LIE ALGEBRAS WITH ABELIAN CENTRALIZERS

IGOR KLEP AND PRIMOŽ MORAVEC

Abstract. We classify all finite dimensional Lie algebras over an algebraically closed field of characteristic 0, whose nonzero elements have abelian centralizers. These algebras are either simple or solvable, where the only simple such Lie algebra is $\mathfrak{sl}_2$. In the solvable case they are either abelian or a one-dimensional split extension of an abelian Lie algebra.

1. Introduction

A Lie algebra L is called *commutative transitive (CT)* if for all $x, y, z \in L \setminus \{0\}$, $[x, y] = [y, z] = 0$ imply $[x, z] = 0$. A similar notion for groups was defined and studied by Weisner [We] in 1925. Wu [Wu] proved in 1998 that finite CT groups are either solvable or simple, thus fixing gaps in Weisner's proof. In the solvable case, they are either abelian or cyclic split extensions of their Fitting subgroups [Wu]. Finite nonabelian simple CT groups had been classified by Suzuki [Su] in 1957. He proved that every finite nonabelian simple CT group is isomorphic to some $\mathrm{PSL}(2, 2^f)$, where $f > 1$. Suzuki's result is considered to have been one of the key stones in the proof of the Odd Order Theorem by Feit and Thompson [FT]. Wu's and Suzuki's arguments use deep techniques from cohomology theory and the theory of group representations.

It is straightforward to see that a Lie algebra is CT if and only if its nonzero elements have abelian centralizers. The property CT is clearly subalgebra closed, yet it is not quotient closed, since every free Lie algebra is CT (Example 10). Therefore we restrict ourselves to finite dimensional Lie algebras. The purpose of this paper is to classify finite dimensional CT Lie algebras over an algebraically closed field of characteristic 0. By Levi's theorem [Ja, p. 91] every such algebra is a semidirect product of its radical and a semisimple subalgebra. We first study the solvable case (Section 2). It is shown that every solvable CT Lie algebra L over a field of characteristic 0 is either abelian or a semidirect product of its nil radical N which is abelian, and an abelian Lie algebra that acts on N fixed-point-freely (see Section 2 for a precise definition). Furthermore, the complements to N are one-dimensional in the algebraically closed case. Conversely, every Lie algebra that is an extension of an abelian Lie algebra by a CT Lie algebra acting fixed-point-freely on it, is CT. In Section 3 we apply the classification theorem of finite dimensional simple Lie algebras over an algebraically closed field of characteristic 0 to show that $\mathfrak{sl}_2$ is the only CT Lie algebra of this type. Finally, in Section 4 we combine the above mentioned results and Levi's theorem to show that every finite dimensional CT Lie algebra over an algebraically closed field of characteristic 0 is either solvable or simple. The paper contains several examples demonstrating that the results obtained are best possible. We emphasize that, unlike in the group theoretical case, our proofs are elementary.

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Throughout the paper k will denote a field of characteristic 0 and L will be a finite dimensional Lie algebra over k. Our basic references for Lie algebras are Humphreys [Hu] and Jacobson [Ja].

2. Solvable CT Lie algebras

In the beginning we mention the following elementary property of CT Lie algebras:

Lemma 1. *Every nonabelian CT Lie algebra is directly indecomposable.*

Proof. Let L be a nonabelian CT Lie algebra and suppose that $L = L_1 \oplus L_2$. Without loss of generality we may assume that L_1 is nonabelian. Let $x, y \in L_1$ be two noncommuting elements. If $L_2 \neq \{0\}$, choose $z \in L_2 \setminus \{0\}$. As $[x, z] = [z, y] = 0$, we get $[x, y] = 0$ by the CT property. This contradicts our assumption. □

Let L be a Lie algebra, N any ideal in L and U a subalgebra in L. Then U acts on N by derivations, that is, $(u, n) \mapsto [u, n]$, where $u \in U$ and $n \in N$. Each action of U induces *conjugation* $(u, n) \mapsto n + [u, n]$. Furthermore, if N an abelian ideal in L, then L/N acts on N by $(x + N, n) \mapsto [x, n]$, where $x \in L$ and $n \in N$.

An action of an algebra U on an ideal N of L is said to be *fixed-point-free* if the stabilizer of any nonzero element of N in U under conjugation is trivial. This is equivalent to saying that $[u, n] = 0$ implies $u = 0$ or $n = 0$ for every $u \in U$ and $n \in N$. Note that if N is an abelian ideal in L, then L/N acts fixed-point-freely on N if and only if $C_L(y) \subseteq N$ for every $y \in N \setminus \{0\}$.

Proposition 2. *Let L be a Lie algebra, N an abelian ideal in L, and suppose that L/N is a CT Lie algebra that acts fixed-point-freely on N. Then L is a CT Lie algebra.*

Proof. Let $x, y, z \in L \setminus \{0\}$ and suppose that $[x, y] = [y, z] = 0$. If $y \in N$, then $x, z \in C_L(y) \subseteq N$, hence clearly $[x, z] = 0$. So we may assume that $y \notin N$. If one of x or z, say x, belongs to N, then $y \in C_L(x) \subseteq N$, a contradiction. Thus we may assume that neither of x, y, z belongs to N. Then we use the fact that L/N is CT to conclude that $[x, z] \in N$. Suppose that $[x, z] \neq 0$. Then the Jacobi identity gives $[[x, z], y] = 0$, hence $y \in C_L([x, z]) \subseteq N$, contrary to our assumption. □

Remark. The notion of CT Lie algebras is thus closely related to fixed-point-free actions. Another evidence of this is the fact that a semidirect product of an abelian Lie algebra N by a simple CT Lie algebra S is CT if and only if the action is fixed-point-free. One direction is an easy consequence of Proposition 2. For the converse we proceed as follows. Suppose $L = N \rtimes_\varphi S$ is CT and φ is not fixed-point-free. Then there are nonzero $x \in S$ and $u \in N$ such that $\varphi(x)u = [u, x] = 0$. By the CT property, $[x, v] = 0$ for all $v \in N$. Hence $\ker \varphi \neq \{0\}$. Since S is simple, this implies $\varphi = 0$, therefore $L = N \oplus S$. This contradicts the assumption by Lemma 1.

On the other hand, a semidirect product of two abelian Lie algebras can be CT even if the action is not fixed-point-free. For example, if $L = k \rtimes_\varphi k^2$ and the action is induced by $\varphi(1, 0) = 0$ and $\varphi(0, 1) = 1$, then L is easily seen to be CT.

In the solvable case, a strong converse of Proposition 2 holds true, as our next result shows.

Theorem 3. *Let L be a finite dimensional solvable CT Lie algebra over k. If L is nonabelian, then L is a semidirect product of its nil radical N which is abelian, and an abelian Lie algebra that acts fixed-point-freely on N. If U and V are two complements to N in L, then there exists $a \in N$ such that $V = (1 + \operatorname{ad} a)(U)$. Furthermore, if k is algebraically closed, then the complements are one-dimensional.*

Proof. As N is nilpotent, it has nontrivial center. If $x \in Z(N) \setminus \{0\}$, then any two elements of N commute with x, hence they commute with each other by the CT property. Thus N is abelian. Since L is solvable, Lie's theorem [Hu, 4.1] implies that $[L, L] \leqslant N$, thus L/N is also abelian, hence L is metabelian. Let us first show that L/N acts fixed-point-freely on N. Let $y \in N \setminus \{0\}$ and suppose that there exists $a \in C_L(y) \setminus N$. Since $[a, y] = 0$ and N is abelian, we conclude that $[a, N] = \{0\}$ by the CT property. Let I be the ideal in L generated by a. For any $x \in L$ and $n \in N$ we get $[[x, a], n] = 0$ by the Jacobi identity, hence $[I, N] = \{0\}$. Using the fact that L is a CT Lie algebra, we conclude that I is abelian, thus $I \subseteq N$. But this implies that $a \in N$, which is a contradiction proving our assertion.

For any $y \in N \setminus \{0\}$ we now get $N \subseteq C_L(y) \subseteq N$, hence $C_L(y) = N$. From here it follows that $C_L(N) = N$. Note also that $[L, u] \subseteq N$ and $[N, u] \neq 0$ for any $u \in L \setminus N$. We claim that $[N, u] = N$ for every $u \in L \setminus N$. So suppose there exists $u \in L \setminus N$ such that $[N, u] \neq N$, and pick $v \in N \setminus [N, u]$. Consider the sequence of elements $v, [v, u], [[v, u], u], \ldots$. Denote $[v,{}_1 u] = [v, u]$ and define $[v,{}_i u]$ inductively by $[v,{}_i u] = [[v,{}_{i-1} u], u]$ for $i > 1$. Let ℓ be the smallest integer such that

$$[v,{}_\ell u] = c_0 v + c_1 [v, u] + \cdots + c_{\ell-1} [v,{}_{\ell-1} u]$$

for some scalars $c_0, \ldots, c_{\ell-1}$. Note that $c_0 = 0$, since otherwise $v \in [N, u]$ which is not the case. Denote $z = [v,{}_{\ell-1} u] - c_1 v - \cdots - c_{\ell-1} [v,{}_{\ell-2} u]$. Then $z \in N$ and $u \in C_L(z)$. If $z \neq 0$, then $u \in C_L(z) \subseteq N$, a contradiction. Thus $z = 0$, but this implies that $[v,{}_{\ell-1} u] = c_1 v + \cdots + c_{\ell-1} [v,{}_{\ell-2} u]$, contrary to the choice of ℓ. Hence $[N, u] = N$ for every $u \in L \setminus N$. Then we have $N = [N, u] \subseteq [L, u] \subseteq N$, whence $[L, u] = N$. Choose now any $x \in L$. Then $[x, u] \in N = [N, u]$, hence there exists $a \in N$ such that $[x, u] = [a, u]$. It follows that $x - a \in C_L(u)$, therefore $L = N + C_L(u)$. Define $C = N + ku$. Then C is a nonabelian ideal in L. If $b \in N \cap C_L(u)$, then $b \in Z(C) = \{0\}$, hence $N \cap C_L(u) = \{0\}$. This shows that $C_L(u)$ is a complement to N in L.

For $u \in L \setminus N$ denote $U = C_L(u)$. Let V be any complement to N in L. We can write $u = b + v$ for some $b \in N$ and $v \in V$. Since $b \in N = [N, u]$, there exists $a \in N$ such that $[a, u] = -b$. Denote $\alpha = 1 + \operatorname{ad} a$. We have that $\alpha(u) = u - b = v$, hence $\alpha(U) \subseteq C_L(v)$. On the other hand, since V is abelian, we also have that $V \subseteq C_L(v)$. Comparing dimensions, we see that $V = C_L(v) = \alpha(U)$. It is now not difficult to see that any two complements to N in L are conjugate.

It remains to show that the complements are one-dimensional in the algebraically closed case. By the above, L is isomorphic to a semidirect product $k^n \rtimes_\varphi k^m$. Let us first show that $\ker \varphi(a) = \{0\}$ for all $a \in k^m \setminus \{0\}$. Otherwise, for some $0 \neq u \in k^n$, $[u, a] = \varphi(a)u = 0$. On the other hand, $[u, u'] = 0$ for all $u' \in k^n$. Hence by the CT property, a commutes with all elements of k^n, so $\varphi(a) = 0$. For all $b \in k^m$, $[a, b] = 0$, so again by the CT property, k^m commutes with k^n elementwise. Hence $\varphi = 0$ leading to a contradiction since L is not abelian. To finish the proof, suppose that $m > 1$ and choose $a, b \in k^m$ linearly independent. The mapping $t \mapsto \det \varphi(a + tb)$ is a nonzero polynomial in t. As k is algebraically closed, this polynomial has a root t_0, i.e., $\ker \varphi(a + t_0 b) \neq \{0\}$. Since $a + t_0 b \neq 0$, this contradicts the above conclusion. Therefore $m = 1$. □

If k is not algebraically closed, then the complements are not necessarily one-dimensional.

Example 4. Let k be a field that is not algebraically closed. Then there exists an irreducible polynomial $p \in k[t]$ of degree $d \geqslant 2$. Let A be the companion matrix of p and B the $d \times d$ identity matrix. Form $L = k^d \rtimes_\varphi k^2$, where $\varphi(1, 0) = A$ and $\varphi(0, 1) = B$. This action is fixed-point-free, since $\det(A - \lambda B) = p(\lambda)$ has no roots in k. Hence L is CT by Proposition 2.

Example 5. We classify all finite dimensional nonabelian solvable CT Lie algebras over an algebraically closed field k of characteristic 0. By Theorem 3, these are isomorphic to $k^n \rtimes_\varphi k$ for some n and φ. Suppose that $\Psi : k^n \rtimes_\varphi k \to k^n \rtimes_\psi k$ is an isomorphism. Then Ψ can be represented by an $(n+1)\times(n+1)$ matrix over k, say

$$\Psi = \begin{bmatrix} A & b \\ c^t & d \end{bmatrix},$$

where $A \in k^{n\times n}$, $b, c \in k^n$ and $d \in k$. Let $u, v \in k^n$ and $\lambda, \mu \in k$ be arbitrary. Since Ψ is a Lie algebra homomorphism,

$$\Psi([u+\lambda, v+\mu]) = [\Psi(u+\lambda), \Psi(v+\mu)] \tag{1}$$

and hence $c^t(\varphi(\mu)u - \varphi(\lambda)v) = 0$. Choosing $v = 0$ and μ any nonzero element of k, we get $c^t\varphi(\mu)u = 0$. Since $\varphi(\mu)$ is invertible (see the proof of Theorem 3), we get that $c = 0$. In particular, A is invertible. Now the equation (1) implies $A\varphi(\mu)u - A\varphi(\lambda)v = d\psi(\mu)Au - d\psi(\lambda)Av$, hence as before we conclude that $A\varphi(\mu) = d\psi(\mu)A$. Since φ and ψ are determined by $\varphi(1)$ and $\psi(1)$, Ψ is an isomorphism if and only $A\varphi(1) = d\psi(1)A$. This is in turn equivalent to the fact that $\varphi(1)$ is similar to a nonzero multiple of $\psi(1)$. In practice, this condition is easily checked by looking at the Jordan canonical form of $\varphi(1)$ and $\psi(1)$.

3. Semisimple CT Lie algebras

In this section we classify all semisimple CT Lie algebras over an algebraically closed field of characteristic 0.

Theorem 6. *If k is algebraically closed, then the only finite dimensional semisimple CT Lie algebra over k is $\mathfrak{sl}_2$.*

Proof. Let us first show that $\mathfrak{sl}_2$ is indeed CT. Recall that $\mathfrak{sl}_2$ is a simple three-dimensional Lie algebra with basis $\{X, Y, H\}$, subject to relations $[X, Y] = H$, $[H, X] = 2X$, $[H, Y] = -2Y$. Given a nonzero $x \in \mathfrak{sl}_2$, its centralizer is of dimension 1 or 2, whence it is abelian. This proves our assertion.

Let L be a finite dimensional semisimple CT Lie algebra over k. Lemma 1 implies that L has to be simple. By the well known classification theorem [Hu, 11.4], the following is an exhaustive list of pairwise nonisomorphic simple Lie algebras over k: the classical algebras A_n $(n \geqslant 1)$, B_n $(n \geqslant 2)$, C_n $(n \geqslant 3)$, D_n $(n \geqslant 4)$, and the exceptional algebras E_6, E_7, E_8, F_4, G_2.

The case of classical Lie algebras follows from a straightforward computation using their well-known matrix representations. For instance, B_n consists of all matrices of the form

$$\begin{bmatrix} 0 & b_1^t & b_2^t \\ -b_2 & x & y \\ -b_1 & z & -x^t \end{bmatrix},$$

where $b_1, b_2 \in k^n$ and $x, y, z \in k^{n\times n}$, $y^t = -y$, $z^t = -z$. The mapping $x \mapsto \begin{bmatrix} 0 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & -x^t \end{bmatrix}$ is an embedding of $\mathfrak{gl}_n$ into B_n. As the center of $\mathfrak{gl}_n$ is nontrivial, it follows that $\mathfrak{gl}_n$ is not CT, hence the same holds true for B_n.

In the exceptional case, we have a chain of embeddings $F_4 \subseteq E_6 \subseteq E_7 \subseteq E_8$. It remains to be seen that F_4 and G_2 are not CT. For a simple Lie algebra L, let H denote its Cartan subalgebra and Φ the corresponding set of roots. For each $\alpha \in \Phi$ denote $L_\alpha = \{x \in L \mid [h, x] = \alpha(h)x \text{ for all } h \in H\}$. We then have the

Cartan decomposition $L = H \oplus \bigoplus_{\alpha\in\Phi} L_\alpha$, cf. [Hu, §8]. For all $\alpha \in \Phi$, $\dim L_\alpha = 1$. Furthermore, for $\alpha, \beta \in \Phi$, $\alpha \neq -\beta$, we have

$$[L_\alpha, L_\beta] = \begin{cases} L_{\alpha+\beta} & : \alpha+\beta \in \Phi, \\ 0 & : \text{otherwise.} \end{cases} \tag{2}$$

F_4: The set of vectors

$$\left\{\pm e_i, e_i \pm e_j, \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4) \mid i \neq j, i, j = 1, \ldots, 4\right\}$$

is a root system of type F_4. Here e_i is the i-th standard basis vector. For each $\alpha \in \Phi$, let z_α be a nonzero element of L_α. By (2), $[z_{\frac{1}{2}(e_1+e_2+e_3+e_4)}, z_{e_1}] = [z_{e_1}, z_{\frac{1}{2}(e_1-e_2-e_3+e_4)}] = 0$, but $[z_{\frac{1}{2}(e_1+e_2+e_3+e_4)}, z_{\frac{1}{2}(e_1-e_2-e_3+e_4)}] \neq 0$. This shows that F_4 is not CT.

G_2: The root system of type G_2 can be described as the set of algebraic integers of the cyclotomic field generated by a cubic root of unity with norm 1 (denoted by ξ) or 3 (denoted by ζ). As before, let z_α be a nonzero element of L_α for $\alpha \in \Phi$. Then it follows from (2) that $[z_{\xi+\zeta}, z_\zeta] = [z_\zeta, z_{2\xi+\zeta}] = 0$, but $[z_{\xi+\zeta}, z_{2\xi+\zeta}] \neq 0$. We conclude that G_2 is not CT. □

4. General case

Theorem 7. *Let k be algebraically closed. Then every finite dimensional CT Lie algebra over k is either solvable or simple.*

Proof. Let L be a finite dimensional CT Lie algebra over k. Suppose that L is neither solvable nor simple. Then by Levi's theorem and Theorems 3 and 6, L can be written as a semidirect product $(k^n \rtimes_\varphi k) \rtimes_\psi \mathfrak{sl}_2$ for some n, φ and ψ.

Let us first determine the derivation algebra of $k^n \rtimes_\varphi k$. Every derivation is a linear mapping and can thus be represented by an $(n+1)\times(n+1)$ matrix over k, say

$$\delta = \begin{bmatrix} A & b \\ c^t & d \end{bmatrix},$$

where $A \in k^{n\times n}$, $b, c \in k^n$ and $d \in k$. Let $u, v \in k^n$ and $\lambda, \mu \in k$ be arbitrary. Since δ is a derivation,

$$\delta([u+\lambda, v+\mu]) = [\delta(u+\lambda), v+\mu] + [u+\lambda, \delta(v+\mu)] \tag{3}$$

and hence $c^t(\varphi(\mu)u - \varphi(\lambda)v) = 0$. Choosing $v = 0$ and μ any nonzero element of k, we get $c^t\varphi(\mu)u = 0$. Since $\varphi(\mu)$ is invertible (see the proof of Theorem 3), we get that $c = 0$. (A straighforward computation also shows that $A\varphi(1) - \varphi(1)A = d\varphi(1)$ must hold.)

Suppose $\psi(x) = \begin{bmatrix} A_x & b_x \\ 0 & d_x \end{bmatrix}$ for $x \in \mathfrak{sl}_2$. Then $x \mapsto A_x$ is an n-dimensional representation of $\mathfrak{sl}_2$. As the latter is simple, it decomposes into a direct sum of irreducible representations. Under each of these $X \in \mathfrak{sl}_2$ is mapped to a nilpotent matrix [Hu, p. 32], hence A_X is nilpotent. So there exists $0 \neq u \in k^n$ with $A_X u = 0$, i.e., $[X, u] = 0$. By the CT property this implies that X commutes with k^n elementwise, thus $A_X = 0$. Similarly, $A_Y = 0$. Hence $\psi(H) = \psi([X,Y]) = \begin{bmatrix} 0 & d_Y b_X - d_X b_Y \\ 0 & 0 \end{bmatrix}$. Using $[H,X] = 2X$ and $[H,Y] = -2Y$, we obtain $d_X = 0$ and $d_Y = 0$. This implies $\psi(H) = 0$.

Since $\mathfrak{sl}_2$ is simple and $\ker\psi \neq \{0\}$, $\psi = 0$. But then $L = (k^n \rtimes_\varphi k) \oplus \mathfrak{sl}_2$ cannot be CT by Lemma 1. □

If k is not algebraically closed, then Theorem 7 fails.

Example 8. Let k be a formally real field and L_0 the three-dimensional vector space k^3 endowed with the operation $[x,y] = x \times y$ (cross product). Then L_0 is a simple Lie algebra as $L_0 \otimes_k \bar{k} \cong \mathfrak{sl}_2(\bar{k})$. (Here $\bar{k}$ denotes the algebraic closure of k.) This also implies that L_0 is CT. Let us construct a Lie algebra homomorphism $\psi : L_0 \to \mathfrak{gl}_4$ as follows:

$$\begin{bmatrix}1\\0\\0\end{bmatrix} \mapsto x = \begin{bmatrix}0 & -\frac{1}{2} & 0 & 0\\ \frac{1}{2} & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{2}\\ 0 & 0 & -\frac{1}{2} & 0\end{bmatrix}, \quad \begin{bmatrix}0\\1\\0\end{bmatrix} \mapsto y = \begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -\frac{1}{4} & 0 & 0 & 0\\ 0 & -\frac{1}{4} & 0 & 0\end{bmatrix}.$$

It follows that

$$\psi \begin{bmatrix}0\\0\\1\end{bmatrix} = z = \begin{bmatrix}0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\\ 0 & -\frac{1}{4} & 0 & 0\\ \frac{1}{4} & 0 & 0 & 0\end{bmatrix}.$$

Since $\det(\lambda_1 x + \lambda_2 y + \lambda_3 z) = \frac{1}{4}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)$, ψ maps nonzero elements of k^3 to invertible matrices. In particular, ψ induces a fixed-point-free action of L_0 on k^4. Hence $L = k^4 \rtimes_\psi L_0$ is CT by Proposition 2.

Observe that the last example also shows that the CT property is not closed under scalar extensions.

Corollary 9. *Let k be algebraically closed. Then the class of all finite dimensional CT Lie algebras over k is quotient closed.*

Proof. Let L be a finite dimensional CT Lie algebra over k. By Theorem 7, L is either solvable or simple. Without loss of generality we may assume that L is neither abelian nor simple. Then L is a semidirect product of its nil radical N by an abelian Lie algebra U that acts fixed-point-freely on N, see Theorem 3. Let I be an ideal in L. It is straightforward to see that either $I \subseteq N$ or $N \subseteq I$. For, suppose that $I \nsubseteq N$ and pick $x \in I \setminus N$. Then it follows from the proof of Theorem 3 that $N = [L,x] \subseteq I$, as required. If $N \subseteq I$, then L/I is abelian. Thus we can assume that $I \subsetneq N$. In this case we clearly have that L/I is isomorphic to a semidirect product of N/I by U. For any $u \in U \setminus \{0\}$ we have that $[I,u] = I$ by a similar argument as in the proof of Theorem 3. Thus, if $[n,u] + I = I$ for some $n \in N$ and $u \in U \setminus \{0\}$, then $[n,u] \in I = [I,u]$, whence there exists $m \in I$ such that $[n-m,u] = 0$. As U acts fixed-point-freely on N, it follows that $n = m \in I$. Therefore we conclude that U also acts fixed-point-freely on N/I. By Proposition 2, L/I is a CT Lie algebra. □

In general, the class of CT Lie algebras is not quotient closed, as the following example implies.

Example 10. Every free Lie algebra is a CT Lie algebra. To see this let L_X be the free Lie algebra on a nonempty set X. Then the universal enveloping algebra $U(L_X)$ is the free associative algebra $k\langle X\rangle$. But it is well-known that the centralizer of a nonscalar element of $k\langle X\rangle$ is a polynomial ring in one variable over k; this is Bergman's centralizer theorem (see e.g. [Co, Theorem 6.7.7]). Hence L_X is CT.

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Univerza v Ljubljani, IMFM, Oddelek za Matematiko, Jadranska 19, SI–1000 Ljubljana, Slovenia

E-mail address: igor.klep@fmf.uni-lj.si, primoz.moravec@fmf.uni-lj.si