*

Theorem 3.2 (Yu-Fen Wu[16]). *Let G be an insoluble locally finite group. Then G is a* CT*-group if and only if it is isomorphic with* $\mathrm{PSL}(2, F)$ *for some locally finite field F of characteristic* 2 *with* $|F| \geqslant 4$.

As in the case of CT-groups, every locally finite $\mathfrak{N}_2$T-group is either soluble or simple (see Delizia, Moravec and Nicotera[2]). For the $\mathfrak{N}_c$T-groups with $c > 2$ this is no longer true. For instance, Bachmuth and Mochizuki[1] constructed an insoluble $\mathfrak{N}(2,3)$-group H of exponent 5. This group H is a locally finite $\mathfrak{N}_3$T-group (therefore a 3-Engel group, and so locally nilpotent). Clearly H is not simple, otherwise H would coincide with the normal closure of any non-trivial element of H, and it would therefore be nilpotent (see, for instance, Kappe and Kappe[4]), a contradiction.

The next two results describe the structure of locally finite $\mathfrak{N}_c$T-groups which are either soluble or not locally soluble. Analogous results hold true for locally finite $\mathfrak{N}$T-groups. To simplify the formulations we allow c to be ∞, and identify $\mathfrak{N}_\infty$T with $\mathfrak{N}$T and $\mathfrak{N}(2, \infty)$ with the class of weakly nilpotent groups, i.e., groups in which every 2-generator subgroup is nilpotent. With these identifications we have the following.

Theorem 3.3 (Delizia, Moravec and Nicotera[2]). *Let $c \in \mathbb{N} \cup \{\infty\}$. Every locally finite soluble $\mathfrak{N}_c$T-group is either an $\mathfrak{N}(2, c)$-group or a Frobenius group whose kernel and complement are both $\mathfrak{N}(2, c)$-groups. Conversely, every locally finite Frobenius group in which kernel and complement are both $\mathfrak{N}(2, c)$-groups is an $\mathfrak{N}_c$T-group.*

Theorem 3.4 (Delizia, Moravec and Nicotera[2]). *Let $c \in \mathbb{N} \cup \{\infty\}$, and let G be a locally finite $\mathfrak{N}_c$T-group which is not locally soluble. Then* *there exists a locally finite field F such that G is isomorphic either to* $\mathrm{PSL}(2,F)$ *or to* $\mathrm{Sz}(F)$*.*

We are now able to complete the classification of locally finite $\mathfrak{N}_c$T-groups and $\mathfrak{N}$T-groups partially given in Delizia, Moravec and Nicotera[2]. The next result namely describes locally finite locally soluble $\mathfrak{N}_c$T-groups that are not soluble.

Theorem 3.5. *Let $c \in \mathbb{N}\cup\{\infty\}$. Then every locally finite $\mathfrak{N}_c$T-group which is insoluble and locally soluble is either an $\mathfrak{N}(2,c)$-group or a Frobenius group whose kernel and complement are both $\mathfrak{N}(2,c)$-groups.*

Proof. Let G be an insoluble, locally soluble and locally finite $\mathfrak{N}_c$T-group. Suppose G is not an $\mathfrak{N}(2,c)$-group. Then there exist elements $a, b \in G$ such that the subgroup $\langle a, b\rangle$ is not in $\mathfrak{N}_c$. Thus Theorem 2.8 yields that $\langle a, b\rangle$ is a finite Frobenius group. By the hypotheses, the same conclusion holds for the subgroup $\langle a, b, x_1, \dots, x_n\rangle$, for all integers n and all choices of elements $x_1, \dots, x_n \in G$. Hence the finite subgroups of G containing a and b are a local system of Frobenius groups for G. Therefore G is a Frobenius group by Proposition 1.J.3 in Kegel and Wehrfritz[6]. Moreover, Theorem 1.J.2 in the same book states that both Frobenius kernel and complement of G have non-trivial center, therefore they are $\mathfrak{N}(2,c)$-groups, since G is an $\mathfrak{N}_c$T-group. $\square$

From Theorem 3.3, 3.4 and 3.5 we readily conclude the following:

Corollary 3.1. *Let $c \in \mathbb{N} \cup \{\infty\}$ and let G be a locally finite $\mathfrak{N}_c$T-group. Then one of the following holds:*

(1) *G is an $\mathfrak{N}(2,c)$-group;*
(2) *G is a Frobenius group whose kernel and complement are both $\mathfrak{N}(2,c)$-groups;*
(3) *$G \cong \mathrm{PSL}(2,F)$ or $G \cong \mathrm{Sz}(F)$, for some locally finite field F.*

Yu-Fen Wu[16] also dealt with polycyclic CT-groups. The following theorem characterizes the abelian-by-finite case.

Theorem 3.6 (Yu-Fen Wu[16]). *Let G be an abelian-by-finite polycyclic group and let F be its Fitting subgroup (which is abelian). Then G is a* CT*-group if and only if one of the following holds:*

(1) *$G = F \rtimes \langle x\rangle$ where $\langle x\rangle$ is a finite fixed-point-free group of automorphisms of F;*

(2) *$Q = G/F$ is a generalized quaternion group of fixed-point-free automorphisms of F and $\mathrm{res}^{Q}_{Q_0}(\alpha) \neq 0$ for every quaternion subgroup Q_0 of Q with order 8, where α is the cohomology class of the extension $F \rightarrowtail G \twoheadrightarrow Q$.*

Polycyclic CT-groups which are not abelian-by-finite groups are in close relationship with algebraic number fields, as the following result shows.

Theorem 3.7 (Yu-Fen Wu[16]). *A polycyclic* CT*-group G which is not abelian-by-finite has a normal subgroup $F \rtimes X$ of finite index, where F is the Fitting subgroup of G, X and F are free-abelian groups, and F has a series in which each factor H_i is X-rationally irreducible with fixed-point-free X-action. Moreover, H_i is isomorphic to an additive subgroup of the ring of algebraic integers of a number field K_i, and X is embedded in the unit group of K_i.*

4. Infinite polycyclic-transitive groups

In this section we study polycyclic-transitive groups. Of course, Theorem 2.6 stases that for finite groups this condition is equivalent to the solubility. Moving on to infinite groups, it is helpful to point out first some basic properties of soluble-transitive groups. We start by establishing a weak form of closure under taking quotients.

Lemma 4.1. *Let G be a soluble-transitive group, and let N be a soluble normal subgroup of G. Then G/N is also soluble-transitive.*

Proof. Let $x, y, z \in G\backslash N$ and suppose that the groups $\langle xN, yN\rangle$ and $\langle yN, zN\rangle$ are soluble. Then the groups $\langle x, y\rangle N/N$ and $\langle y, z\rangle N/N$ are soluble. As N is soluble, it follows that $\langle x, y\rangle$ and $\langle y, z\rangle$ are soluble. Since G is soluble-transitive, it follows that $\langle x, z\rangle$ is soluble, hence so is the group $\langle xN, zN\rangle$. □

Theorem 4.1. *Let G be a soluble-by-finite soluble-transitive group. Then G is soluble.*

Proof. Let N be a soluble normal subgroup of G such that $|G : N|$ is finite. By Lemma 4.1 the factor group G/N is soluble-transitive. Thus G/N is soluble by Theorem 2.6. Therefore G is soluble. □

By a result of Lennox[7], polycyclicity is bigenetic within the class of all finitely generated hyper(abelian-by-finite) groups. This means that a finitely generated hyper(abelian-by-finite) group is polycyclic provided that every its 2-generator subgroup is. This fact motivates us to study infinite polycyclic-transitive groups in the class of finitely generated hyper(abelian-by-finite) groups.

Proposition 4.1. *Let G be a finitely generated hyper(abelian-by-finite) group. If G is soluble-transitive, then G every 2-generator subgroup of G is soluble.*

Proof. By definition there exists a non-trivial normal subgroup N of G that is abelian-by-finite. Using Theorem 4.1, we see that N has to be soluble. It follows that $\langle N, g\rangle$ is soluble for all $g \in G$. Let $x, y \in G$ and let a be a non-trivial element of N. Since $\langle a, x\rangle$ and $\langle a, y\rangle$ are soluble, and G is soluble-transitive, it follows that $\langle x, y\rangle$ is soluble. □

Corollary 4.1. *Let G be a finitely generated hyper(abelian-by-finite) soluble-transitive group which is linear over a field of characteristic k. Then G is soluble-by-periodic. Moreover, if $k = 0$, then G is soluble.*

Proof. By Proposition 4.1, every 2-generator subgroup of G is soluble. Then G is soluble-by-periodic by a result of Wehrfritz[14]. Moreover, in the case $k = 0$, the group G is soluble-by-finite by Platonov's theorem[8], so G is soluble by Theorem 4.1. □

Note that a similar argument as in the proof of Theorem 4.1 shows that every polycyclic-by-finite polycyclic-transitive group is polycyclic. Moving on to hyper(abelian-by-finite) groups, we can prove the following result.

Theorem 4.2. *Let G be a finitely generated hyper(abelian-by-finite) group. If G is polycyclic-transitive, then G is either polycyclic or polycyclic-semisimple.*

Proof. Suppose there exists a non-trivial normal polycyclic subgroup N of G. Then for all $g \in G$ the group $N\langle g\rangle/N$ is cyclic, hence $N\langle g\rangle$ is polycyclic. Let x and y in G. Choose an element $a \in N \setminus \{1\}$. Since $\langle a, x\rangle$ and $\langle a, y\rangle$ are polycyclic, and G is polycyclic-transitive, it follows that $\langle x, y\rangle$ is polycyclic. Thus every 2-generator subgroup of G is polycyclic. Therefore G is polycyclic by the above mentioned result of Lennox[7]. □

Of course, the hypothesis that G is finitely generated cannot be removed in Theorem 4.2. For, every infinitely generated abelian group is a polycyclic-transitive group which is neither polycyclic nor polycyclic-semisimple.

The following example shows that there exist finitely generated hyper(abelian-by-finite) polycyclic-transitive groups which are not polycyclic.

Example 4.1. Consider

$$A = \left\{ \frac{a}{2^n} : a \in \mathbb{Z}, n \geqslant 0 \right\}$$

as an additive subgroup of $\mathbb{Q}$. Note that A is torsion-free and locally cyclic. The map $x \mapsto x/2$ induces an automorphism ϕ of A. Let $G = A \rtimes \langle \phi \rangle$. Clearly we have that $G = \langle 1/2, \phi \rangle$. Let $x = (\phi^k, a/2^n)$ and $y = (\phi^l, b/2^m)$ be arbitrary elements of G. Then calculation shows that

$$[x, y] = \left(1, -\frac{a}{2^n} - \frac{b}{2^{m+k}} + \frac{a}{2^{n+l}} + \frac{b}{2^m} \right).$$

At first we show that G is a CT-group. To this end, let $x = (\phi^k, a/2^n)$, $y = (\phi^l, b/2^m)$ and $z = (\phi^j, a/2^p)$ be non-trivial elements of G and suppose that $[x, y] = [y, z] = 1$. Then the above argument yields the following relations:

$$\begin{aligned} 2^{-n}a(2^{-l} - 1) &= 2^{-m}b(2^{-k} - 1), \\ 2^{-m}b(2^{-j} - 1) &= 2^{-p}c(2^{-l} - 1). \end{aligned}$$

Note that if $l = 0$, then $k = j = 0$, as $b \neq 0$. In this case it is clear that $[x, z] = 1$. So we can suppose that $l \neq 0$. If $b = 0$, then also $a = c = 0$, and again we conclude that $[x, z] = 1$. So we can additionally assume that $b \neq 0$. If both k and j are zero, then we also have $[x, z] = 1$, thus we can assume without loss of generality that $j \neq 0$. Then we obtain

$$2^{-n}a(2^{-j} - 1) = \frac{2^{-n-p}ac(2^{-l} - 1)}{2^{-m}b} = 2^{-p}c(2^{-k} - 1),$$

hence $[x, z] = 1$, as required.

To show that G is polycyclic-transitive, it clearly suffices to prove that a 2-generator subgroup of G is polycyclic if and only if it is abelian. Let $x = (\phi^k, a/2^n)$ and $y = (\phi^l, b/2^m)$ be elements of G. Suppose that $[x, y] \neq 1$. We have that $[x, y] = (1, c)$, where $c = -a/2^n - b/2^{m+k} + a/2^{n+l} + b/2^m$ is a non-trivial element of A. Consider the group $H = \langle [x, y] \rangle^{\langle (\phi, 0) \rangle}$. It is easy to see that $(\phi, 0)^{-u}[x, y]^v(\phi, 0)^u = (1, 2^{-u}vc)$ for all $u, v \in \mathbb{Z}$, hence H is generated by the set $\{(1, 2^{-u}c) : u \geqslant 0\}$, and it cannot be generated by a smaller set. This shows that if $[x, y] \neq 1$ then $\langle x, y \rangle$ is not polycyclic.

The group G in Example 4.1 is soluble of finite rank. The structure of finitely generated soluble polycyclic-transitive groups of finite rank is subject to the following restrictions, that make them near to be polycyclic.

Theorem 4.3. *Let G be a finitely generated soluble group of finite rank which is polycyclic-transitive. Then G is residually finite. Moreover the Fitting subgroup F of G is nilpotent, and G/F is polycyclic. Finally, G is nilpotent-by-abelian-by-finite, and it has no infinite subgroups satisfying the minimal condition on subgroups.*

Proof. First, the group G is minimax (see, for instance, Robinson's book[10], Part 2, Theorem 10.38). Moreover, by Theorem 4.2, G is either polycyclic or polycyclic-semisimple. Every polycyclic group is residually finite by a theorem due to Hirsch (see, for instance, Theorem 5.4.17 in Robinson[11]). Moreover, a well-known result of Mal'cev states that every polycyclic group has a normal subgroup of finite index whose derived subgroup is nilpotent (see, for instance, Theorem 15.1.6 in Robinson[11]). So clearly we can assume that G is polycyclic-semisimple. If A is a normal abelian subgroup of G satisfying the minimal condition on subgroups then A is trivial, otherwise A has a non-trivial characteristic polycyclic subgroup, a contradiction since G is polycyclic-semisimple. Let R be the finite residual of G. Then R is the direct product of finitely many quasicyclic subgroups of G (see Robinson[10], Part 2, Theorem 10.33). It follows that $R = 1$ and G is residually finite. So G has no infinite subgroups satisfying the minimal condition on subgroups. Moreover F is nilpotent, and G/F is polycyclic and abelian-by-finite. □

The above theorem shows that the Fitting subgroup F plays a decisive role in finitely generated soluble groups G of finite rank that are polycyclic-transitive. Namely, if F is finitely generated, then G is polycyclic by Theorem 4.2. If however F is not finitely generated, then Theorem 4.3, together with Theorem 4.2, shows that G is polycyclic-semisimple.

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