GROUPS WITH ALL CENTRALIZERS SUBNORMAL OF DEFECT AT MOST TWO

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Abstract. We study the class of all groups in which the centralizer of each element is a subnormal subgroup. In particular, we focus on the case when the defect of every centralizer is at most 2. We show that a group without involutions satisfies this property if and only if it is 3-Engel.

1. Introduction

A group G is said to be n-*Engel* if it satisfies the identity $[x, {}_n y] = 1$ for all $x, y \in G$. More generally, G is said to be an *Engel group* if for all $x, y \in G$ there exists a non-negative integer $n = n(x, y)$ such that $[x, {}_n y] = 1$. These classes of groups have been widely studied, we refer to a paper by Traustason [10] for a recent survey. There are several well-known characterizations of 2-Engel groups; see, for instance, [6]. One of them says that a group G is 2-Engel if and only if for every $x \in G$ the centralizer $C_G(x)$ is normal in G. A natural generalization of 2-Engel groups is therefore the class $\mathfrak{C}_n$ of all groups G in which for every $x \in G$ the centralizer $C_G(x)$ is subnormal in G of defect $\leq n$. Furthermore, we denote by $\mathfrak{C}$ the class of all groups G in which for every $x \in G$ the centralizer $C_G(x)$ is subnormal in G.

The purpose of this paper is to study the groups belonging to $\mathfrak{C}_n$, with emphasis on the case $n = 2$. At first we note that every group in $\mathfrak{C}_n$ (respectively, in $\mathfrak{C}$) is a locally nilpotent $(n+1)$-Engel group (respectively, Engel group). The main part of the paper consists of a detailed description of $\mathfrak{C}_2$-groups. The starting point is a characterization of $\mathfrak{C}_2$-groups in terms of vanishing of a certain commutator word. Based on that, we prove that if G is a $\mathfrak{C}_2$-group, then every abelian subgroup of G is subnormal of defect ≤ 3. We also show that the class $\mathfrak{C}_2$ strictly contains the variety of all nilpotent groups having nilpotency class ≤ 3, and is strictly contained in the variety of all 3-Engel groups. Finally, we deal with the question as to which 3-Engel groups belong to $\mathfrak{C}_2$. Our main result in this direction is the following.

Theorem 1.1. *Let G be a group without involutions. Then G is a $\mathfrak{C}_2$-group if and only if it is a 3-Engel group.*

As a consequence, we show that $\mathfrak{C}_2$-groups need not to be solvable. Since there exist n-generator 3-Engel 5-groups which are nilpotent of class exactly $2n-1$ (see [4]), Theorem 1.1 also implies that there is no bound for the nilpotency class of nilpotent groups in $\mathfrak{C}_2$.

In order to prove the main result, we use the detailed description of 3-Engel groups obtained by Gupta and Newman [4]. As in the latter case, our arguments are guided by computer calculations with GAP [2]. In particular, we used GAP to determine the exact bound for the nilpotency class in Proposition 3.3, as well

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as to find interactions between various commutators appearing in the proof. We point out that our arguments are completely hand proved, and hence they do not depend on these calculations. We also mention that some of our techniques could be extended to cover the classes $\mathfrak{C}_n$ for $n > 2$, but this will not be addressed here. The structure of groups in $\mathfrak{C}_n$ essentially depends on the structure of locally nilpotent $(n+1)$-Engel groups, and this is, apart from the case $n \leq 3$, still not well determined.

2. General properties

For the unexplained notions throughout the paper we refer to [8] and [9]. We begin with the following observation.

Proposition 2.1. *If G is nilpotent of class $\leq n$, then $G \in \mathfrak{C}_n$.*

Proof. This is obvious, since every subgroup of G is subnormal of defect $\leq n$. □

Recall that a group G is said to be *n-Baer* if for every $x \in G$ the cyclic subgroup $\langle x \rangle$ is subnormal in G of defect $\leq n$. Also, G is said to be a *Baer group* if for every $x \in G$ the cyclic subgroup $\langle x \rangle$ is subnormal in G. We refer to [9] for the basic properties of these groups. In particular, it is well-known that Baer groups are locally nilpotent. The following results point out connections between $\mathfrak{C}$-groups, Engel groups and Baer groups.

Proposition 2.2. *Every $\mathfrak{C}_n$-group is a $(n+1)$-Baer group.*

Proof. Let $G \in \mathfrak{C}_n$ and $x \in G$. Then $\langle x \rangle \triangleleft C_G(x) \triangleleft^n G$, hence G is $(n+1)$-Baer. □

Since n-Baer groups are $(n+1)$-Engel, it follows from Proposition 2.2 that every $\mathfrak{C}_n$-group is a $(n+2)$-Engel group. Actually, this can be improved.

Proposition 2.3. *Every $\mathfrak{C}_n$-group is a $(n+1)$-Engel group.*

Proof. Let $G \in \mathfrak{C}_n$ and $x, y \in G$. Write $H = C_G(x)$. By [8, Proposition 1.1.1], we have $[y, {}_n x] \in [G, {}_n H] \leq H$, so $[y,_{n+1} x] = 1$. Therefore G is $(n+1)$-Engel. □

Proposition 2.4. *Every $\mathfrak{C}$-group is a Baer group and an Engel group.*

Proof. This follows by a similar argument as in the proof of Proposition 2.2 and Proposition 2.3. □

Corollary 2.5. *$\mathfrak{C}$-groups are locally nilpotent.*

In the sequel, the following well-known commutator identities will be used without further reference:

$$[a, b] = [b, a]^{-1},$$
$$[ab, c] = [a, c][a, c, b][b, c],$$
$$[a, bc] = [a, c][a, b][a, b, c],$$
$$[a^{-1}, b] = [a, b, a^{-1}]^{-1}[b, a],$$
$$[c, [b, a]] = [c, b, a][c, a, b]^{-1}u$$

where u is a product of commutators with entry set $\{a, b, c\}$ and weight at least 4. We note that the last identity listed above is a version of the Hall-Witt identity (see for instance [4]).

Lemma 2.6. *A group G is in $\mathfrak{C}_2$ if and only if the relation $[g, h_1, h_2, x] = 1$ holds for all $g, x \in G$ and for all $h_1, h_2 \in C_G(x)$.*

Proof. Note that the relation $[g, h_1, h_2, x] = 1$ holds for all $g, x \in G$ and for all $h_1, h_2 \in H = C_G(x)$ if and only if $[g, h_1, h_2] \in H$ for all $h_2 \in H$, which is further equivalent to $[g, h_1] \in N_G(H)$ for all $h_1 \in H$. This is equivalent to $[G, H] \leq N_G(H)$, that is, $[G, H, H] \leq H$. It is well-known (see for instance [8], Proposition 1.1.1) that this is equivalent to the fact that H is subnormal in G of defect at most 2. This completes the proof. □

Proposition 2.7. *Let $G \neq 1$ be a group and suppose that $G = AB$, where the subgroups A and B are abelian and A is normal in G. If $G \in \mathfrak{C}_2$, then $Z(G) \neq 1$.*

Proof. Arguing by contradiction, we assume that $Z(G) = 1$. Since $G \in \mathfrak{C}_2$ and B is abelian, Lemma 2.6 implies that $[a, b_1, b_2, b] = 1$ for all $a \in A$ and for all $b_1, b_2, b \in B$. This means that $[a, b_1, b_2] \in C_G(B)$, and since $[a, b_1, b_2] \in A$, where A is abelian, we have $[a, b_1, b_2] \in Z(G)$. Thus $[a, b_1, b_2] = 1$ for every $a \in A$, $b_1, b_2 \in B$. So $[a, b_1] \in C_G(B) \cap A$, and as above we conclude that $[a, b_1] = 1$ for every $a \in A$ and for every $b_1 \in B$. But this means that G is abelian, a contradiction. □

Proposition 2.8. *Let $G \in \mathfrak{C}_2$. Then every abelian subgroup of G is subnormal of defect ≤ 3.*

Proof. Let A be a maximal abelian subgroup of G. Then for every $g \in G$ and for every $a_1, a_2, a_3 \in A$ we have that $[g, a_1, a_2, a_3] = 1$, since $G \in \mathfrak{C}_2$ and $A \subseteq C_G(a_3)$. By the maximality of A this implies that $[g, a_1, a_2] \in A$. Now we will prove that $[G, A, A] = \langle [k, a] : k \in [G, A], a \in A \rangle \leq A$.

Put $C = \{[g, a] : g \in G, a \in A\}$. If $c = [g, a] \in C$, then for every $a_1 \in A$ we have $[c, a_1] = [g, a, a_1] \in A$. If $c = [a, g] \in C^{-1}$, then $[c, a_1] = [a, g, a_1] = [g, a^{-1}, a_1]^a = [g, a^{-1}, a_1] \in A$. This means that $[c, a_1] \in A$ for all $c \in C \cup C^{-1}$ and for all $a_1 \in A$. Moreover, $a^c = a[a, c] \in A$ for all $a \in A$ and for all $c \in C \cup C^{-1}$. By a similar argument as in the proof of Lemma 2.6 we conclude that $[k, a] \in A$ for every $k \in [G, A]$ and for every $a \in A$. So $[G, A, A] \leq A$, therefore A is subnormal in G of defect ≤ 2.

Now let H be an abelian subgroup of G. Then there exists a maximal abelian subgroup A of G such that $H \leq A$. Clearly, H is normal in A, so H is subnormal in G of defect ≤ 3. □

The converse of Proposition 2.8 is not true. Garrison and Kappe [3, Example 5.2] constructed a finite 3-Baer group G that is not 3-Engel, hence it does not belong to $\mathfrak{C}_2$. On the other hand, one can compute all conjugacy classes of abelian subgroups of G using GAP [2], and verify that the representatives of these classes are always subnormal in G of defect at most 3.

Now we note that there exist 3-Engel groups not belonging to $\mathfrak{C}_2$. For example, if G is the standard wreath product of a cyclic group of order 2 and a countably infinite elementary abelian 2-group, then G is 3-Engel (see [9], p. 48). On the other hand, since $Z(G) = 1$, by Proposition 2.7 the group G is not in $\mathfrak{C}_2$.

Our next example exhibits a finite metabelian 3-Engel group of class 4 that does not belong to $\mathfrak{C}_2$.

Example 2.9. Let E be an elementary abelian 2-group of order 512 generated by $e_1, \ldots, e_9$. Form a semidirect product $G_1 = E \rtimes \langle a \rangle$, where a is of order 2 and acts upon E in the following way: $[e_1, a] = e_6 e_8$ and $[e_i, a] = 1$ for $i \geq 2$. Next, let $G_2 = G_1 \rtimes \langle c \rangle$, where c is of order 2, acting on G_1 by $[a, c] = e_4$, $[e_2, c] = e_7$, $[e_3, c] = e_6$, $[e_5, c] = e_9$, and $[e_i, c] = 1$ otherwise. Now define $G_3 = G_2 \rtimes \langle b \rangle$, where $|b| = 2$, and the action is given by $[c, b] = e_1$, $[a, b] = e_3$, $[e_2, b] = e_5$, $[e_4, b] = e_8$, $[e_7, b] = e_9$, and $[e_i, b] = 1$ otherwise. Finally, let $G = G_3 \rtimes \langle d \rangle$, where d is an automorphism of G_3 of order 2 acting as follows: $[b, d] = [c, d] = 1$, $[a, d] = e_2$, $[e_3, d] = e_5$, $[e_4, d] = e_7$, $[e_6, d] = [e_8, d] = e_9$, and $[e_i, d] = 1$ otherwise. It is straightforward to verify that G is a metabelian group of class 4, and of order 2^{13}. The relations of G show that G/E is elementary abelian. Let $x, y \in G$ and set $z = [x, y] \in E$. Then $[y, x, x, x] = [z, x, x] = z^{(-1+x)^2} = z^{1-2x+x^2} = z^{2-2x} = 1$, hence G is a 3-Engel group. On the other hand, the defining relations imply that $[b, d] = [c, d] = 1$, but $[a, b, c, d] = e_9 \neq 1$, hence G does not belong to $\mathfrak{C}_2$ by Lemma 2.6.

3. Three-Engel groups without involutions

The aim of this section is to prove that every 3-Engel group without involutions is a $\mathfrak{C}_2$-group. The basic information on 3-Engel groups we start with is the following. In [5] Heineken proved that every 2-generator 3-Engel group without involutions is nilpotent of class ≤ 3. Gupta and Newman [4] showed that 3-generator 3-Engel groups are nilpotent of class ≤ 5. We will prove that such groups have nilpotency class ≤ 4 provided that two of the generators commute, and there are no involutions. This, together with Lemma 2.6, will be the key step towards proving Theorem 1.1.

At first we prove the following auxiliary result.

Lemma 3.1. *Let $x_1, \ldots, x_n, a, b$ be elements of a group G, with $[a, b] = 1$. Then $[x_1, \ldots, x_n, a, b] \equiv [x_1, \ldots, x_n, b, a]$ modulo $\gamma_{2(n+1)}(G)$.*

Proof. From $[x_1, \ldots, x_n, ab] = [x_1, \ldots, x_n, ba]$ we obtain

$$[x_1, \ldots, x_n, b][x_1, \ldots, x_n, a][x_1, \ldots, x_n, a, b] = [x_1, \ldots, x_n, a][x_1, \ldots, x_n, b][x_1, \ldots, x_n, b, a],$$

and the result follows. □

We proceed with listing the properties of 3-Engel groups we need due course. Gupta and Newman [4] proved that, for $n > 2$, every n-generator 3-Engel group is nilpotent of class $\leq 2n - 1$. In particular, Lemma 3.4 of [4] states that if G is a 3-Engel group without involutions, then in every commutator of elements of G, a pair of neighboring equal entries can be moved past any other entry without changing the value of the commutator modulo commutators of higher weight:

$$[\ldots, a, a, b, \ldots] \equiv [\ldots, b, a, a, \ldots].$$

As in [4], we will refer to this fact as the *shifting* property. Moreover, for $s \geq 3$, the commutator $[a, b_1, \ldots, b_s, a]$ can be written as a product of commutators of weight $s + 2$ with the same entry set and neighboring a's and of commutators of higher weight (*closing* property).

By Lemma 3.3 of [4], given elements a, b, c, d in a 3-Engel group G having no involutions, we get the following fundamental congruences modulo $\gamma_6(G)$:

$$[c, b, b, a, d] \equiv [c, a, b, b, d] \equiv [c, a, d, b, b], \tag{1}$$

$$[c, b, a, b, d] \equiv [c, a, b, d, b] \equiv [c, a, d, b, b]^3. \tag{2}$$

Lemma 3.2. *Let $G = \langle a, b, c \rangle$ be a 3-Engel group without involutions. If $[b, c] = 1$, then the nilpotency class of G is ≤ 4.*

Proof. We have that $\gamma_6(G) = \{1\}$ by [4, Theorem 2.4]. The subgroup $\gamma_5(G)$ is generated by the commutators $[a, b, b, c, c]$, $[b, a, a, c, c]$ and $[c, a, a, b, b]$ (see [4], Theorem 3.5). Since $[b, c] = 1$, the shifting property gives $\gamma_5(G) = \langle [a, b, b, c, c] \rangle$. Replacing a and d by b, b by c, c by a in (1) and (2), we obtain $[a, c, c, b, b] = [a, b, b, c, c]$ and

$[a,c,b,c,b] = [a,b,b,c,c]^3$. Thus by Lemma 3.1 we get $[a,b,b,c,c]^2 = 1$, and the result follows, since G has no involutions. □

In what follows, we will often use the fact that in a 3-Engel group all commutators with a triple entry are trivial (see [7] and [4]).

Proposition 3.3. *Let $G = \langle a,b,c,d \rangle$ be a 3-Engel group without involutions. If $[b,d] = 1 = [c,d]$, then the nilpotency class of G is ≤ 5.*

Proof. Suppose $z \in G$ has the property that $z^2 \in \gamma_7(G)$. By [4, Theorem 4.4] we have $\gamma_5(G)^5 = 1$, hence $z^5 = 1$. It follows that $z = z^{-4}$ belongs to $\gamma_7(G)$, hence the group $G/\gamma_7(G)$ has no involutions. Passing to this quotient group, we may assume without loss of generality that $\gamma_7(G) = \{1\}$. Our aim is to prove that every commutator of weight 6 having entry set $\{a,b,c,d\}$ is trivial. Notice that every commutator of weight 6 lies in the centre of G, so it can be expanded linearly. Moreover, $[x_2,x_1,x_3,x_4,x_5,x_6] = [x_1,x_2,x_3,x_4,x_5,x_6]^{-1}$. By our assumptions, and since the roles of b and c are symmetric, we only have to deal with commutators $[c,a,x_3,x_4,x_5,x_6]$ (we call these commutators of type 1), $[c,b,x_3,x_4,x_5,x_6]$ (type 2) and $[d,a,x_3,x_4,x_5,x_6]$ (type 3), where x_3,x_4,x_5,x_6 run through the set $\{a,b,c,d\}$. Throughout this proof, the commutators will be always listed in lexicographic order.

We start by considering commutators $w = [c,a,x_3,x_4,x_5,x_6]$ of type 1.

If $x_3 = a$, then we can assume x_4, x_5 and x_6 to be different from a (otherwise $w = 1$, since w has a triple entry). Moreover, we may assume $d \in \{x_4,x_5,x_6\}$, otherwise $w = 1$, since 3-generator 3-Engel groups are nilpotent of class ≤ 5. Then we always get $w = 1$ by using the shifting property and Lemma 3.1. Therefore all commutators of type 1 with $x_3 = a$ are trivial.

If $x_3 = b$ and $x_4 = a$, we have to consider the commutators $w_1 = [c,a,b,a,b,d]$, $w_2 = [c,a,b,a,c,d]$, $w_3 = [c,a,b,a,d,b]$, $w_4 = [c,a,b,a,d,c]$, $w_5 = [c,a,b,a,d,d]$. Now we obtain from (2), by setting $d = a$, that

$$[c,b,a,b,a] \equiv [c,a,b,a,b] \equiv [c,a,a,b,b]^3 \mod \gamma_6(G). \tag{3}$$

From (3) we obtain that w_1 is the cube of a commutator of type 1 with $x_3 = a$, so $w_1 = 1$. By interchanging a and b and replacing d by c in (2), we obtain $w_2 = [c,b,c,a,a,d]^3$; thus by the shifting property, w_2 is the cube of a commutator of type 1 with $x_3 = a$, so $w_2 = 1$. By Lemma 3.1, $w_3 = w_1 = 1$ and $w_4 = w_2 = 1$. Finally, $w_5 = 1$ by the shifting property. Therefore all commutators of type 1 with $x_3 = b$ and $x_4 = a$ are trivial.

If $x_3 = b = x_4$, we need to consider the commutators $w_6 = [c,a,b,b,a,d]$, $w_7 = [c,a,b,b,c,d]$, $w_8 = [c,a,b,b,d,a]$, $w_9 = [c,a,b,b,d,c]$, $w_{10} = [c,a,b,b,d,d]$. From (1), by setting $d = a$, we obtain that w_6 is a commutator of type 1 with $x_3 = a$, so $w_6 = 1$. Using Lemma 3.1 and the equations (2) and (1), we get $[c,a,b,b,d] \equiv [c,a,d,b,b]^3 \equiv [c,a,b,b,d]^3$ modulo $\gamma_6(G)$. Since G has no involutions, this implies that $[c,a,b,b,d] \in \gamma_6(G)$. Thus $w_7 = w_8 = w_9 = w_{10} = 1$. Therefore all commutators of type 1 with $x_3 = b = x_4$ are trivial.

If $x_3 = b$ and $x_4 = c$, we have to consider the commutators $w_{11} = [c,a,b,c,a,d]$, $w_{12} = [c,a,b,c,b,d]$, $w_{13} = [c,a,b,c,d,a]$, $w_{14} = [c,a,b,c,d,b]$, $w_{15} = [c,a,b,c,d,d]$. Interchanging a and b and replacing d by c in (2), we get $[c,a,b,c,a] \equiv [c,b,c,a,a]^3$ modulo $\gamma_6(G)$. By the shifting property, w_{11} is the cube of a commutator of type 1 with $x_3 = a$, so $w_{11} = 1$. By expanding $[c,a,bc,bc,bc,d] = 1$ we obtain $1 = [c,a,b,b,c,d]w_{12}[c,a,c,b,b,d]$, so $w_{12} = 1$ by the shifting property, and since all commutators of type 1 with $x_3 = b = x_4$ are trivial. We also get $w_{13} = [a,c,b,c,d,a]^{-1}$. By the closing property, $[a,c,b,c,d,a]$ is a product of certain commutators u_i of weight 6 with entries b, c, c, d and neighboring a's. By the shifting property, each u_i can be written as a commutator with $x_2 = a = x_3$. Since $[b, d] = 1 = [c, d]$, Lemma 3.1 and the shifting property yield $u_i = [b, d, \dots]$ or $u_i = [c, d, \dots]$, so $u_i = 1$. Thus $w_{13} = 1$. Finally, $w_{14} = w_{12} = 1$ by Lemma 3.1, and $w_{15} = 1$ by the shifting property. Therefore all commutators of type 1 with $x_3 = b$ and $x_4 = c$ are trivial.

If $x_3 = b$ and $x_4 = d$, we need to consider the commutators $w_{16} = [c, a, b, d, a, b]$, $w_{17} = [c, a, b, d, a, c]$, $w_{18} = [c, a, b, d, a, d]$, $w_{19} = [c, a, b, d, b, a]$, $w_{20} = [c, a, b, d, b, c]$, $w_{21} = [c, a, b, d, b, d]$, $w_{22} = [c, a, b, d, c, a]$, $w_{23} = [c, a, b, d, c, b]$, $w_{24} = [c, a, b, d, c, d]$, $w_{25} = [c, a, b, d, d, a]$, $w_{26} = [c, a, b, d, d, b]$, $w_{27} = [c, a, b, d, d, c]$. By the shifting property we get $w_{25} = w_{26} = w_{27} = 1$. By Lemma 3.1 and the shifting property we also have $w_{21} = w_{24} = 1$. Moreover, since all commutators of type 1 with $x_3 = b$ and $x_4 = c$ are trivial, Lemma 3.1 gives $w_{22} = w_{23} = 1$. Analogously, since all commutators of type 1 with $x_3 = b = x_4$ are trivial, it follows from 3.1 that $w_{19} = w_{20} = 1$. By expanding $[c, a, b, ad, ad, ad] = 1$ we easily obtain $w_{18} = 1$. Now, in order to expand $[c, a, b, abd, abd, abd] = 1$, we only need to consider commutators with $x_4 = d$ and $x_5 = a$. Thus we get $w_{16} = w_{18} = 1$. Finally, modulo $\gamma_6(G)$, $[a, c, b, d, a]$ is a product of commutators of weight 5 with entries b, c, d and neighboring a's by the closing property. By the shifting property and Lemma 3.1, each of these commutators can be written modulo $\gamma_6(G)$ as $[b, d, \dots]$ or $[c, d, \dots]$, so it is trivial. Thus $w_{17} = 1$. Therefore all commutators of type 1 with $x_3 = b$ are trivial.

If $x_3 = c$, we need to consider the following commutators: $w_{28} = [c, a, c, b, a, d]$, $w_{29} = [c, a, c, b, b, d]$, $w_{30} = [c, a, c, b, d, a]$, $w_{31} = [c, a, c, b, d, b]$, $w_{32} = [c, a, c, b, d, d]$, $w_{33} = [c, a, c, d, a, b]$, $w_{34} = [c, a, c, d, b, a]$, $w_{35} = [c, a, c, d, b, b]$, $w_{36} = [c, a, c, d, b, d]$, $w_{37} = [c, a, c, d, d, b]$. Since all commutators of type 1 with $x_3 = b$ are trivial, Lemma 3.1 and the shifting property give $w_{29} = w_{31} = w_{32} = w_{35} = w_{36} = w_{37} = 1$. By the closing property, $[a, c, c, b, a]$ is, modulo $\gamma_6(G)$, a product of commutators of weight 5 with entries b, c and neighboring a's. By the shifting property and Lemma 3.1, each of these commutators can be written modulo $\gamma_6(G)$ in the form $[b, d, \dots]$ or $[c, d, \dots]$, so it is trivial. Thus $w_{28} = 1$. Now $w_{30} = [a, c, c, b, d, a]^{-1} = [a, b, c, c, d, a]^{-1}$ by the shifting property, so $w_{30} = [b, a, c, c, d, a]$. Interchanging b and c in $w_8 = 1$, we obtain $w_{30} = 1$. Furthermore, by Lemma 3.1 we get $w_{34} = w_{30} = 1$. Finally, $[c, a, c, d, a] \in \gamma_6(G)$ by the closing property, so $w_{33} = 1$. Therefore all commutators of type 1 with $x_3 = c$ are trivial.

If $x_3 = d$ and $x_4 = a$, we have to consider the commutators $w_{38} = [c, a, d, a, b, b]$, $w_{39} = [c, a, d, a, b, c]$, $w_{40} = [c, a, d, a, b, d]$, $w_{41} = [c, a, d, a, c, b]$, $w_{42} = [c, a, d, a, d, b]$. First, $w_{38} = w_8 = 1$ by the shifting property. Moreover, Lemma 2.2 (v) of [4] gives $[c, a, d, a]^4 \equiv [c, d, a, a]^2$ modulo $\gamma_5(G)$. In our case this means $[c, a, d, a] \in \gamma_5(G)$, so $w_{39} = w_{40} = w_{41} = w_{42} = 1$. Therefore all commutators of type 1 with $x_3 = d$ and $x_4 = a$ are trivial.

If $x_3 = d$ and $x_4 = b$, we have to consider the commutators $w_{43} = [c, a, d, b, a, b]$, $w_{44} = [c, a, d, b, a, c]$, $w_{45} = [c, a, d, b, a, d]$, $w_{46} = [c, a, d, b, b, a]$, $w_{47} = [c, a, d, b, b, c]$, $w_{48} = [c, a, d, b, b, d]$, $w_{49} = [c, a, d, b, c, a]$, $w_{50} = [c, a, d, b, c, b]$, $w_{51} = [c, a, d, b, c, d]$, $w_{52} = [c, a, d, b, d, a]$, $w_{53} = [c, a, d, b, d, b]$, $w_{54} = [c, a, d, b, d, c]$. Modulo $\gamma_6(G)$, $[a, c, d, b, a]$ is a product of commutators of weight 5 with entries b, c, d and neighboring a's by the closing property. By the shifting property and Lemma 3.1, each of these commutators can be written modulo $\gamma_6(G)$ in the form $[b, d, \dots]$ or $[c, d, \dots]$, so it is trivial. Thus $w_{43} = w_{44} = w_{45} = 1$. Using the shifting property and the fact that all commutators of type 1 with $x_3 = 1 = x_4$ are trivial, we obtain

$w_{46} = w_{47} = w_{48} = 1$. Using the shifting property, together with Lemma 3.1, we easily get $w_{51} = w_{52} = w_{53} = w_{54} = 1$. Expansion of $[c, a, bd, bd, c, bd] = 1$ (by using the shifting property and the fact that all commutators of type 1 with $x_3 = b$ are trivial) gives $w_{50}w_{51} = 1$, so $w_{50} = 1$. Finally, $w_{49} = 1$ by the closing property. Therefore all commutators of type 1 with $x_3 = d$ and $x_4 = b$ are trivial.

If $x_3 = d$ and $x_4 = c$, then we have to consider commutators of the form $[c, a, d, c, x_5, x_6]$. Using Lemma 2.2 (v) of [4] together with the shifting property and Lemma 3.1, we obtain $[c, a, d, c]^4 \equiv [a, c, d, c]^{-4} \equiv [a, d, c, c]^{-2} \equiv [a, c, d, d]^{-2} \equiv [c, a, c, d]^2 \equiv [c, a, d, c]^2$ modulo $\gamma_5(G)$. Since G has no involutions, this means that $[c, a, d, c] \in \gamma_5(G)$. Therefore all commutators of type 1 with $x_3 = d$ and $x_4 = c$ are trivial.

If $x_3 = d = x_4$, then $w = 1$ by the shifting property. Therefore all commutators of type 1 are trivial.

Now we consider commutators $w = [c, b, x_3, x_4, x_5, x_6]$ of type 2.

If $x_3 = a = x_4$, we must consider commutators of the form $w = [c, b, a, a, x_5, x_6]$, with $d \in \{x_5, x_6\}$. By the shifting property we have $w = [c, a, a, b, x_5, x_6]$, with $d \in \{x_5, x_6\}$. Applying Lemma 3.1, we get $w = [c, d, a, a, b, , x_6] = 1$. Therefore all commutators of type 2 with $x_3 = a = x_4$ are trivial.

If $x_3 = a$ and $x_4 = b$, we have to consider the commutators $w_{55} = [c, b, a, b, a, d]$, $w_{56} = [c, b, a, b, c, d]$, $w_{57} = [c, b, a, b, d, a]$, $w_{58} = [c, b, a, b, d, c]$, $w_{59} = [c, b, a, b, d, d]$. From (3) we obtain that $w_{55} = w_1 = 1$. Moreover, the equation (2) yields $w_{57} = w_{19} = 1$, $w_{58} = w_{20} = 1$, $w_{59} = w_{21} = 1$. Finally, $w_{56} = w_{58} = 1$ by Lemma 3.1. Therefore all commutators of type 2 with $x_3 = a$ and $x_4 = b$ are trivial.

If $x_3 = a$ and $x_4 = c$, we must consider commutators $w = [c, b, a, c, x_5, x_6]$. We have that $w^{-1} = [b, c, a, c, x_5, x_6]$. Since the roles of b and c are symmetric, we conclude that $w^{-1} = 1$, as all commutators of type 2 with $x_3 = a$ and $x_4 = b$ are trivial. Therefore all commutators of type 2 with $x_3 = a$ and $x_4 = c$ are trivial.

If $x_3 = a$ and $x_4 = d$, we have to consider commutators $w = [c, b, a, d, x_5, x_6]$. If $x_5 = a$ then, by interchanging a and b in (2), we obtain $w = [c, a, b, a, d, x_6]$, so $w = 1$, since all commutators of type 1 are trivial. Moreover, if $x_5 \in \{b, c\}$, then $w = 1$ by Lemma 3.1, since all commutators of type 2 with $x_3 = a$ and $x_4 = b$ or $x_4 = c$ are trivial. Finally, if $x_5 = d$, then $w = 1$ by the shifting property. Therefore all commutators of type 2 with $x_3 = a$ are trivial.

If $x_3 = b$, then we may assume $x_4 = a$. In this case we only have to consider the commutators $w_{60} = [c, b, b, a, a, d]$, $w_{61} = [c, b, b, a, c, d]$, $w_{62} = [c, b, b, a, d, a]$, $w_{63} = [c, b, b, a, d, c]$, $w_{64} = [c, b, b, a, d, d]$. Clearly, $w_{60} = 1 = w_{64}$ by the shifting property. Setting $d = c$ in (1), we obtain $w_{61} = w_7 = 1$. From (1) we also get $w_{62} = w_8 = 1$. Finally, $w_{63} = w_{61} = 1$ by Lemma 3.1. Therefore all commutators of type 2 with $x_3 = b$ are trivial.

If $x_3 = c$, then we have to consider commutators $w = [c, b, c, x_4, x_5, x_6] = [b, c, c, x_4, x_5, x_6]^{-1}$. Thus, by the symmetry between b and c, $w = 1$ since all commutators of type 2 with $x_3 = b$ are trivial. It is also clear that if $x_3 = d$, then $w = 1$. Therefore all commutators of type 2 are trivial.

We are left with the commutators $w = [d, a, x_3, x_4, x_5, x_6]$ of type 3.

Let $x_3 = a$. If $x_4 = a$, then $w = 1$, since G is 3-Engel. If $x_4 \in \{b, c, d\}$, then $w = 1$ by the shifting property. Therefore all commutators of type 3 with $x_3 = a$ are trivial.

If $x_3 = b$ and $x_4 = a$, we have to consider the commutators $w_{65} = [d, a, b, a, b, c]$, $w_{66} = [d, a, b, a, c, b]$, $w_{67} = [d, a, b, a, c, c]$, $w_{68} = [d, a, b, a, c, d]$, $w_{69} = [d, a, b, a, d, c]$. By setting $c = d$ in Lemma 2.3 (v) of [4], we obtain $w_{65} = [d, a, a, b, b, c]^3$, so $w_{65} = 1$, since all commutators of type 3 with $x_3 = a$ are trivial. From (2), by interchanging c by d and replacing b by a, we get $[d, a, b, a, c] = 1$. Thus $w_{66} = w_{67} = w_{68} = 1$. Of course, $w_{69} = w_{68}$ by Lemma 3.1. Therefore all commutators of type 3 with $x_3 = b$ and $x_4 = a$ are trivial.

Clearly, all commutators of type 3 with $x_3 = b = x_4$ are trivial by the shifting property.

Let $x_3 = b$ and $x_4 = c$. If $x_5 = a$, then $w = [d, a, b, c, a, x_6]$. By the closing property, $[a, d, b, c, a]$ is, modulo $\gamma_6(G)$, a product of commutators of weight 5 with entries b, c, d and neighboring a's. By the shifting property and Lemma 3.1, each of these commutators can be written modulo $\gamma_6(G)$ in the form $[b, d, \dots]$ or $[c, d, \dots]$, so it is trivial. It follows that $w = 1$. The same argument shows that if $w_6 = a$, then $w = 1$. Thus we only need to consider the commutators $w_{70} = [d, a, b, c, b, c]$, $w_{71} = [d, a, b, c, b, d]$, $w_{72} = [d, a, b, c, c, b]$, $w_{73} = [d, a, b, c, c, d]$, $w_{74} = [d, a, b, c, d, b]$, $w_{75} = [d, a, b, c, d, c]$. Clearly, $w_{72} = w_{73} = 1$ by the shifting property. From (1), by replacing c by d, d by b and b by bc, we easily obtain $[d, a, b, bc, bc] \equiv [d, bc, bc, a, b] = 1$ modulo $\gamma_6(G)$. By Lemma 3.1 it follows that $[d, a, b, c, b] \in \gamma_6(G)$. Hence $w_{70} = w_{71} = 1$. Using (1) again, and replacing c by d, d by b and b by cd therein, we obtain $[d, a, b, cd, cd] \equiv [d, cd, cd, a, b] = 1$ modulo $\gamma_6(G)$. From Lemma 3.1 it follows that $[d, a, b, c, d][d, a, b, d, c] \in \gamma_6(G)$, that is, $[d, a, b, c, d]^2 \in \gamma_6(G)$. Then $[d, a, b, c, d] \in \gamma_6(G)$, since G has no involutions, so $w_{74} = w_{75} = 1$. Therefore all commutators of type 3 with $x_3 = b$ and $x_4 = c$ are trivial.

Let $x_3 = b$ and $x_4 = d$. If $a \in \{x_5, x_6\}$, then, as before, $w = 1$ by the closing property. Thus we only need to consider the commutators $w_{76} = [d, a, b, d, b, c]$, $w_{77} = [d, a, b, d, c, b]$, $w_{78} = [d, a, b, d, c, c]$. By Lemma 3.1 and the shifting property we easily get $w_{76} = 1$. Moreover, $w_{77} = w_{76} = 1$ by Lemma 3.1, and $w_{78} = 1$ by the shifting property. Therefore all commutators of type 3 with $x_3 = b$ are trivial.

If $x_3 = c$, then we have to deal with commutators $w = [d, a, c, x_4, x_5, x_6]$. By the symmetry between b and c, $w = 1$ since all commutators of type 3 with $x_3 = b$ are trivial.

If $x_3 = d$, then we may assume that $x_4 \in \{b, c\}$, since 2-generator 3-Engel groups are nilpotent of class ≤ 3. If $a \in \{x_5, x_6\}$, then, as before, $w = 1$ by the closing property. Thus we only need to consider the commutators $w_{79} = [d, a, d, b, b, c]$, $w_{80} = [d, a, d, b, c, b]$, $w_{81} = [d, a, d, b, c, c]$, $w_{82} = [d, a, d, c, b, b]$, $w_{83} = [d, a, d, c, b, c]$, $w_{84} = [d, a, d, c, c, b]$. Clearly, $w_{79} = w_{81} = w_{82} = w_{84} = 1$ by the shifting property. By the same property we also have $[d, a, d, b, bc, bc] = 1$, since $[b, d] = 1[c, d]$. Expanding this equality using the shifting property, we obtain $w_{80} = 1$. Finally, $w_{83} = 1$ by the symmetry between b and c. Therefore all commutators of type 3 are trivial.

This completes the proof. □

Lemma 3.4. *Let $G = \langle a, b, c, d \rangle$ be a 3-Engel group without involutions, and suppose that $[b, d] = 1 = [c, d]$. If $\{b, c, d\} \subseteq \{x_1, x_2, x_3, x_4, x_5\} \subseteq \{a, b, c, d\}$, then $[x_1, x_2, x_3, x_4, x_5] = 1$.*

Proof. This easily follows from Lemma 3.6 of [4]. □

Lemma 3.5. *Let $G = \langle a, b, c, d \rangle$ be a 3-Engel group without involutions, and suppose that $[b, d] = 1 = [c, d]$. If $\{x_1, x_2, x_3, x_4\} = \{a, b, c, d\}$, then*

$$[x_1, x_2, x_3, x_4] = [x_3, x_2, x_1, x_4][x_3, x_1, x_2, x_4]^{-1}.$$

Proof. Using Proposition 3.3 and Lemma 3.4, we obtain

$$\begin{aligned}[x_1, x_2, x_3, x_4] &= [[x_2, x_1]^{-1}, x_3, x_4] \\ &= [x_3, [x_2, x_1], x_4] \\ &= [[x_3, x_2, x_1][x_3, x_1, x_2]^{-1}, x_4] \\ &= [x_3, x_2, x_1, x_4][[x_3, x_1, x_2]^{-1}, x_4] \\ &= [x_3, x_2, x_1, x_4][x_3, x_1, x_2, x_4]^{-1},\end{aligned}$$

as required. □

Now we are ready to prove our main result.

Proof of Theorem 1.1. Let G be a 3-Engel group without involutions. Let $a, d \in G$ and $b, c \in C_G(d)$. By Lemma 3.2, the subgroup $\langle a, c, d\rangle$ of G has nilpotency class ≤ 4. Thus Lemma 2.2 (*iv*) of [4] gives $[d, a, a, c] = 1$. Replacing a by ab, we also obtain $[d, ab, ab, c] = 1$. By expanding the latter commutator using Lemma 3.4, we get $[d, a, b, c] = 1$. Hence Lemma 3.5 implies $[b, a, d, c] = 1$. By Lemma 3.1, this means $[b, a, c, d] = 1$. It now follows from Lemma 3.4 that $[a, b, c, d] = 1$, and the result follows by Lemma 2.6. □

Corollary 3.6. *There exist non-solvable $\mathfrak{C}_2$-groups.*

Proof. By Theorem 1 of [1] there exists a 3-Engel group of exponent 5 which is not solvable. Therefore the result follows by Theorem 1.1. □

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