ON THE SCHUR MULTIPLIERS OF FINITE p-GROUPS OF GIVEN COCLASS

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Abstract. In this paper we obtain bounds for the order and exponent of the Schur multiplier of a p-group of given coclass. These are further improved for p-groups of maximal class. In particular, we prove that if G is p-group of maximal class, then $|H_2(G,\mathbb{Z})| < |G|$ and $\exp H_2(G,\mathbb{Z}) \leq \exp G$. The bound for the order can be improved asymptotically.

1. Introduction and main results

The *coclass* of a finite p-group of order p^n and class c is defined to be $\mathrm{cc}(G) = n - c$. This invariant was introduced by Leedham-Green and Newman [14], and has proved to be one of the most promising tools in the attempt to classify or at least better understand the structure of finite p-groups.

Eick [5, 6] recently considered Schur multipliers of finite p-groups of fixed coclass and pro-p groups of finite coclass. Here the *Schur multiplier* $M(G)$ of an abstract group G is defined to be the second homology group $H_2(G,\mathbb{Z})$, where $\mathbb{Z}$ is considered as a trivial G-module, and if G is a pro-p group, then its Schur multiplier $M(G)$ is defined to be $H_2(G,\mathbb{Z}_p)$, where $\mathbb{Z}_p$ denotes the p-adic integers. Eick showed that for odd primes p there are finitely many p-groups G of fixed coclass r with $M(G)$ of prescribed order. Furthermore, if $p > 2$ and G is a pro-p group of finite coclass, then $|M(G)| = \infty$.

A fundamental aim of the theory of Schur multipliers has been to determine the order, rank and exponent of $M(G)$, where G is a finite group. In this paper we obtain new results on the structure of p-groups of fixed coclass and their Schur multipliers. At first we apply Shalev's structure theorem on p-groups of given coclass [21] to obtain bounds for the order and exponent of the nonabelian exterior square $G \wedge G$ of a p-group G of coclass r. These bounds can be expressed in terms of $\exp G$ and p and r only, and yield bounds for the order and exponent of $M(G)$. As a consequence we prove the following result.

Theorem 1.1. *Let G be a finite p-group of coclass r. Then $|M(G)| \leq |\gamma_2(G)|^r p^{r^2+2r}$.*

The proof of Theorem 1.1 given here was suggested to us by Avinoam Mann. It actually implies that $|M(G)| \leq |\gamma_2(G)|^{d-1} p^{kd-\binom{d}{2}}$, where $d = \mathrm{d}(G)$ is the minimal number of generators of G and $k = \log_p |G^{\mathrm{ab}}|$.

There is a classical result of Green [10] stating that if G is a group of order p^n, then $|M(G)| \leq p^{n(n-1)/2}$. If G is a p-group of coclass r, then $|G| \leq |G'|p^{r+1}$. Denoting $|G'| = p^\ell$, we obtain from Green's result that $\log_p |M(G)| \leq (\ell + r + 1)(\ell + r)/2$ which is quadratic in $\ell + r$. On the other hand, Theorem 1.1 implies

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that $\log_p |M(G)| \leq r^2 + 2r + \ell r$ which is only linear in ℓ. Thus the above result may be seen as an improvement of Green's bound (this was pointed out to us by the referee).

Finite p-groups of coclass 1 are also known as *p-groups of maximal class*. They were first studied by Blackburn [1] in 1958. Here we study Schur multipliers of p-groups of maximal class. At first we obtain the following generic bound.

Theorem 1.2. *Let G be a p-group of maximal class. Then $|M(G)| < |G|$.*

According to some computer experiments with small groups it seems quite probable that the order of $M(G)$ is somewhat close to $\sqrt{|G|}$. This is confirmed by the following result.

Theorem 1.3. *Let G be a p-group of maximal class, $|G| = p^n$ and $n > p+1$. Then*

$$|M(G)| \leq p^{\frac{p+1}{2}\lceil \frac{n-1}{p-1} \rceil}.$$

As for the exponent, our main result in this direction is the following.

Theorem 1.4. *Let G be a p-group of maximal class. Then the exponent of $G \wedge G$, and therefore the exponent of $M(G)$, divides* $\exp G$.

This bound is tight, even for the exponent of $M(G)$. As the referee observed, the estimate also improves a general bound obtained by Schur [22] stating that if G is a finite group, then $(\exp M(G))^2$ divides $|G|$. Namely, if G is a p-group of maximal class and order p^n, then its exponent is roughly $p^{n/(p-1)}$, which is at most $p^{n/2}$ if p is odd.

It had been conjectured for a long time that $\exp M(G)$ divides $\exp G$ for every finite group G. The conjecture is known to be false in general, see e.g. [18] for a counterexample. This leads to the following question.

Question 1.5. *Given a prime p, what is the smallest $r = r(p)$ such that there exists a p-group G of coclass r with* $\exp M(G) \nmid \exp G$*? Does there exist such r for $p > 2$?*

In the case $p = 2$, there exists a group G of order 2^{11} and coclass 5 with $\exp G = 4$ and $\exp M(G) = 8$. A construction is given in [18]. The question is whether there are examples of this kind having smaller coclass. The second part of Question 1.5 is related to the existence of a group G of odd order with $\exp M(G) > \exp G$. This still appears to be unknown.

The methods of the paper rely on the notion of the nonabelian tensor square of a group, a construction introduced by Brown and Loday [3]. Here we extend this notion to fit into the category of pro-p groups. We show that the complete nonabelian tensor square $G \mathbin{\hat{\otimes}} G$ of a pro-p group G is just the completion of the (abstract) nonabelian tensor square $G \otimes G$ equipped with a natural topology. Then we focus on complete nonabelian exterior squares $G \mathbin{\hat{\wedge}} G$. Extending Miller's work [17], we show that $M(G)$ is isomorphic to the kernel of the commutator map $G \mathbin{\hat{\wedge}} G \to G'$. Using the structure of pro-p groups of finite coclass, we derive bounds for the rank and $\mathbb{Z}_p$-length of $S \mathbin{\hat{\wedge}} S$, where S is an infinite pro-p group of finite coclass r.

2. The nonabelian tensor square of a pro-p group

Let G be a pro-p group. Define $G \mathbin{\hat{\otimes}} G$ to be the pro-p group (topologically) generated by the symbols $g \mathbin{\hat{\otimes}} h$, where $g, h \in G$, subject to the following relations that define $G \mathbin{\hat{\otimes}} G$ as a pro-p group:

$$g_1 g \mathbin{\hat{\otimes}} h = (g_1^g \mathbin{\hat{\otimes}} h^g)(g \mathbin{\hat{\otimes}} h) \quad \text{and} \quad g \mathbin{\hat{\otimes}} h_1 h = (g \mathbin{\hat{\otimes}} h)(g^h \mathbin{\hat{\otimes}} h_1^h), \tag{2.0.1}$$

for all $g, g_1, h, h_1 \in G$. The group $G \hat{\otimes} G$ is said to be the *(complete) nonabelian tensor square* of G. Denote $\nabla(G) = \overline{\langle x \hat{\otimes} x \mid x \in G \rangle}$. Then the group $G \hat{\wedge} G = (G \hat{\otimes} G)/\nabla(G)$ is called the *(complete) nonabelian exterior square* of G. Note that if G is abelian, then we have that $G \hat{\otimes} G \cong G^{\mathrm{ab}} \hat{\otimes}_{\mathbb{Z}_p} G^{\mathrm{ab}}$ and $G \hat{\wedge} G \cong G^{\mathrm{ab}} \hat{\wedge}_{\mathbb{Z}_p} G^{\mathrm{ab}}$.

The complete tensor square of a pro-p group has the following universal property. Let G and K be pro-p groups. A continuous map $\phi : G \times G \to K$ is said to be a *crossed pairing* if $(g_1 g, h)^\phi = (g_1^g, h^g)^\phi (g, h)^\phi$ and $(g, h_1 h)^\phi = (g, h)^\phi (g^h, h_1^h)^\phi$ for all $g, g_1, h, h_1 \in G$. If ϕ further satisfies $(x, x)^\phi = 1$ for all $x \in G$, then we say that ϕ is a *diagonal crossed pairing*. It is now evident that every crossed pairing $\phi : G \times G \to K$ induces a unique homomorphism $\psi : G \hat{\otimes} G \to K$ such that $(g \hat{\otimes} h)^\psi = (g, h)^\phi$ for all $g, h \in G$. A similar conclusion holds for diagonal crossed pairings and nonabelian exterior squares. Note that, when proving that a certain pro-p group T is isomorphic to $G \hat{\otimes} G$, it suffices to show that T satisfies the universal property for crossed pairings. Furthermore, since K is the inverse limit of its finite quotients, it suffices to assume that K is a finite p-group.

If G is a finite p-group (or an abstract group in general), then we write $\otimes$ and $\wedge$ instead of $\hat{\otimes}$ and $\hat{\wedge}$ throughout, respectively. In this case, the above notions are constistent with those of [3].

Theorem 2.1. *Let G be a pro-p group. If $G = \varprojlim G_i$, where $\{G_i \mid i \in I\}$ is an inverse system of finite p-groups, then $G \hat{\otimes} G = \varprojlim (G_i \otimes G_j)$ and $G \hat{\wedge} G = \varprojlim (G_i \wedge G_j)$, where the mixed terms $G_i \otimes G_j$ and $G_i \wedge G_j$, where $i \neq j$, are considered as the 'usual' abelian tensor (exterior) products over $\mathbb{Z}$.*

Proof. We only prove the assertion for the complete nonabelian tensor square, the exterior version follows along the same lines. Let $T = \varprojlim (G_i \otimes G_j)$, where (i, j) runs through $I \times I$. At first we note that $G \times G = \varprojlim (G_i \times G_j)$. We have canonical crossed pairings $G_i \times G_j \to G_i \otimes G_j$ given by $(g_i, g_j) \mapsto g_i \otimes g_j$. Taking inverse limits, we get a crossed pairing $\iota : G \times G \to T$. For $a, b \in G$ denote $a \hat{\otimes} b = (a, b)^\iota$. Suppose now K is a finite p-group and $\varphi : G \times G \to K$ a crossed pairing. For $i, j \in I$ denote by π_{ij} the projection $G \times G \to G_i \times G_j$. There exists a crossed pairing $\varphi_{ij} : G_i \times G_j \to K$ such that φ factors through it, i.e., $\varphi = \pi_{ij} \varphi_{ij}$; this follows from [23, Proposition 1.1.6], by suitable adaptation of the proof. By the universal property, there exist homomorphisms $\bar{\varphi}_{ij} : G_i \otimes G_j \to K$ such that $(g_i \otimes g_j)^{\bar{\varphi}_{ij}} = (g_i, g_j)^{\varphi_{ij}}$ for all $g_i \in G_i$ and $g_j \in G_j$. The homomorphism $\bar{\varphi} : T \to K$ defined via $\bar{\varphi}_{ij}$ now satisfies $(a \hat{\otimes} b)^{\bar{\varphi}} = (a, b)^\varphi$ for all $a, b \in G$. By a remark above, T has the universal property for crossed pairings, therefore $T \cong G \hat{\otimes} G$. □

A consequence of this theorem is that if G is a pro-p group, $G \hat{\otimes} G$ is the completion of $G \otimes G$ having the topology for which a fundamental system of neighborhoods consists of the kernels of the canonical maps $G \otimes G \to G_i \otimes G_j$. A similar statement holds true for $G \hat{\wedge} G$.

The commutator maps $\kappa_G^{\hat{\otimes}} : G \hat{\otimes} G \to \gamma_2(G)$ and $\kappa_G^{\hat{\wedge}} : G \hat{\wedge} G \to \gamma_2(G)$, defined by $g \hat{\otimes} h \mapsto [g, h]$ and $g \hat{\wedge} h \mapsto [g, h]$, respectively, are surjective homomorphisms of pro-p groups. In the abstract setting, these are closely related to the theory of Schur multipliers of groups and homotopy groups. For instance, Miller [17] proved that $M(G)$ is isomorphic to $\ker \kappa_G^{\wedge}$ for any group G. Furthermore, if a group G is given by a presentation $G = F/R$, then $G \wedge G \cong F'/[F, R]$. Analogs of these results also hold true for pro-p groups. They can be proved by suitably adjusting Miller's arguments, so we leave out the details. The result is as follows.

Proposition 2.2 (cf. Miller [17])**.** *Let G be a pro-p group given by a pro-p presentation $G = F/R$. Then $G \hat{\wedge} G \cong F'/[F, R]$, and $M(G) \cong \ker \kappa_{G,G}^{\hat{\wedge}}$.*

3. Pro-p groups of finite coclass

An infinite pro-p group S is said to have *coclass* r if $|S : \gamma_i(S)| = p^{i+r-1}$ for all sufficiently large i. In this section we will further explore the complete nonabelian exterior square of a pro-p group S of finite coclass. Its interest lies in the fact that $M(S) \cong \ker(S \mathbin{\hat\wedge} S \to S')$ by Proposition 2.2.

Let S be a pro-p-group of finite coclass r. By [13, Corollary 7.4.13], S has the structure of a *uniserial p-adic pre-space group*. More precisely, S is an extension of a $\mathbb{Z}_pP$-lattice $T \cong \mathbb{Z}_p^d$ by a finite p-group P, and P acts *uniserially* on T, i.e., $|T : [T, {}_iP]| = p^i$ for all $i \geq 0$. Without loss of generality we can take $T = \gamma_j(S)$ for some large enough $j \in \mathbb{N}$, and therefore $|\gamma_i(S) : \gamma_{i+1}(S)| = p$ for $i \geq j$. Furthermore, we have $d = (p-1)p^s$ for some s with $0 \leq s \leq r-1$ if p is odd, and $0 \leq s \leq r+1$ if $p = 2$. The integer d is said to be the *dimension* of S.

Let $S = F/R$ be a free pro-p presentation of S. Then we can identify $M(S) = (F' \cap R)/[R, F]$ by the Hopf formula, and $S \mathbin{\hat\wedge} S = F'/[R, F]$ by Proposition 2.2. The factor group $S_j = S/\gamma_j(S)$ has a free presentation $S_j = F/\gamma_j(F)R$, and thus we use the Hopf formula again to further identify $M(S_j) = (F' \cap \gamma_j(F)R)/[R, F]\gamma_{j+1}(F)$ and $S_j \wedge S_j = F'/[R, F]\gamma_{j+1}(F)$. Eick [5] showed that the 5 term homology sequence gives rise to the following exact sequence:

$$M(S) \longrightarrow M(S_j) \longrightarrow C_p \longrightarrow 1.$$

In view of the above identifications, all maps in the above sequence are the canonical ones. Let K_j be the kernel of the map $M(S) \to M(S_j)$. We have that $K_j = (R \cap [R, F]\gamma_{j+1}F)/[R, F]$ by [5]. Besides, we have a canonical map $S \mathbin{\hat\wedge} S \to S_j \wedge S_j$ given by $[x, y][R, F] \mapsto [x, y][R, F]\gamma_{j+1}(F)$. Its kernel is given by $L_j = [R, F]\gamma_{j+1}(F)/[R, F]$, and thus we have a natural embedding $K_j \hookrightarrow L_j$. From the presentations it becomes apparent that there is a natural homomorphism $L_j \to \gamma_j(S)$ given by $x[R, F] \to xR$ for $x \in [R, F]\gamma_{j+1}F$. The kernel of this map is precisely K_j. Its image is $[R, F]\gamma_{j+1}(F)R/R = \gamma_{j+1}(F)R/R = \gamma_{j+1}(S)$. Since $|\gamma_j(S) : \gamma_{j+1}(S)| = p$ by the uniseriality, and because of Proposition 2.2, we get the following result.

Theorem 3.1. *Let S be a pro-p group of finite coclass. Denote $S_j = S/\gamma_j(S)$, $K_j = \ker(M(S) \to M(S_j))$, and $L_j = \ker(S \mathbin{\hat\wedge} S \to S_j \wedge S_j)$. For every $j \gg 1$ we have the following commutative diagram with exact rows and columns:*

(3.1.1)

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & K_j & \longrightarrow & M(S) & \longrightarrow & M(S_j) & \longrightarrow & C_p & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & L_j & \longrightarrow & S \mathbin{\hat\wedge} S & \longrightarrow & S_j \wedge S_j & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \gamma_j(S) & \longrightarrow & S' & \longrightarrow & S'_j & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & C_p & & 1 & & 1 & & \\
 & & \downarrow & & & & & & \\
 & & 1 & & & & & &
\end{array}$$

Theorem 3.2. *Let $p > 2$ and let S be an infinite pro-p group of coclass r. Let d be the dimension of S. Then* $\mathrm{rk}(S \mathbin{\hat\wedge} S) \leq d + q^2(1 + \lceil \log_2 q \rceil) - 1$, *where $q = 2p^r + r - 1$.*

Proof. We have that $\gamma_j(S) \cong \mathbb{Z}_p^d$ for some large enough j. Thus it follows from the commutative diagram (3.1.1) that $\mathrm{rk}(S \mathbin{\hat\wedge} S) \leq d + \mathrm{d}(M(S)) + \mathrm{rk}(S_j \wedge S_j)$, where

$S_j = S/\gamma_j(S)$. Suppose that $M(S) \neq 1$. Using [5, Theorem 10], we get $\mathrm{d}(M(S)) = \mathrm{d}(M(S_j)) - 1$. As S_j is a p-group of coclass r, its rank is bounded by q [16, Corollary 9]. From [15, Theorem 2.3] we thus get $\mathrm{d}(M(S_i)) \leq \binom{q}{2} + q^2\lceil \log_2 q \rceil$, whereas [20, Proposition 3.3] implies that $\mathrm{rk}(S_i \wedge S_i) \leq \binom{q+1}{2} + q^2\lceil \log_2 q \rceil$. Combining these estimates, we get the required bound. □

Let S be a soluble pro-p group of finite rank. Then there exists a series $1 = S_0 \trianglelefteq S_1 \trianglelefteq \cdots \trianglelefteq S_n = S$ such that all factors S_i/S_{i-1} are procyclic. The number of factors isomorphic to $\mathbb{Z}_p$ is independent of the choice of series. It is called the *torsion-free rank* (or the $\mathbb{Z}_p$*-length*) of S. It is denoted by $\mathrm{tf}(S)$.

Suppose now that p is odd and that S is an infinite pro-p group of finite coclass. Denote with N the hypercenter of S, let T/N be the Fitting subgroup of S/N, and put $C/T = Z(S/T)$. The group C/T is cyclic of order p^t for some $t \in \mathbb{N}$. According to [6], we call t the *central exponent* of S.

Theorem 3.3. *Let $p > 2$ and let S be an infinite pro-p group of coclass r. Let d be the dimension of S and t the central exponent of S. Then* $\mathrm{tf}(S \mathbin{\hat{\wedge}} S) \leq d + p^{t-1}(p-1)/2$.

Proof. Since S is soluble by Theorem C of the coclass theorems [13], the group $S \mathbin{\hat{\wedge}} S$ is soluble too by diagram (3.1.1), hence $S \mathbin{\hat{\wedge}} S$ has finite torsion-free rank. As in the proof of Theorem 3.2 let $j \in \mathbb{N}$ be such that $\gamma_j(S) \cong \mathbb{Z}_p^d$, and put $S_j = S/\gamma_j(S)$. We get $\mathrm{tf}(S \mathbin{\hat{\wedge}} S) \leq d + \mathrm{tf}(M(S)) + \mathrm{tf}(S_j \wedge S_j)$. Note that $\mathrm{tf}(S_j \wedge S_j) = 0$, and [6, Theorem A] implies that $\mathrm{tf}(M(S)) = p^{t-1}(p-1)/2$. This gives the result. □

4. Finite p-groups of fixed coclass

Let G be a group of order p^n and nilpotency class $\mathrm{cl}(G) = c$. The *coclass* of G is defined to be $\mathrm{cc}(G) = n - c$. We refer to [7, 13] for accounts on coclass theory. One of the most illustrative notions here is that of the *coclass graph* $\mathcal{G}(p, r)$. The vertices of $\mathcal{G}(p, r)$ correspond to the isomorphism types of p-groups of coclass r. Two vertices G and H are joined by a directed edge from G to H if and only if $G \cong H/\gamma_{\mathrm{cl}(H)}(H)$. Every infinite pro-$p$ group S of coclass r determines a *maximal coclass tree* $\mathcal{T}(S)$ in $\mathcal{G}(p, r)$, namely, the subtree of $\mathcal{G}(p, r)$ consisting of all descendants of $S/\gamma_i(S)$, where i is minimal such that $S/\gamma_i(S)$ has coclass r and $S/\gamma_i(S)$ is not a quotient of another infinite pro-p group R of coclass r not isomorphic to S. The coclass theorems [13] imply that $\mathcal{G}(p, r)$ consists of finitely many maximal coclass trees and finitely many groups lying outside these trees.

The structure of maximal coclass trees can be further described as follows. Let S be an infinite pro-p group of coclass r and denote $S_j = S/\gamma_j(S)$. Then there exists i large enough such that $\mathcal{T}(S)$ contains a unique infinite path $S_i, S_{i+1}, \ldots$ starting from the root S_i of $\mathcal{T}(S)$. This path is called the *main line* of $\mathcal{T}(S)$. For $j \geq i$ we define $\mathcal{B}_j(S)$ to be the subtree of $\mathcal{T}(S)$ with the root S_j and containing all descendants of S_j that are not descendants of S_{j+1}. Each $\mathcal{B}_j(S)$ is a finite subtree of $\mathcal{T}(S)$, and it is called a *branch*. The subtree $\mathcal{T}(S, k)$ of $\mathcal{T}(S)$ containing all groups of distance at most k from the main line is called a *shaved tree*. We denote its branches by $\mathcal{B}_j(S, k)$.

In this section we derive bounds for the order and exponent of the nonabelian exterior square of a p-group of given coclass. Note that if G is a finite group, then $|G \wedge G| = |G'| \cdot |M(G)|$ and $\exp(G \wedge G) \geq \exp M(G)$, so these bounds will yield corresponding estimates for the order and exponent of $M(G)$.

At first we state the following elementary lemma that is probably well known.

Lemma 4.1. *Let G be a finite p-group of coclass r. Then* $\mathrm{d}(G) \leq r + 1$.

Proof. Let $|G| = p^n$, $\mathrm{d}(G) = d$ and $\mathrm{cl}(G) = c$. Then $|G : \Phi(G)| = p^d$. On the other hand, $|G : \Phi(G)| \leq |G : G'| \leq p^{n-(c-1)} = p^{r+1}$, as required. □

The following result will be fundamental in our considerations.

Lemma 4.2 (cf Ellis [8])**.** *Let G be a group and K and M normal subgroups of G, $K \leq M$. Then the sequence*

$$K \wedge G \longrightarrow M \wedge G \longrightarrow M/K \wedge G/K \longrightarrow 1,$$

where all the maps are canonical, is exact.

A notion closely related to coclass theory is that of uniserial actions. Let G be a finite p-group and N a normal subgroup of G. Denote $|N| = p^n$ and define $N_i = [N, {}_iG]$ for $i \geq 0$. We say that G acts *uniserially* on N if $|N_i : N_{i+1}| = p$ for $1 \leq i \leq n$.

Proposition 4.3. *Let G be a p-group acting uniserially on a normal subgroup N. Let $|N| = p^n$ and $\mathrm{d}(G) = t$. Then $|G \wedge G| \leq p^{tn}|G/N \wedge G/N|$ and $\exp(G \wedge G) \leq p^n \exp(G/N \wedge G/N)$.*

Proof. Define $N_0 = N$ and $N_{i+1} = [N_i, G]$ for $i \geq 0$. We have a central series $N = N_0 > N_1 > \cdots > N_n = 1$ with $|N_i : N_{i+1}| = p$ for all $i = 0, \ldots, n-1$. From the exact sequences

$$N \wedge G \longrightarrow G \wedge G \longrightarrow G/N \wedge G/N \longrightarrow 1$$

and

$$N_{i+1} \wedge G \longrightarrow N_i \wedge G \longrightarrow N_i/N_{i+1} \wedge G/N_{i+1} \longrightarrow 1,$$

where $i = 1, \ldots, n-1$, we obtain the following inequality:

$$|G \wedge G| \leq |G/N \wedge G/N| \prod_{i=1}^{n} |N_i/N_{i+1} \wedge G/N_{i+1}| \tag{4.3.1}$$

and

$$\exp(G \wedge G)| \text{ divides } \exp(G/N \wedge G/N) \prod_{i=1}^{n} \exp(N_i/N_{i+1} \wedge G/N_{i+1}). \tag{4.3.2}$$

Note that $N_i/N_{i+1} \cong C_p$ for all $i = 0, \ldots, n-1$. In addition to that, $N_i/N_{i+1} \leq Z(G/N_{i+1})$, hence G/N_{i+1} and N_i/N_{i+1} act trivially upon each other. Therefore, $N_i/N_{i+1} \wedge G/N_{i+1} \cong N_i/N_{i+1} \wedge (G/N_{i+1})^{\mathrm{ab}} \cong N_i/N_{i+1} \otimes (G/N_{i+1})^{\mathrm{ab}}/\langle n \otimes n(G/N_{i+1})^{\mathrm{ab}} \mid n \in N_i/N_{i+1} \rangle$, in other words, the group $N_i/N_{i+1} \wedge G/N_{i+1}$ is either isomorphic to $C_p \otimes G^{\mathrm{ab}}$ or to a quotient of this group by a subgroup of order p. In fact, if $i > 0$, then $N_i/N_{i+1} \leq G'/N_{i+1}$, hence $N_i/N_{i+1} \wedge G/N_{i+1} \cong C_p \otimes G^{\mathrm{ab}}$ for $i = 1, \ldots, n-1$. It follows from here that $\exp(N_i/N_{i+1} \wedge G/N_{i+1}) = p$ and $|N_i/N_{i+1} \wedge G/N_{i+1}| \leq p^t$, where $t = \mathrm{d}(G^{\mathrm{ab}}) = \mathrm{d}(G)$. Applying these to (4.3.1) and (4.3.2), we get the result. □

At this stage we require some general bounds for the order and exponent of the nonabelian exterior square of a given group.

Lemma 4.4 (Ellis, [8], Jones, [12])**.** *Let G be a finite p-group. If $|G| = p^n$ and $\mathrm{d}(G) = d$, then $p^{\binom{d}{2}} \leq |G \wedge G| \leq p^{nd-\binom{d+1}{2}}$.*

Lemma 4.5 ([19])**.** *Let G be a finite p-group of exponent p^e and nilpotency class $c \geq 2$. Then $\exp(G \wedge G)$ divides $p^{2e\lfloor \log_2 c \rfloor}$.*

We also mention the following result proved by Shalev [21].

Lemma 4.6 (Shalev, [21])**.** *Let G be a p-group of coclass r.*

(a) *Let p be odd and $|G| \geq p^{2p^r+r}$. Let $m = p^r - p^{r-1}$. Then $\gamma_m(G)$ is powerful with $d = (p-1)p^s$ generators, where $0 \leq s \leq r-1$.*
(b) *Let $p = 2$ and $|G| \geq 2^{2^{r+3}+r}$. Let $m = 2^{r+2}$. Then $\gamma_m(G)$ is powerful with $d = 2^s$ generators, where $0 \leq s \leq r+1$.*

In both cases, G acts uniserially on $\gamma_m(G)$ and $\mathfrak{V}_1(\gamma_i(G)) = \gamma_{i+d}(G)$ for all $i \geq m$.

Theorem 4.7. *Let G be a finite p-group of coclass r. Denote $d_1 = \mathrm{d}(G)$ and suppose that $\exp \gamma_m(G) = p^e$, and $\exp G/\gamma_m(G) = p^f$. With notations and assumptions of Lemma 4.6, we have $|G \wedge G| \leq p^{dd_1e+d_1(r+m-1)-\binom{d_1+1}{2}}$, and $\exp(G \wedge G)$ divides $p^{de+2f\lfloor \log_2(m-1)\rfloor}$.*

Proof. By Lemma 4.6, G acts uniserially on $\gamma_m(G)$. Using Proposition 4.3, we obtain $|G \wedge G| \leq |\gamma_m(G)|^{d_1}|G/\gamma_m(G) \wedge G/\gamma_m(G)|$, and that $\exp(G \wedge G)$ divides $|\gamma_m(G)| \exp(G/\gamma_m(G) \wedge G/\gamma_m(G))$. As $\gamma_m(G)$ is a powerful p-group, it follows that $|\gamma_m(G)| \leq p^{de}$ by a result of Lubotzky and Mann [15]. On the other hand, the group $G/\gamma_m(G)$ has coclass $\leq r$ and nilpotency class $\leq m-1$. It follows from here that $|G : \gamma_m(G)| \leq p^{r+m-1}$. By Lemma 4.4, $|G/\gamma_m(G) \wedge G/\gamma_m(G)| \leq p^{d_1(r+m-1)-\binom{d_1+1}{2}}$, whereas Lemma 4.5 implies that $\exp(G/\gamma_m(G) \wedge G/\gamma_m(G))$ divides $p^{2f\lfloor \log_2(m-1)\rfloor}$. From here the assertion follows readily. □

Using Theorem 4.7 and Lemma 4.1, we obtain the following bounds:

Corollary 4.8. *Let G be a p-group of coclass r and exponent p^e. With notations and assumptions of Lemma 4.6, we have $|G \wedge G| \leq p^{(r+1)^2e+(r+1)(r+m-1)-1}$, and $\exp(G \wedge G)$ divides $(\exp G)^{r+1+2\lfloor \log_2(m-1)\rfloor}$.*

We now proceed to the proof of Theorem 1.1. The following lemma was suggested to us by Avinoam Mann.

Lemma 4.9. *Let G be a finite p-group and denote $d = \mathrm{d}(G)$. Then $|M(G)| \leq |G'|^{d-1}|M(G^{\mathrm{ab}})|$.*

Proof. Let $G = F/R$ be a free presentation of G. Let $N = S/R$ be a subgroup of G of order p and contained in $\gamma_2(G)$. Then a result of Blackburn and Evens [2, Theorem 1.1] implies that there is an exact sequence

$$1 \longrightarrow [S,F]/[R,F] \longrightarrow M(G) \longrightarrow M(G/N) \longrightarrow C_p \longrightarrow 1.$$

Since $S/[R,F] \leq Z_2(F/[R,F])$, and S/R has order p, we see that $S/[R,F]$ centralizes $(\gamma_2(F)F^p)/[R,F]$, i.e., $[\gamma_2(F)F^p, S] = [R,F]$. Since $p^d = |G : \Phi(G)| = |F : \gamma_2(F)F^pR|$, we conclude that $|[S,F]/[R,F]| \leq p^d$. This, together with the above exact sequence, implies $|M(G)| \leq p^{d-1}|M(G/N)|$, and iterating this yields $|M(G)| \leq |\gamma_2(G)|^{d-1}|M(G/\gamma_2(G))|$, as required. □

Proof of Theorem 1.1. If $d = \mathrm{d}(G)$ and $|G^{\mathrm{ab}}| = p^k$, then $d \leq k \leq r+1$. By Lemma 4.4 we have that $|M(G^{\mathrm{ab}})| = |G^{\mathrm{ab}} \wedge G^{\mathrm{ab}}| \leq p^{kd-\binom{d}{2}}$. From Lemma 4.9 we therefore obtain $|M(G) \leq |\gamma_2(G)|^{d-1}p^{kd-\binom{d}{2}} \leq |\gamma_2(G)|^r p^{r^2+2r}$. □

In the rest of the section we focus on the Schur multipliers of p-groups of coclass r, according to their positions in the coclass trees induced by infinite pro-p groups S of coclass r. At first we deal with groups belonging to a given branch of $\mathcal{T}(S)$.

Proposition 4.10. *Let S be a pro-p group of coclass r and $G \in \mathcal{B}_j(S)$, where j is large enough. Then $|G \wedge G| \leq |\gamma_j(G)|^{\mathrm{d}(G)}|S_j \wedge S_j|$, and $\exp(G \wedge G)$ divides $|\gamma_j(G)| \exp(S_j \wedge S_j)$.*

Proof. This follows directly from Proposition 4.3, as every $G \in \mathcal{B}_j(S)$ is a uniserial extension of $\gamma_j(G)$ by S_j. □

If we consider only shaved trees, we obtain the following estimates that extend a related result proved by Eick for $M(G)$, see [5].

Proposition 4.11. *Let S be a pro-p group of coclass r and $G \in \mathcal{B}_j(S,k)$ for large enough j. Then $|G \wedge G| \leq p^{k\,\mathrm{d}(G)}|S_j \wedge S_j|$, and $\exp(G \wedge G)$ divides $p^k \exp(S_j \wedge S_j)$.*

Proof. Suppose without loss of generality that k is the distance of G from the main line of $\mathcal{T}(S)$. We prove the assertion by induction on k, the case $k = 0$ being obvious. Suppose that $k > 0$. Let $H \in \mathcal{B}_j(S, k-1)$ be the ancestor of G in $\mathcal{T}(S)$. Then $H \cong G/N$, where N is the last nontrivial term of the lower central series of G. As $N \cong C_p$ and $N \leq G'$, we conclude that $N \wedge G \cong C_p \otimes G^{\mathrm{ab}} \cong C_p^{\mathrm{d}(G)}$. From the exact sequence

$$N \wedge G \longrightarrow G \wedge G \longrightarrow H \wedge H \longrightarrow 1$$

we obtain $|G \wedge G| \leq p^{\mathrm{d}(G)}|H \wedge H|$, and that $\exp(G \wedge G)$ divides $p \exp(H \wedge H)$. This concludes the proof by the induction assumption. □

Finally we consider the main line groups of $\mathcal{T}(S)$ and obtain the following.

Proposition 4.12. *Let S be a pro-p group of coclass r and j large enough. Let $d_j = \mathrm{d}(S_j)$ and $\exp S_j = p^{e_j}$. Then $|S_j \wedge S_j| \leq p^{d_j(r+j-1)-\binom{d_j}{2}}$, and $\exp(S_j \wedge S_j)$ divides $p^{2e_j\lfloor \log_2(j-1)\rfloor}$.*

Proof. As $|S_j| = p^{r+j-1}$ and $\mathrm{cl}(S_j) = j-1$, this is a consequence of Lemma 4.4 and Lemma 4.5. □

5. Finite p-groups of maximal class

Finite p-groups of coclass 1 are also known as *p-groups of maximal class*. For $p = 2$ these are known to be either dihedral, semidihedral or quaternion groups [13, Theorem 3.3.10]. We have that $M(D_{2^n}) = C_2$, $M(SD_{2^n}) = 1$ and $M(Q_{2^n}) = 1$ for every n. Thus we assume for the rest of this section that p is odd, unless otherwise stated.

Finite p-groups of maximal class have a tight power-commutator structure, as opposed to the p-groups of larger coclass. Thus the asymptotic results obtained in Section 4 that hold for any coclass do not seem to reflect the whole structure of p-groups of maximal class. Here we obtain relatively sharp bounds for the order and exponent of the Schur multiplier of a p-group G of maximal class. At first we deal with the order and prove Theorem 1.2 stating that $|M(G)| < |G|$. This will follow from a bound obtained for a larger class of p-groups, called the *ECF-groups* according to Blackburn [1]. Let p be a prime and m and n integers, $3 \leq m \leq n$. Then $\mathrm{ECF}(m,n,p)$ is defined to be the set of all groups of order p^n and class $m-1$ for which G^{ab} is elementary abelian and $|\gamma_i(G) : \gamma_{i+1}(G)| = p$ for $2 \leq i \leq m-1$. It is clear that every p-group of maximal class and order p^n belongs to $\mathrm{ECF}(n,n,p)$. Here we prove the following.

Proposition 5.1. *If $G \in \mathrm{ECF}(m,n,p)$, then $|M(G)| \leq p^{(n-m+1)(n+m-2)/2}$.*

Proof. By definition we have that $|G'| = p^{m-2}$ and hence $G^{\mathrm{ab}} \cong C_p^{n-m+2}$. Therefore $M(G^{\mathrm{ab}}) \cong C_p^{(n-m+2)(n-m+1)/2}$ and $\mathrm{d}(G) = n-m+2$. Now the result follows from Lemma 4.9. □

Theorem 1.2 now follows from Proposition 5.1 applied to the case $m = n$.

Next we deal with the exponent of $M(G)$. Our aim is to prove Theorem 1.4. We require the following lemma on commutator calculus.

Lemma 5.2. *Let G be a group and N a normal subgroup of G. Then the following hold:*

(a) $[\mathfrak{V}_i(N), G] \equiv \mathfrak{V}_i([N, G]) \mod \prod_{r=1}^{i} \mathfrak{V}_{i-r}([G, {}_{p^r}N])$,

(b) $[\mathfrak{V}_i(N), G] \equiv \mathfrak{V}_i([N, G]) \mod \prod_{r=1}^{i} [\mathfrak{V}_{i-r}(N), {}_{r(p-1)+1}G]$.

Proof. The congruence (a) is a well known consequence of the Hall-Petrescu identity and can be proved by suitably modifying the proof of [4, Lemma 11.9]. In order to prove (b), we use induction on i, the cases $i = 0$ and $i = 1$ being obvious. Now let $1 \leq r \leq i$. Then the induction assumption implies

$$[\mathfrak{V}_{i-r}(N), {}_{p^r}G] \equiv \mathfrak{V}_{i-r}([N, {}_{p^r}G]) \mod \prod_{t=1}^{i-r} [\mathfrak{V}_{i-r-t}(N), {}_{t(p-1)+p^r}G].$$

As $p^r \geq r(p-1)+1$, we get $\mathfrak{V}_{i-r}([N, {}_{p^r}G]) \leq \prod_{r=1}^{i} [\mathfrak{V}_{i-r}(N), {}_{r(p-1)+1}G]$. Thus

$$\prod_{r=1}^{i} \mathfrak{V}_{i-r}([G, {}_{p^r}N]) \leq \prod_{r=1}^{i} \mathfrak{V}_{i-r}([N, {}_{p^r}G]) \leq \prod_{r=1}^{i} [\mathfrak{V}_{i-r}(N), {}_{r(p-1)+1}G].$$

By (a) the result follows. □

Proof of Theorem 1.4. Let $|G| = p^n$ and suppose that $\exp G = p^e$. Let H be a covering group of G. Then there exists $Z \leq Z(H) \cap H'$ such that $H/Z \cong G$ and $Z \cong M(G)$. As G is nilpotent of class $n-1$, we conclude that $\gamma_{n+1}(H) = 1$. As $G \wedge G \cong H'$, it suffices to show that $\mathfrak{V}_e(H') = 1$. By our assumption we have

$$[\mathfrak{V}_e(H), H] = 1. \tag{5.2.1}$$

If $n < p+1$, then Lemma 5.2 (a), together with (5.2.1), implies $\mathfrak{V}_e(\gamma_2(H)) \leq \prod_{r=1}^{e} \mathfrak{V}_{e-r}(\gamma_{p^r+1}(H))$, hence $\mathfrak{V}_e(\gamma_2(H)) = 1$. Suppose now that $n = p+1$. Then G is not regular, and we have $\exp \gamma_2(G) = \exp G/\gamma_p(G) = p$ [13]. Therefore $\exp G = p^2$. It follows from here that $[\mathfrak{V}_2(H), H] = 1$ and $\mathfrak{V}_1(H) \leq \gamma_p(H)Z$. The latter implies that $[\mathfrak{V}_1(H), H, H] = 1$. From Lemma 5.2 (a) we now obtain $\mathfrak{V}_2([H, H]) \leq \mathfrak{V}_1(\gamma_{p+1}(H))$. As $\gamma_{p+1}(H) \leq Z$, we conclude $\mathfrak{V}_1(\gamma_{p+1}(H)) = [\mathfrak{V}_1(H), {}_pH] = 1$, thus $\mathfrak{V}_2([H, H]) = 1$, as required.

From here on we assume that $n > p+1$. Define $P_1 = C_G(\gamma_2(G)/\gamma_4(G))$ and $P_i = \gamma_i(G)$ for $2 \leq i \leq n$. By [13] we have a chief series $G > P_1 > P_2 > \cdots > P_{n-1} > P_n = 1$. Denote $\exp P_i = p^{e_i}$ for $1 \leq i \leq n$. Furthermore, we have that $\Omega_1(P_i) = P_{n-p+1}$ and $\mathfrak{V}_1(P_i) = P_{i+p-1}$ for every i with $1 \leq i \leq n-p+1$. As P_1 is regular by [1, p. 69, Corollary 1], we conclude from here that if $i \geq 1$, then $\mathfrak{V}_k(P_i) = P_{i+k(p-1)}$ for $i + k(p-1) \leq n$, and $\mathfrak{V}_k(P_i) = 1$ otherwise. It follows that $\exp P_i = p^{e_i}$, where $e_i = \lceil (n-i)/(p-1) \rceil$. Clearly we have $e_1 \geq 2$. Choose any $x \in G \setminus P_1$. Then $G = \langle x \rangle P_1$. For any $g \in G$ we have $g = x^\alpha y$ for some $y \in P_1$ and $\alpha \in \mathbb{Z}$. If α is divisible by p, then $x^\alpha \in P_{n-1} \leq Z(G)$ by [1], therefore $g^{p^{e_1}} = 1$, and $g^p = y^p \in \mathfrak{V}_1(P_1) = P_p$. Assume now that $\gcd(\alpha, p) = 1$. This clearly gives $x^\alpha \notin P_1$. Replacing x by a suitable power, we may assume without loss of generality that $\alpha = 1$. If $y \in P_2$, we get by [1, p. 64, Corollary 2] that xy is conjugate to x, and $(xy)^p = x^p \in P_{n-1} \leq P_p$. This yields $g^{p^2} = 1$, hence also $g^{p^{e_1}} = 1$. Suppose now that $y \notin P_2$. Then it follows from [1, Lemma 3.3] that $(xy)^p \equiv x^p \mod P_{p+1}$. Thus $g^p = x^p z$ for some $z \in P_{p+1}$. As $x^p \in P_{n-1} \leq Z(G)$, it follows that $g^p \in P_p$ and $g^{p^{e_1}} = z^{p^{e_1-1}}$. By the above we have that $e_{p+1} = 1$ if $n \leq 2p$, whereas $e_{p+1} = e_2 - 1 \leq e_1 - 1$ if $n > 2p$. In both cases we conclude that $g^{p^{e_1}} = 1$. This shows that $\exp G = \exp P_1$, and we have also proved along the way that $\mathfrak{V}_1(G) \leq P_p$.

Write $P_i = Q_i/Z$ for $i = 1, \ldots n$. Then we have a series $H > Q_1 > Q_2 > \cdots > Q_{n-1} > Q_n = Z$. From the above argument we get that $[\mathfrak{V}_{e_1}(H), H] = 1$

and $[\mathfrak{U}_{e_i}(Q_i), H] = 1$ for $i = 1, \dots, n$. In addition to that, we have $Q_i = \gamma_i(H)Z$ for $i \geq 2$, hence $\gamma_2(H) = Q_2$ and $\gamma_{i+1}(H) = [Q_i, H]$ for $i \geq 2$. This implies $[\mathfrak{U}_k(Q_i), H] = \gamma_{i+k(p-1)+1}(H)$ for $i \geq 1$. Furthermore, since $H/Q_p \cong G/P_p$, we conclude that $\mathfrak{U}_1(H) \leq Q_p$ by the above. By Lemma 5.2 (b) we have

$$\mathfrak{U}_{e_1}([H,H]) \leq \prod_{r=1}^{e_1} [\mathfrak{U}_{e_1-r}(H), {}_{r(p-1)+1}H]. \tag{5.2.2}$$

If $r = e_1$, then $[\mathfrak{U}_{e_1-r}(H), {}_{r(p-1)+1}H] = \gamma_{e_1(p-1)+2}(H) \leq \gamma_{n+1}(H) = 1$. Assume now that $r < e_1$. Then $\mathfrak{U}_{e_1-r}(H) \leq \mathfrak{U}_{e_1-r-1}(\mathfrak{U}_1(H)) \leq \mathfrak{U}_{e_1-r-1}(Q_p)$. This inclusion yields that $[\mathfrak{U}_{e_1-r}(H), {}_{r(p-1)+1}H] \leq [[\mathfrak{U}_{e_1-r-1}(Q_p), H], {}_{r(p-1)}H] = \gamma_{p+(e_1-r-1)(p-1)+1+r(p-1)}(H) = \gamma_{2+e_1(p-1)}(H) \leq \gamma_{n+1}(H) = 1$. Therefore the equation (5.2.2) becomes $\mathfrak{U}_{e_1}([H,H]) = 1$, and this concludes the proof. □

Let G be a p-group of maximal class with $|G| = p^n$. In the course of proof of Theorem 1.4 we considered three separate cases: $n < p+1$, $n > p+1$, and $n = p+1$. In all these, there are examples showing that the bound for $\exp(G \wedge G)$ given by Theorem 1.4 is tight in general.

Example 5.3. Let G be the 26th group in the GAP [9] library of groups of order 3^5. Then G is of maximal class and $\exp(G) = \exp M(G) = 9$.

Similiarly, let G be the 7th group of order 5^4 in the GAP library of small groups. Then $\operatorname{cl} G = 3$ and $\exp G = \exp M(G) = 5$.

Lastly, let G be the 630th group of order 5^6 in the GAP library of small groups. We have that G is of maximal class, $\exp G = \exp(G \wedge G) = 25$, but $\exp M(G) = 5$.

On the other hand, we have been able to find examples with $\exp M(G) = \exp G$ only in the cases when $n \neq p+1$. If $n = p+1$, then $\exp G = p^2$. Computer experiments with GAP [9] suggest that $M(G)$ is always an elementary abelian p-group in this case. The methods of the proof of Theorem 1.4 do not seem to be strong enough to decide whether or not this is true.

Returning to the question of the order of $M(G)$, we prove Theorem 1.3 and thus improve the bound of Theorem 1.2 for groups G with $\log_p |G| > p+1$.

Proof of Theorem 1.3. By the Universal Coefficient Theorem we have that the group $H^2(G, \mathbb{Z}/p\mathbb{Z})$ is isomorphic to $\operatorname{Ext}(G^{\mathrm{ab}}, \mathbb{Z}/p\mathbb{Z}) \oplus \operatorname{Hom}(\mathbb{Z}/p\mathbb{Z}, M(G))$, hence $H^2(G, \mathbb{Z}/p\mathbb{Z}) \cong (G^{\mathrm{ab}} \otimes \mathbb{Z}/p\mathbb{Z}) \oplus (M(G) \otimes \mathbb{Z}/p\mathbb{Z})$. Therefore we conclude that $\mathrm{d}(M(G)) = \dim_{\mathbb{F}_p} H^2(G, \mathbb{Z}/p\mathbb{Z}) - \mathrm{d}(G)$. Since G is a p-group of maximal class, we have that $\mathrm{d}(G) = 2$ by Lemma 4.1. Let $r(G)$ be the minimal number of relations needed in a presentation of G as a factor group F/R of a free group F of rank 2 by R. By [11, Proposition 7.2.4] we have that $H^2(G, \mathbb{Z}/p\mathbb{Z}) \cong \operatorname{Hom}(R/R^p[F,R], \mathbb{Z}/p\mathbb{Z})$, hence $\dim_{\mathbb{F}_p} H^2(G, \mathbb{Z}/p\mathbb{Z}) \leq r(G)$. From [13, Exercise 3.3.4] it follows that $r(G) \leq (p+5)/2$. This yields $\mathrm{d}(M(G)) \leq (p+1)/2$. By Theorem 1.4 we have that $\log_p \exp M(G) \leq \lceil (n-1)/(p-1) \rceil$. This gives the required result. □

References

[1] N. Blackburn, *On a special class of p-groups*, Acta Math. **100** (1958), 49–92.

[2] N. Blackburn, and L. Evens, *Schur multipliers of p-groups*, J. Reine Angew. Math. **309** (1979), 100–113.

[3] R. Brown, and J.-L. Loday, *Van Kampen theorems for diagrams of spaces*, Topology **26** (1987), no. 3, 311–335.

[4] J. D. Dixon, M. P. F. du Sautoy, A. Mann, and D. Segal, *Analytic pro-p groups*, 2nd Edition, Cambridge University Press, 1999.

[5] B. Eick, *Schur multiplicators of infinite pro-p-groups with finite coclass*, Israel J. Math. **166** (2008), 147–156.

[6] B. Eick, *Schur multiplicators of finite p-groups with fixed coclass*, Israel J. Math. **166** (2008), 157–166.

[7] B. Eick, and C. Leedham-Green, *On the classification of prime-power groups by coclass*. Bull. Lond. Math. Soc. **40** (2008), no. 2, 274–288.
[8] G. Ellis, *On the relation between upper central quotients and lower central series of a group*, Trans. Amer. Math. Soc. **353** (2001), no. 10, 4219–4234.
[9] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.4.12*; 2008, (`http://www.gap-system.org`).
[10] J. A. Green, *On the number of automorphisms of finite groups*, Proc. Roy. Soc. London Ser. A **237** (1956), 574–581.
[11] K. W. Gruenberg, *Cohomological topics in group theory*, Springer Verlag, New York, 1970.
[12] M. R. Jones, *Some inequalities for the multiplicator of a finite group*, Proc. Amer. Math. Soc. **39** (1973), 450–456.
[13] C. R. Leedham-Green, and S. McKay, *The structure of groups of prime power order*, Oxford University Press, 2002.
[14] C. R. Leedham-Green, and M. F. Newman, *Space groups and groups of prime-power order I*, Arch. Math. **35** (1980), 193–203.
[15] A. Lubotzky, and A. Mann, *Powerful p-groups. I. Finite groups*, J. Algebra **105** (1987), 484–505.
[16] A. Mann, *Space groups and groups of prime power order VII. Powerful p-groups and uncovered p-groups*, Bull. London Math. Soc. **24** (1992), 271–276.
[17] C. Miller, *The second homology of a group*, Proc. Amer. Math. Soc. **3** (1952), 588–595.
[18] P. Moravec, *Schur multipliers and power endomorphisms of groups*, J. Algebra **308** (2007), no. 1, 12–25.
[19] P. Moravec, *The exponents of nonabelian tensor products of groups*, J. Pure Appl. Algebra **212** (2008), no. 7, 1840–1848.
[20] P. Moravec, *Groups of prime power order and their nonabelian tensor squares*, Israel J. Math., to appear.
[21] A. Shalev, *The structure of finite p-groups: effective proof of the coclass conjectures*, Invent. Math. **115** (1994), 315–345.
[22] I. Schur, *Über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen*, J. für Math. **127** (1904), 20–50.
[23] J. S. Wilson, *Profinite groups*, Clarendon Press, Oxford, 1998.

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