GROUPS IN WHICH THE BOUNDED NILPOTENCY OF TWO-GENERATOR SUBGROUPS IS A TRANSITIVE RELATION

COSTANTINO DELIZIA, PRIMOŽ MORAVEC AND CHIARA NICOTERA

Abstract. In this paper we describe the structure of locally finite groups in which the bounded nilpotency of two-generator subgroups is a transitive relation. We also introduce the notion of (nilpotent of class c)-transitive kernel. Our results generalize several known results related to the groups in which commutativity is a transitive relation.

1. Introduction

Let c be a positive integer and let $\mathfrak{N}_c$ denote the class of all groups which are nilpotent of class $\leq c$. A group G is said to be an $\mathfrak{N}_cT$-*group* if for all $x, y, z \in G\backslash\{1\}$ the relations $\langle x, y\rangle \in \mathfrak{N}_c$ and $\langle y, z\rangle \in \mathfrak{N}_c$ imply $\langle x, z\rangle \in \mathfrak{N}_c$. In the case $c = 1$ these groups are known as commutative-transitive groups (also CT-groups or CA-groups) and have been studied by several authors [2, 3, 4, 8, 11, 14, 15]. It is not difficult to see that CT-groups are precisely the groups in which centralizers of non-identity elements are abelian. The study of these groups was initiated by Weisner [14] in 1925, but there are some fallacies in his proofs. Nevertheless, it turns out that finite CT-groups are either soluble or simple. Finite nonabelian simple CT-groups have been classified by Suzuki [11]. He proved that every finite nonabelian simple CT-group is isomorphic to some $\mathrm{PSL}(2, 2^f)$, where $f > 1$. The complete description of finite soluble CT-groups has been given by Wu [15] (see also a paper of Lescot [8]), who has also obtained information on locally finite CT-groups and polycyclic CT-groups. At roughly the same time Fine et al. [4] introduced the notion of the commutative-transitive kernel of a group. This topic has been further explored by the first and the third author; see [2] and [3].

Passing to finite $\mathfrak{N}_cT$-groups with $c > 1$ we first note that in these groups centralizers of non-identity elements are nilpotent. The converse is not true, however, as the example of $\mathrm{PSL}(2, 9)$ shows (see Proposition 4.5). Compared to the CT-case, this may seem to be a certain disadvantage at first glance, but

2000 *Mathematics Subject Classification.* 20E15, 20D25.

Key words and phrases. nilpotent-transitive groups, (nilpotent of class c)-transitive kernel, locally finite groups, Frobenius groups.

The second author was partially supported by G.N.S.A.G.A. He also wishes to thank the Department of Mathematics and Informatics at the University of Salerno for its excellent hospitality.

nevertheless we obtain satisfactory information on the structure of locally finite $\mathfrak{N}_c T$-groups. We show that soluble locally finite $\mathfrak{N}_c T$-groups are either Frobenius groups or belong to the class of groups in which every two-generator subgroup is nilpotent of class $\leq c$. Furthermore, we prove that finite $\mathfrak{N}_c T$-groups are either soluble or simple. This provides a generalization of results in [15]. Additionally, we show that the groups $\mathrm{PSL}(2, 2^f)$, where $f > 1$, and Suzuki groups $\mathrm{Sz}(q)$, with $q = 2^{2n+1} > 2$, are the only finite nonabelian simple $\mathfrak{N}_c T$-groups for $c > 1$. This result is probably the strongest evidence showing the gap between CT-groups and $\mathfrak{N}_c T$-groups with $c > 1$. We also show that locally finite $\mathfrak{N}_c T$-groups are either locally soluble or simple. In the latter case we give a classification of these groups.

Another notion closely related to CT-groups is the commutative-transitive kernel of a group. Given a group G, we can construct a characteristic subgroup $T(G)$ as the union of a chain $1 = T_0(G) \leq T_1(G) \leq \dots$ in such way that $G/T(G)$ is a CT-group [4]. In [2] it is proved that if G is locally finite, then $T(G) = T_1(G)$. Similar results have also been obtained in [3] for other classes of groups, such as supersoluble groups. In analogy with this we introduce the notion of the $\mathfrak{N}_c$-transitive kernel of a group and prove that it has similar properties like the commutative-transitive kernel.

In the final section we present some examples of $\mathfrak{N}_2 T$-groups. In particular, we present Frobenius $\mathfrak{N}_2 T$-groups with nonabelian kernel and Frobenius $\mathfrak{N}_2 T$-groups with noncyclic complement. We also show that some finite linear groups with nilpotent centralizers are in a certain sense far from being $\mathfrak{N}_c T$-groups.

2. $\mathfrak{N}_c T$-Groups

In this section we investigate the structure of locally finite $\mathfrak{N}_c T$-groups. In the beginning we exhibit some basic properties of these groups. For positive integers $r > 1$ and n denote by $\mathfrak{N}(r, n)$ the class of all groups in which every r-generator subgroup is nilpotent of class $\leq n$. Every finite $\mathfrak{N}(r, n)$-group is nilpotent by Zorn's theorem (see Theorem 12.3.4 in [10]). It is now clear that every locally nilpotent $\mathfrak{N}_c T$-group is also an $\mathfrak{N}(2, c)$-group. In fact, every $\mathfrak{N}_c T$-group with nontrivial center is an $\mathfrak{N}(2, c)$-group. On the other hand, the property $\mathfrak{N}_c T$ behaves badly under taking quotients and forming direct products. For, it is known that every free (soluble) group is a CT-group [15]. Moreover if G and H are $\mathfrak{N}_c T$-groups and there exist $x, y \in G$ such that $\langle x, y \rangle$ is not nilpotent, then it is easy to see that $G \times H$ is not an $\mathfrak{N}_d T$-group for any $d \in \mathbb{N}$.

Our first result shows that the classes of $\mathfrak{N}_c T$-groups form a chain.

Proposition 2.1. *Let c and d be integers, $c \geq d \geq 1$. Then every $\mathfrak{N}_d T$-group is also an $\mathfrak{N}_c T$-group.*

Proof. Let G be an $\mathfrak{N}_d T$-group. Let $x, y, z \in G \backslash \{1\}$ and suppose that the groups $\langle x, y \rangle$ and $\langle y, z \rangle$ are nilpotent of class $\leq c$. By the above remarks $\langle x, y \rangle$ and

$\langle y, z\rangle$ are nilpotent of class $\leq d$. As G is an $\mathfrak{N}_d T$-group, it follows that $\langle x, z\rangle$ is nilpotent of class $\leq d$, hence it is nilpotent of class $\leq c$. □

The following lemma is crucial for the description of soluble locally finite $\mathfrak{N}_c T$-groups.

Lemma 2.2. *Let G be a locally finite $\mathfrak{N}_c T$-group with nontrivial Hirsch-Plotkin radical H. Then the factor group G/H acts fixed-point-freely on H by conjugation.*

Proof. As the Hirsch-Plotkin radical H is a locally nilpotent $\mathfrak{N}_c T$-group, it is also an $\mathfrak{N}(2, c)$-group. Let y be a nontrivial element in H. Suppose there exists $a \in C_G(y)\backslash H$. Since the group $\langle a, y\rangle$ is abelian and H is an $\mathfrak{N}(2, c)$-group, we conclude that the group $\langle a, h\rangle$ is nilpotent of class $\leq c$ for every $h \in H$, since G is an $\mathfrak{N}_c T$-group. By conjugation we get that $\langle a^g, h\rangle$ is also nilpotent of class $\leq c$ for all $g \in G$ and $h \in H$. As G is an $\mathfrak{N}_c T$-group, this implies that the group $\langle a, a^g\rangle$ is nilpotent of class $\leq c$ for every $g \in G$. In particular, we have $1 = [a^g, {}_c a] = [a, g, {}_c a]$ for all $g \in G$, hence a is a left $(c + 1)$-Engel element of G. As G is locally finite, this implies that $a \in H$ (see, for instance, Exercise 12.3.2 of [10]), which is a contradiction. □

Theorem 2.3. *Every locally finite soluble $\mathfrak{N}_c T$-group is either an $\mathfrak{N}(2, c)$-group or a Frobenius group whose kernel and complement are both $\mathfrak{N}(2, c)$-groups. Conversely, every locally finite Frobenius group in which kernel and complement are both $\mathfrak{N}(2, c)$-groups is an $\mathfrak{N}_c T$-group.*

Proof. Let G be a locally finite soluble $\mathfrak{N}_c T$-group and suppose G is not in $\mathfrak{N}(2, c)$. Let N be its Hirsch-Plotkin radical. As N is also an $\mathfrak{N}_c T$-group, it is an $\mathfrak{N}(2, c)$-group. By Lemma 2.2 G/N acts fixed-point-freely on N, hence G is a Frobenius group with the kernel N and a complement H; see, for instance, Proposition 1.J.3 in [7]. Since H has a nontrivial center [7, Theorem 1.J.2], we have that $H \in \mathfrak{N}(2, c)$. Besides, N is nilpotent by the same result from [7].

Conversely, let G be a locally finite Frobenius group with the kernel N and a complement H and suppose that both N and H are $\mathfrak{N}(2, c)$-groups. Let $x, y, z \in G\backslash\{1\}$ and let the groups $\langle x, y\rangle$ and $\langle y, z\rangle$ be nilpotent of class $\leq c$. Suppose $x \in N$ and $y \notin N$. Then the equation $[x, {}_c y] = 1$ implies $[x, {}_{c-1} y] = 1$, since H acts fixed-point-freely on N. By the same argument we get $x = 1$, which is not possible. This shows that if $x \in N$ then $y \in N$ and similarly also $z \in N$. But in this case $\langle x, z\rangle$ is clearly nilpotent of class $\leq c$, since N is an $\mathfrak{N}(2, c)$-group. Thus we may assume that $x, y, z \notin N$. Let $x \in H^g$ and $y \in H^k$ for some $g, k \in G$ and suppose $H^g \neq H^k$. We clearly have $C_G(x) \leq H^g$ and $C_G(y) \leq H^k$. Let α be any simple commutator of weight c with entries in $\{x, y\}$. As $\langle x, y\rangle$ is nilpotent of class $\leq c$, we have $\alpha \in C_G(x) \cap C_G(y) = 1$. This implies that $\langle x, y\rangle$ is nilpotent of class $\leq c - 1$. Continuing with this process, we end at $x = y = 1$ which is impossible. Hence we conclude that $\langle x, y\rangle \leq H^g$ and similarly also $\langle y, z\rangle \leq H^g$. Therefore we have $\langle x, z\rangle \leq H^g$. But H^g is an $\mathfrak{N}(2, c)$-group, hence the group $\langle x, z\rangle$ is nilpotent of class $\leq c$. This concludes the proof. □

Theorem 2.3 can be further refined when we restrict ourselves to finite groups.

Theorem 2.4. *Let G be a finite group. Then G is a soluble $\mathfrak{N}_cT$-group if and only if it is either an $\mathfrak{N}(2,c)$-group or a Frobenius group with the kernel which is an $\mathfrak{N}(2,c)$-group and a complement which is nilpotent of class $\leq c$.*

Proof. By Theorem 2.3 we only need to show that if G is a finite soluble $\mathfrak{N}_cT$-group which is not an $\mathfrak{N}(2,c)$-group, then every complement H of the Frobenius kernel N of G is nilpotent of class $\leq c$. Suppose N is not abelian. Then the order of H is odd, hence all Sylow subgroups of H are cyclic. This implies that H is cyclic. Assume now that N is abelian. Then all the Sylow p-subgroups of H are cyclic for $p \neq 2$, whereas the Sylow 2-subgroup is either cyclic or a generalized quaternion group Q_{2^n} [5]. Moreover, since $H \in \mathfrak{N}(2,c)$, we obtain $n \leq c+1$. As H is nilpotent and all its Sylow subgroups are nilpotent of class $\leq c$, the nilpotency class of H does not exceed c. □

Let G be a finite $\mathfrak{N}_cT$-group and suppose $G \notin \mathfrak{N}(2,c)$. If the Fitting subgroup of G is nontrivial, then Lemma 2.2 together with Theorem 2.4 shows that G is soluble and so its structure is completely determined by Theorem 2.4. The complete classification of finite insoluble $\mathfrak{N}_cT$-groups is described in our next result. Note that it has been shown in [11] that the groups $\mathrm{PSL}(2,2^f)$, where $f > 1$, are the only finite insoluble $\mathfrak{N}_1T$-groups. Passing to finite $\mathfrak{N}_cT$-groups with $c > 1$, we obtain an additional family of simple groups.

Theorem 2.5. *Let G be a finite $\mathfrak{N}_cT$-group with $c > 1$. Then G is either soluble or simple. Moreover, G is a nonabelian simple $\mathfrak{N}_cT$-group if and only if it is isomorphic either to $\mathrm{PSL}(2,2^f)$, where $f > 1$, or to $\mathrm{Sz}(q)$, the Suzuki group with parameter $q = 2^{2n+1} > 2$.*

Proof. It is easy to see that in every finite $\mathfrak{N}_cT$-group G the centralizers of nontrivial elements are nilpotent, i.e., G is an CN-group. Suppose that G is not soluble. By a result of Suzuki [12, Part I, Theorem 4], G is a CIT-group, i.e., the centralizer of any involution in G is a 2-group. Let P and Q be any Sylow p-subgroups of G and suppose that $P \cap Q \neq 1$. Since P and Q are $\mathfrak{N}(2,c)$-groups and G is an $\mathfrak{N}_cT$-group, we conclude that $\langle P, Q\rangle$ is an $\mathfrak{N}(2,c)$-group, hence it is nilpotent. This shows that $\langle P, Q\rangle$ is a p-group, which implies $P = Q$. Therefore Sylow subgroups of G are independent. Combining Theorem 1 in Part I and Theorem 3 in Part II of [12], we conclude that G has to be simple. Additionally, we also obtain that G is a ZT-group, that is, G is faithfully represented as a doubly transitive permutation group of odd degree in which the identity is the only element fixing three distinct letters. The structure of these groups is described in [13]. It turns out that G is isomorphic either to $\mathrm{PSL}(2,2^f)$, where $f > 1$, or to $\mathrm{Sz}(q)$ with $q = 2^{2n+1} > 2$.

It remains to prove that $\mathrm{PSL}(2,2^f)$ and $\mathrm{Sz}(q)$ are $\mathfrak{N}_cT$-groups. For projective special linear groups this has been done in [11]. Now, let $G = \mathrm{Sz}(q)$ where

$q = 2^{2n+1} > 2$. By Theorem 3.10 c) in [6] G has a nontrivial partition $(G_i)_{i\in I}$, where for every $i \in I$ the group G_i is either cyclic or nilpotent of class ≤ 2. Moreover, the proof of result 3.11 in [6] implies that for all $g \in G\backslash\{1\}$ the relation $g \in G_i$ implies that $C_G(g) \leq G_i$. Let $x, y, z \in G\backslash\{1\}$ and suppose that the groups $\langle x, y\rangle$ and $\langle y, z\rangle$ are nilpotent of class ≤ 2. Let a and b be nontrivial elements in $Z(\langle x, y\rangle)$ and $Z(\langle y, z\rangle)$, respectively, and suppose that $a \in G_i$ and $b \in G_j$ for some $i, j \in I$. Then $y \in C_G(a) \cap C_G(b) \leq G_i \cap G_j$, hence $i = j$. But now we get $x, z \in G_i$ and since G_i is nilpotent of class ≤ 2, the same is true for the group $\langle x, z\rangle$. Hence G is an $\mathfrak{N}_2T$-group. By Proposition 2.1 G is an $\mathfrak{N}_cT$-group for every $c > 1$. □

It is proved in [15] that every locally finite insoluble CT-group is isomorphic to $\mathrm{PSL}(2, F)$ for some locally finite field F. For $\mathfrak{N}_cT$-groups, where $c > 1$, we have the following result.

Theorem 2.6. *Let $c > 1$ and let G be a locally finite $\mathfrak{N}_cT$-group which is not locally soluble. Then there exists a locally finite field F such that G is isomorphic either to $\mathrm{PSL}(2, F)$ or to $\mathrm{Sz}(F)$.*

Proof. Let G be a locally finite $\mathfrak{N}_cT$-group and suppose that G is not locally soluble. Then G contains a finite insoluble subgroup, hence every finite subgroup of G is contained in some finite insoluble subgroup of G. Using Theorem 2.5, we conclude that every finitely generated subgroup of G has a faithful representation of degree 4 over some field of even characteristic. By Mal'cev's representation theorem [7, Theorem 1.L.6], G has a faithful representation of the same degree over a field which is an ultraproduct of some finite fields. Hence G is a linear periodic group. It is not difficult to see that G has to be simple. Namely, the set of all finite nonabelian simple subgroups of G is a local system of G. By a theorem of Winter [7] the group G is countable. Thus we obtain a chain $(G_i)_{i\in\mathbb{N}}$ of nonabelian finite simple subgroups in G such that G is the union of this chain. By Theorem 2.5 we have either $G_i \cong \mathrm{PSL}(2, F_i)$ or $G_i \cong \mathrm{Sz}(F_i)$ for suitable finite fields F_i, $i \in \mathbb{N}$. On the other hand, $PSL(2, F)$ does not contain any Suzuki group as a subgroup and vice versa (this follows from [13] and Dickson's theorem in [5]). Therefore we either have $G_i \cong \mathrm{PSL}(2, F_i)$ for all $i \in \mathbb{N}$ or $G_i \cong \mathrm{Sz}(F_i)$ for all $i \in \mathbb{N}$. By a theorem of Kegel [7, Theorem 4.18] there exists a locally finite field F such that either $G \cong \mathrm{PSL}(2, F)$ or $G \cong \mathrm{Sz}(F)$. □

Let the group G be locally finite and locally soluble. If G is an $\mathfrak{N}_2T$-group, then Theorem 2.5 implies that every finitely generated subgroup of G is either a 2-Engel group or a Frobenius group with the kernel which is a 2-Engel group and a complement which is nilpotent of class ≤ 2. As every 2-Engel group is nilpotent of class ≤ 3 (see [9, p. 45]), the derived length of finitely generated subgroups of G is bounded, so G is actually soluble. Therefore we have:

Corollary 2.7. *Let G be a locally finite $\mathfrak{N}_2T$-group. Then G is either soluble or simple.*

The structure of locally finite $\mathfrak{N}_c T$-groups, where $c > 2$, is more complicated. Namely, Bachmuth and Mochizuki [1] constructed an insoluble $\mathfrak{N}(2,3)$-group of exponent 5. This is a locally finite $\mathfrak{N}_3 T$-group in which all finite subgroups are nilpotent. Therefore the result of Corollary 2.7 is no longer true for $\mathfrak{N}_c T$-groups with $c > 2$.

3. $\mathfrak{N}_c$-Transitive Kernel

Let G be a group and let c be a positive integer. Put $T_0^{(c)}(G) = 1$ and let $T_1^{(c)}(G)$ be the group generated by all commutators $[x_1, x_2, \ldots, x_{c+1}]$ for $x_i \in \{a, b\}$, where a and b are nontrivial elements of G such that there exist $t \in \mathbb{N}_0$ and $y_1, \ldots, y_t \in G \backslash \{1\}$ with $\langle a, y_1\rangle \in \mathfrak{N}_c, \langle y_1, y_2\rangle \in \mathfrak{N}_c, \ldots, \langle y_t, b\rangle \in \mathfrak{N}_c$. It is clear that $T_1^{(c)}(G)$ is a characteristic subgroup of G. For $n > 1$ we define $T_n^{(c)}(G)$ inductively by $T_n^{(c)}(G)/T_{n-1}^{(c)}(G) = T_1^{(c)}(G/T_{n-1}^{(c)}(G))$. So we get a chain $1 = T_0^{(c)}(G) \le T_1^{(c)}(G) \le \ldots \le T_n^{(c)}(G) \le \ldots$ of characteristic subgroups of the group G. We define

$$T^{(c)}(G) = \bigcup_{n \in \mathbb{N}_0} T_n^{(c)}(G)$$

to be the *(nilpotent of class c)-transitive kernel* or, shorter, *$\mathfrak{N}_c$-transitive kernel* of the group G. In the case $c = 1$ this definition coincides with the usual definition of the commutative-transitive kernel given in [4]. From the definition it also follows that $T^{(c)}(G)$ is a characteristic subgroup of G and that $T^{(c)}(G) = 1$ if and only if G is an $\mathfrak{N}_c T$-group. Moreover, $G/T^{(c)}(G)$ is an $\mathfrak{N}_c T$-group for every group G. Additionally, notice that $T^{(c)}(G) = T_n^{(c)}(G)$ for some $n \in \mathbb{N}_0$ if and only if $G/T_n^{(c)}(G)$ is an $\mathfrak{N}_c T$-group. We use the notation $\Gamma_t(G) = \langle \gamma_t(\langle a, b\rangle) \mid a, b \in G\rangle$. It is easy to see that $T^{(c)}(G) \le \Gamma_{c+1}(G)$.

In [2] it is proved that if G is a locally finite group, then $T^{(1)}(G) = T_1^{(1)}(G)$. In this section we shall show that we have an analogous result for the $\mathfrak{N}_c$-transitive kernel.

Proposition 3.1. *Let G be a group and H a subgroup of G. Let c be a positive integer and suppose that the set $\mathcal{S} = \{h \in H \mid \langle h, k\rangle \in \mathfrak{N}_c$ for all $k \in H\}$ contains a nontrivial element. Then the group $HT_1^{(c)}(G)/T_1^{(c)}(G)$ is an $\mathfrak{N}(2,c)$-group.*

Proof. Let $z \in \mathcal{S} \backslash \{1\}$. For all $a, b \in H \backslash \{1\}$ we have $\gamma_{c+1}(\langle a, b\rangle) \le T_1^{(c)}(H)$, since the groups $\langle a, z\rangle$ and $\langle z, b\rangle$ are nilpotent of class $\le c$. This implies that $\Gamma_{c+1}(H) = T_1^{(c)}(H) \le T_1^{(c)}(G)$, so $HT_1^{(c)}(G)/T_1^{(c)}(G)$ is an $\mathfrak{N}(2,c)$-group. □

Note that Proposition 3.1 implies that if G is a finite group, then every Sylow subgroup of $G/T_1^{(c)}(G)$ is an $\mathfrak{N}(2,c)$-group. In particular, if G is finite then the Fitting subgroup of $G/T_1^{(c)}(G)$ is an $\mathfrak{N}(2,c)$-group.

Proposition 3.2. *The class of finite $\mathfrak{N}_c T$-groups is closed under taking quotients.*

Proof. By Theorem 2.5 it suffices to consider finite soluble $\mathfrak{N}_cT$-groups. So suppose that G is a finite soluble $\mathfrak{N}_cT$-group. If $G \in \mathfrak{N}(2,c)$, then we are done. Otherwise, G is a Frobenius group with the kernel $F = \mathrm{Fitt}(G)$ which is an $\mathfrak{N}(2,c)$-group and a complement H which is nilpotent of class $\leq c$ by Theorem 2.4. If N is a normal subgroup of G, then we have either $N \leq F$ or $F \leq N$. If $F \leq N$, then G/N is nilpotent of class $\leq c$, hence it is an $\mathfrak{N}_cT$-group. Assume now that N is a proper subgroup of F. Then $G/N = F/N \rtimes H$, where the action of H on F/N is induced by the conjugation on F with elements of H. Since the subgroup N is invariant under the action of H, we conclude that H acts fixed-point-freely on F/N by Satz 8.10 in [5]. Therefore G/N is an $\mathfrak{N}_cT$-group by Theorem 2.4. □

The following result is a generalization of Theorem 3 in [2]:

Theorem 3.3. *Let G be a finite group. Then $T^{(c)}(G) = T_1^{(c)}(G)$ for every positive integer c.*

Proof. If $T_1^{(c)}(G) = 1$ or $T_1^{(c)}(G) = \Gamma_{c+1}(G)$, then we have nothing to prove. So we may assume that $1 \neq T_1^{(c)}(G) < \Gamma_{c+1}(G)$. Additionally, we may suppose that $T^{(c)}(H) = T_1^{(c)}(H)$ for every proper subgroup H of G. Let $\mathcal{F} = \{1 \neq H \lhd G \mid \Gamma_{c+1}(H) \leq T_1^{(c)}(G)\}$. Then this set is not empty since $T_1^{(c)}(G) \in \mathcal{F}$. So $\mathcal{F}$ has a maximal element N. First of all, it is clear that $N \neq G$, since $T_1^{(c)}(G) \neq \Gamma_{c+1}(G)$. Furthermore, since $NT_1^{(c)}(G)/T_1^{(c)}(G)$ is an $\mathfrak{N}(2,c)$-group, the group $NT_1^{(c)}(G)$ also belongs to $\mathcal{F}$, so we have $T_1^{(c)}(G) \leq N$ by the maximality of N. Let $F/T_1^{(c)}(G)$ be the Fitting subgroup of $G/T_1^{(c)}(G)$. Since $N/T_1^{(c)}(G)$ is an $\mathfrak{N}(2,c)$-group, it is nilpotent, hence $N/T_1^{(c)}(G) \leq F/T_1^{(c)}(G)$. On the other hand, since $F/T_1^{(c)}(G)$ is an $\mathfrak{N}(2,c)$-group, we have that $\Gamma_{c+1}(F) \leq T_1^{(c)}(G)$. Thus $F \in \mathcal{F}$, hence $F = N$ by the maximality of N in $\mathcal{F}$. Consider now the set $\mathcal{S} = \{h \in N \mid \langle h, k\rangle \in \mathfrak{N}_c \text{ for all } k \in N\}$. Here we have to consider the following two cases.

Case 1. Suppose that $\mathcal{S} \neq \{1\}$ and let h be a nontrivial element of $\mathcal{S}$. Let $y \in N\backslash\{1\}$ and let $a \in C_G(y)$. For every $b \in N$ we have $\gamma_{c+1}(\langle a, b\rangle) \leq T_1^{(c)}(G)$, since $\langle a, y\rangle$, $\langle y, h\rangle$ and $\langle h, b\rangle$ are in $\mathfrak{N}_c$. Additionally we have that $\langle a^g, y^g\rangle$, $\langle y^g, h\rangle$, $\langle h, y^k\rangle$ and $\langle y^k, a^k\rangle$ are in $\mathfrak{N}_c$ for all $g, k \in G$. Hence $\gamma_{c+1}(\langle a^g, a^k\rangle) \leq T_1^{(c)}(G)$ for all $g, k \in G$. In particular, this implies that $aT_1^{(c)}(G)$ is a left $(c+1)$-Engel element of the group $G/T_1^{(c)}(G)$, hence it is contained in the Fitting subgroup of $G/T_1^{(c)}(G)$ by Theorem 12.3.7 in [10]. This gives that $a \in N$. By Satz 8.5 in [5] G is a Frobenius group and N is its kernel. Let A be a complement of N in G. Since $T_1^{(c)}(A) \leq A \cap T_1^{(c)}(G) \leq A \cap N = 1$, it follows that A is an $\mathfrak{N}_cT$-group. Moreover the center of A is nontrivial by [5, Satz 8.18], so A is an $\mathfrak{N}(2,c)$-group. Therefore G is soluble. If the nilpotency class of N does not exceed c, then G is an $\mathfrak{N}_cT$-group by Theorem 2.3 and $T_1^{(c)}(G) = 1$, which is a contradiction. Hence we may suppose that the nilpotency class of N is greater than c. Consider the group $G/T_1^{(c)}(G) = N/T_1^{(c)}(G) \rtimes AT_1^{(c)}(G)/T_1^{(c)}(G)$. This is a Frobenius group with the kernel $N/T_1^{(c)}(G) \in \mathfrak{N}(2,c)$ and complement $AT_1^{(c)}(G)/T_1^{(c)}(G)$ which is also an $\mathfrak{N}(2,c)$-group. By Theorem 2.3 the group $G/T_1^{(c)}(G)$ is an $\mathfrak{N}_cT$-group, hence $T^{(c)}(G) = T_1^{(c)}(G)$ in this case.

CASE 2. Suppose now that $\mathcal{S} = \{1\}$. Let $\Phi(G)$ be the Frattini subgroup of G. If $T_1^{(c)}(G) \le \Phi(G)$, then the nilpotency of the group $N/T_1^{(c)}(G)$ implies that N is nilpotent, which is a contradiction. Hence $T_1^{(c)}(G) \not\le \Phi(G)$, so there exists a maximal subgroup M of G such that $T_1^{(c)}(G) \not\le M$. Then $G = MT_1^{(c)}(G)$ and $T_1^{(c)}(M) = T^{(c)}(M)$ since $M < G$. From $T_1^{(c)}(M) \le T_1^{(c)}(G) \cap M$ we now obtain that $G/T_1^{(c)}(G)$ is an $\mathfrak{N}_cT$-group, since it is a homomorphic image of the $\mathfrak{N}_cT$-group $M/T_1^{(c)}(M)$. So $T^{(c)}(G) = T_1^{(c)}(G)$, as required. □

Corollary 3.4. *Let G be a locally finite group. Then $T^{(c)}(G) = T_1^{(c)}(G)$ for every positive integer c.*

Proof. It suffices to show that if G is locally finite, then $G/T_1^{(c)}(G)$ is an $\mathfrak{N}_cT$-group. Let $x, y, z \in G \backslash T_1^{(c)}(G)$ and suppose that the groups $\langle x, y\rangle T_1^{(c)}(G)/T_1^{(c)}(G)$ and $\langle y, z\rangle T_1^{(c)}(G)/T_1^{(c)}(G)$ are nilpotent of class $\le c$. This means that $\gamma_{c+1}(\langle x, y\rangle) \le T_1^{(c)}(G)$ and $\gamma_{c+1}(\langle y, z\rangle) \le T_1^{(c)}(G)$. Let $\{\alpha_1, \dots, \alpha_r\}$ and $\{\overline{\alpha}_1, \dots, \overline{\alpha}_{r'}\}$ be the sets of all simple commutators of weight $c+1$ with entries from $\{x, y\}$ and $\{y, z\}$, respectively. For every $i = 1, \dots, r$ we have

$$\alpha_i = \prod_{t=1}^{n_i} [x_{i,t,1}, \dots, x_{i,t,c+1}]^{\epsilon_{i,t}},$$

where $\epsilon_{i,t} = \pm 1$, $x_{i,t,j} \in \{a_{i,t}, b_{i,t}\}$ for some $a_{i,t}, b_{i,t} \in G$ for which there exist $y_{i,t,1}, \dots, y_{i,t,s_{i,t}}$ in G such that $\langle a_{i,t}, y_{i,t,1}\rangle, \langle y_{i,t,1}, y_{i,t,2}\rangle, \dots, \langle y_{i,t,s_{i,t}}, b_{i,t}\rangle$ are nilpotent of class $\le c$, for all $i = 1, \dots, r$, $j = 1, \dots, c+1$ and $t = 1, \dots, n_i$. Similarly,

$$\overline{\alpha}_{i'} = \prod_{t'=1}^{m_{i'}} [\overline{x}_{i',t',1}, \dots, \overline{x}_{i',t',c+1}]^{\overline{\epsilon}_{i',t'}},$$

where $\overline{\epsilon}_{i',t'} = \pm 1$, $\overline{x}_{i',t',j} \in \{\overline{a}_{i',t'}, \overline{b}_{i',t'}\}$ for some $\overline{a}_{i',t'}, \overline{b}_{i',t'} \in G$ for which there exist $\overline{y}_{i',t',1}, \dots, \overline{y}_{i',t',s'_{i',t'}}$ in G such that $\langle \overline{a}_{i',t'}, \overline{y}_{i',t',1}\rangle, \langle \overline{y}_{i',t',1}, \overline{y}_{i',t',2}\rangle, \dots, \langle \overline{y}_{i',t',s'_{i',t'}}, \overline{b}_{i',t'}\rangle$ are nilpotent of class $\le c$, for all $i' = 1, \dots, r'$, $j = 1, \dots, c+1$ and $t' = 1, \dots, m_{i'}$. Let H be the subgroup of G generated by all

$$x,\ y,\ z,\ x_{i,t,j},\ \overline{x}_{i',t',j},\ a_{i,t},\ \overline{a}_{i',t'}, y_{i,t,k},\ \overline{y}_{i',t',k'},$$

where $i = 1, \dots, r$, $i' = 1, \dots, r'$, $t = 1, \dots, n_i$, $t' = 1, \dots, m_{i'}$, $j = 1, \dots, c+1$, $k = 1, \dots, s_{i,t}$ and $k' = 1, \dots, s'_{i',t'}$. Then $\gamma_{c+1}(\langle x, y\rangle) \le T_1^{(c)}(H)$ and $\gamma_{c+1}(\langle y, z\rangle) \le T_1^{(c)}(H)$. Since $H/T_1^{(c)}(H)$ is an $\mathfrak{N}_cT$-group by Theorem 3.3, we have $\gamma_{c+1}(\langle y, z\rangle) \le T_1^{(c)}(H) \le T_1^{(c)}(G)$. This concludes the proof. □

Remark 3.5. Let G be a locally nilpotent group, and let $c \geq 1$ be any positive integer. It easily follows from Proposition 3.1 that $T_1^{(c)}(G) = T^{(c)}(G) = \Gamma_{c+1}(G)$.

Remark 3.6. Let G be a supersoluble group. It is proved in [3] that $T^{(1)}(G) = T_1^{(1)}(G)$. It is to be expected that the same holds true for $\mathfrak{N}_c$-transitive kernel where $c > 1$, and that the proofs require only suitable modifications of those in [3].

4. Examples and Non-examples

Theorem 2.4 completely describes the structure of finite soluble $\mathfrak{N}_cT$-groups. At least in the case $c \leq 2$ we are able to obtain more detailed information about these groups, using the descriptions of fixed-point-free actions on finite abelian groups obtained by Zassenhaus [16].

Example 4.1. Let G be a finite soluble $\mathfrak{N}_1T$-group (or CT-group) which is not abelian. Then $G = F \rtimes \langle x \rangle$ where F is abelian and $\langle x \rangle$ acts fixed-point-freely on F (see Theorem 2.4 or Theorem 10 of [15]). Suppose $F = \bigoplus_{i=1}^m F_i$ where $F_i \cong \mathbb{Z}_{p_i^{e_i}}^{n_i}$ and $e_i \neq e_j$ if $p_i = p_j$. Let k be the order of $\langle x \rangle$. Then it follows from [16] that $x = (x_1, \ldots, x_m)$ where $\langle x_i \rangle$ is a fixed-point-free automorphism group of order k on G_i for all $i = 1, \ldots, m$. Conversely, for every x with this property the group $\langle x \rangle$ acts fixed-point-freely on F. Note also that a necessary and sufficient condition for the existence of a fixed-point-free automorphism on F is given in Theorem 2 of [15].

As the class of $\mathfrak{N}(2,2)$-groups coincides with the variety of 2-Engel groups, Theorem 2.4 implies that a finite soluble $\mathfrak{N}_2T$-group is either 2-Engel or it is a Frobenius group with the kernel F which is 2-Engel and a complement H which is nilpotent of class ≤ 2. Thus it follows from Levi's theorem (see [9, p. 45]) that F is nilpotent of class ≤ 3. Moreover, if $|H|$ is even, then F is abelian. In this case, H is either a cyclic group or the quaternion group Q_8 of order 8 or $C_m \times Q_8$ where m is odd. Our next example shows that there is essentially only one possibility of having a Frobenius $\mathfrak{N}_2T$-group with the prescribed kernel and a complement isomorphic to Q_8.

Example 4.2. Let F be a finite abelian group and $F = \bigoplus_{i=1}^m F_i$ where $F_i \cong \mathbb{Z}_{p_i^{e_i}}^{n_i}$ and $e_i \neq e_j$ if $p_i = p_j$. Then it follows from [16] that F admits a quaternion fixed-point-free automorphism group H of order 8 if and only if $2 \nmid p_i$ and $2|n_i$ for all $i = 1, \ldots, m$. In this case, H is conjugated to the group $\langle x, y \rangle$ where the restrictions of x and y on F_i can be presented by matrices

$$A_i = \bigoplus_{j=1}^{n_i/2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B_i = \bigoplus_{j=1}^{n_i/2} \begin{pmatrix} \alpha_i & \beta_i \\ \beta_i & -\alpha_i \end{pmatrix},$$

where $i = 1, \ldots, m$ and $\alpha_i^2 + \beta_i^2 \equiv -1 \mod p_i^{e_i}$ for all $i = 1, \ldots, m$.

In the following example we present a Frobenius group G with abelian kernel F and a complement H which is isomorphic to $C_p \times Q_8$, where p is an arbitrary odd prime. Of course, in this case G is an $\mathfrak{N}_2T$-group.

Example 4.3. Let q be a prime such that $p|(q-1)$ and let $F = C_q^2$. Let $a, b \in \mathbb{Z}_q$ be such that $a^2 + b^2 + 1 \equiv 0 \mod q$. Consider the automorphisms of C_q^2 represented by the following matrices over $\mathbb{Z}_q$:

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ , \ B = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \ , \ X = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta \end{pmatrix}.$$

Here ζ is a primitive p-th root modulo q. Then we have $\langle A, B, X\rangle \cong C_p \times Q_8$ and it can be verified that $H = \langle A, B, X\rangle$ acts fixed-point-freely on F. The corresponding Frobenius group $F \rtimes H$ is an $\mathfrak{N}_2T$-group, but it is not an $\mathfrak{N}_1T$-group.

On the other hand, if the order of H is odd, then H is cyclic and the group F may be nonabelian. In the next example we show that this is indeed so.

Example 4.4. Let $D = \langle x_1\rangle \times \langle x_2\rangle \times \langle x_3\rangle \times \langle x_4\rangle$ be an elementary group of order 16. Put $D_1 = D \rtimes \langle a\rangle$, where a is an element of order 2 acting on D in the following way: $[x_1, a] = x_3x_4$, $[x_2, a] = x_4$, $[x_3, a] = [x_4, a] = 1$. We make another split extension $F = D_1 \rtimes \langle b\rangle$, where b induces an automorphism of order 2 on D_1 in the following way: $[x_1, b] = x_3$, $[x_2, b] = x_3x_4$ and $[x_3, b] = [x_4, b] = [a, b] = 1$. The group F is nilpotent of class 2 and $|F| = 64$. Consider the following map on F:

$$x_1^\alpha = x_2 \ , \ x_2^\alpha = x_1x_2 \ , \ x_3^\alpha = x_4 \ , \ x_4^\alpha = x_3x_4 \ , \ a^\alpha = ab \ , \ b^\alpha = a.$$

It can be verified that α is an automorphism of order 3 on F. Moreover, α acts fixed-point-freely on F. The corresponding split extension $G = F \rtimes \langle\alpha\rangle$ is an $\mathfrak{N}_2T$-group of order 192 with the kernel F. One can verify that this is the smallest example of a non-nilpotent soluble $\mathfrak{N}_2T$-group having the nonabelian Frobenius kernel.

Finite simple groups with nilpotent centralizers are classified in [12] and [13]. It turns out that every finite nonabelian simple CN-group is of one of the following types:

(i) $\mathrm{PSL}(2, 2^f)$, where $f > 1$;
(ii) $\mathrm{Sz}(q)$, the Suzuki group with parameter $q = 2^{2n+1} > 2$;
(iii) $\mathrm{PSL}(2, p)$, where p is either a Fermat prime or a Mersenne prime;
(iv) $\mathrm{PSL}(2, 9)$;
(v) $\mathrm{PSL}(3, 4)$.

By Theorem 2.5 only groups listed under (i) and (ii) are $\mathfrak{N}_cT$-groups for $c > 1$. Our aim is to show that in groups (iii)-(v) we can always find such nontrivial elements x, y and z that the groups $\langle x, y\rangle$ and $\langle y, z\rangle$ are nilpotent of class ≤ 2, yet the group $\langle x, z\rangle$ is not even nilpotent. We call such a triple of elements a *bad triple*.

Proposition 4.5. *In the groups* PSL(2, 9) *and* PSL(3, 4) *there exist bad triples of elements.*

Proof. First we want to show that our proposition holds true for PSL(3, 4). To this end, consider the matrices

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

over the Galois field GF(4). It is easy to see that A, B and C belong to SL(3, 4). Besides, these matrices are not in the center of SL(3, 4) and a straightforward calculation shows that $[A, B] = [B, C, C] = [C, B, B] = 1$. Let $\overline{A}$, $\overline{B}$ and $\overline{C}$ be the homomorphic images of A, B and C, respectively, under the canonical homomorphism $\mathrm{SL}(3, 4) \to \mathrm{PSL}(3, 4)$. Then the group $\langle \overline{A}, \overline{B} \rangle$ is abelian and $\langle \overline{B}, \overline{C} \rangle$ is nilpotent of class 2. On the other hand, $\langle \overline{A}, \overline{C} \rangle$ is not nilpotent, since $[A, C], [A, C, C] \notin Z(\mathrm{SL}(3, 4))$ and $[A, C, C, C] = [A, C, C]$.

A similar argument also works for the group PSL(2, 9). In this case, we have to consider the following matrices in SL(2, 9):

$$A = \begin{pmatrix} \zeta^3 & 0 \\ 0 & \zeta^5 \end{pmatrix} \quad , \quad B = \begin{pmatrix} \zeta^2 & 0 \\ 0 & \zeta^6 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 & \zeta^4 \\ \zeta^4 & \zeta^4 \end{pmatrix}.$$

Here ζ is a generator of the multiplicative group of GF(9). If $\overline{A}$, $\overline{B}$ and $\overline{C}$ are the corresponding elements of PSL(2, 9), then it is a routine to verify that the group $\langle \overline{A}, \overline{B} \rangle$ is abelian and $\langle \overline{B}, \overline{C} \rangle$ is nilpotent of class 2, but $\langle \overline{A}, \overline{C} \rangle$ is not nilpotent. □

Finally we consider the groups PSL(2, p) where p is a Fermat prime or a Mersenne prime. If $p = 5$, then $\mathrm{PSL}(2, 5) \cong \mathrm{PSL}(2, 4)$ is an $\mathfrak{N}_1T$-group by [11]. For $p > 5$ the situation is completely different.

Proposition 4.6. *If p is a Fermat prime or a Mersenne prime and $p \neq 5$, then* PSL(2, p) *contains a bad triple of elements.*

Proof. First we cover the case of Fermat primes. For this we need the following number-theoretical result:

Claim 1. If p is a Fermat prime, then there exists $x \in \mathbb{Z}_p$ such that $2x^2 \equiv -1 \mod p$.

Proof of Claim 1. Let $p = 2^{2^n} + 1$ for some $n > 1$. It is enough to show that $2^{2^n - 1}$ is a quadratic residue modulo p. Let P be the set of all integers $a \in \{0, \dots, p-1\}$ which are primitive roots modulo p and let Q be the set of all $a \in \{0, \dots, p-1\}$ which are not quadratic residues modulo p. We shall show that $P = Q$. First, if $a \notin Q$, then there exists an integer t such that $t^2 \equiv a \mod p$. By Euler's theorem, $a^{\phi(p)/2} \equiv t^{\phi(p)} \equiv 1 \mod p$, hence a is not a primitive root modulo p (here ϕ is the Euler function). This shows that $P \subseteq Q$. To prove the converse inclusion, note that p has exactly $\phi(\phi(p))$ incongruent primitive roots and exactly $(p-1)/2$ quadratic non-residues. Hence

$$|P| = \phi(\phi(p)) = \phi(p-1) = \phi(2^{2^n}) = 2^{2^n-1} = \frac{p-1}{2} = |Q|$$

and therefore $P = Q$. Since $2^{2^n-1} \notin P = Q$, we have that $2^{2^n-1} \equiv x^2 \mod p$ for some $x \in \mathbb{Z}_p$, hence $2x^2 \equiv -1 \mod p$, as desired.

Now we are ready to finish the proof. Let $c, x \in \mathbb{Z}_p$ be such that $c^2 \equiv -1 \mod p$, $c \not\equiv -c \mod p$ and $2x^2 \equiv -1 \mod p$ (such x exists by Claim 1). Let

$$A = \begin{pmatrix} 2x & 0 \\ 0 & -x \end{pmatrix} \ , \ B = \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} x & x \\ x & -x \end{pmatrix}$$

be matrices in $\mathrm{SL}(2,p)\backslash Z(\mathrm{SL}(2,p))$. It is clear that A and B commute, and a short calculation shows that $[B,C,C]$ and $[C,B,B]$ belong to $Z(\mathrm{SL}(2,p))$. To prove that $\mathrm{PSL}(2,p)$ is not an $\mathfrak{N}_c T$-group for any $c > 1$ it suffices to show that $[C, {}_n A] \notin Z(\mathrm{SL}(2,p))$ for any $n \in \mathbb{N}$. More precisely, we shall prove that

$$[C, {}_n A] = x^{3 \cdot 2^n - 2} \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix},$$

where $a_n, b_n, c_n, d_n \in \mathbb{Z}_p$ are such that at least one of b_n, c_n and at least one of a_n, d_n are not zero. First note that this is true for $n = 1$, hence we may assume that $n > 1$. Then

$$[C, {}_{n+1} A] = x^{3 \cdot 2^{n+1} - 2} \begin{pmatrix} a_{n+1} & b_{n+1} \\ c_{n+1} & d_{n+1} \end{pmatrix},$$

where $a_{n+1} = -2a_n d_n - 4b_n c_n$, $b_{n+1} = 3b_n d_n$, $c_{n+1} = 2a_n c_n$ and $d_{n+1} = b_n c_n - 2a_n d_n$. If both b_{n+1} and c_{n+1} are zero, then $a_n = d_n = 0$ which is not possible by the induction assumption. Similarly, if $a_{n+1} = d_{n+1} = 0$, then $a_n d_n = -2b_n c_n$ and $b_n c_n = 2a_n d_n$, hence $5b_n c_n = 0$, a contradiction since $p > 5$. This concludes the proof for Fermat primes.

Assume now that p is a Mersenne prime. In this case we need the following auxiliary result:

CLAIM 2. If p is a Mersenne prime, then there exist $x, y \in \mathbb{Z}_p$ such that $x^2 - x + 1 \equiv 0 \mod p$ and $xy^4 \equiv 2y^2 + 1 \mod p$.

Proof of Claim 2. First note that since p is a Mersenne prime, $p-1$ is divisible by 6. The congruence equation $x^3 \equiv -1 \mod p$ is clearly solvable, hence it has $\gcd(3, p-1) = 3$ incongruent solutions. This shows that the equation $x^2 - x + 1 = 0$ is solvable in $\mathbb{Z}_p$. Let x_1 and x_2 be its solutions. Then $x_2 = x_1^{-1} = 1 - x_1$. We claim that at least one of $1 + x_1$, $1 + x_2$ is a quadratic residue modulo p. For this note that since $(p-1)/2$ is odd, Euler's criterion implies that for every $a \in \mathbb{Z}_p\backslash\{0\}$ we have that precisely one of a and $-a$ is a quadratic residue modulo p. Furthermore, since $\gcd(2^k, p-1) = \gcd(2, p-1)$, every quadratic residue modulo p is also a 2^k-power residue modulo p. Suppose $1 + x_1$ is not a square residue modulo p. Then $-1 - x_1$ is a quadratic residue modulo p and $1 + x_2 = 2 - x_1 = 1 - x_1^2 = x_1^2(-1 - x_1)$ is a square residue modulo p. So from now on we assume x is such that $1 - x + x^2 \equiv 0 \mod p$ and $1 + x$ is a square residue modulo p. Then the equation $xt^2 - 2t - 1 = 0$ has two solutions in $\mathbb{Z}_p$, namely $t_{1,2} = x^{-1}(1 \pm c) = x^2(-1 \mp c)$, where $c^2 = 1 + x$ in $\mathbb{Z}_p$. In order to ensure the existence of y it suffices to prove that $-1 \mp c$ are square residues modulo p. Since $(-1+c)(-1-c) = -x = x^4$, we have that $-1+c$ and $-1-c$ are either both squares or both non-squares in $\mathbb{Z}_p$. Assume that they are not squares. Then $1+c$ and $1-c$ are squares in $\mathbb{Z}_p$. For every square q in $\mathbb{Z}_p$ denote by $\sqrt{q}$ the square in $\mathbb{Z}_p$ for which $(\sqrt{q})^2 = q$. Let $u = \sqrt{1-c}$ and $v = \sqrt{1+c}$. Then $(u+v)^2 = u^2 + v^2 + 2uv = 2(1 + \sqrt{1-c^2}) = 2(1 + \sqrt{-x}) = 2(1+x^2)$. Since $p \equiv -1 \mod 8$, 2 is a square residue modulo p, hence $1 + x^2$ is a square in $\mathbb{Z}_p$. On the other hand, $-1 - x^2 = -x = x^4$ is also a square in $\mathbb{Z}_p$. This leads to a contradiction, hence our claim is proved.

Let x and y be as above and let

$$A = \begin{pmatrix} 0 & x^2 \\ x & 0 \end{pmatrix} \quad , \quad B = \begin{pmatrix} x & x \\ -1 & -x \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & y \\ -y^{-1} & 0 \end{pmatrix}$$

be matrices in $\mathrm{SL}(2,p) \backslash Z(\mathrm{SL}(2,p))$. It is not difficult to check that $[A,B] = -1$, hence $[A,B] \in Z(\mathrm{SL}(2,p))$. Beside that, we have $[B,C,C] = (a_{ij})_{i,j}$, and a straightforward calculation shows that $a_{11} - a_{22} = x - x^2y^4 - 2x^4y^2 = 0$ by Claim 2. Similarly, we obtain $a_{21} = a_{12} = 0$, hence $[B,C,C]$ belongs to $Z(\mathrm{SL}(2,p))$. Furthermore, it can be checked that the same holds true for $[C,B,B]$. On the other hand, an induction argument shows that

$$[A, {}_nC] = \begin{pmatrix} y^{(-2)^n} x^{2^{n+1}} & 0 \\ 0 & y^{-(-2)^n} x^{2^n} \end{pmatrix}$$

for every $n \in \mathbb{N}$. If $[A, {}_nC] \in Z(\mathrm{SL}(2,p))$ for some $n \in \mathbb{N}$, then $y^{(-2)^m} x^{2^{m+1}} = y^{-(-2)^m} x^{2^m} = 1$ in $\mathbb{Z}_p$ for every $m > n$. Besides we have that x^{2^k} is either $x - 1$ or $-x$, depending on whether k is odd or even, respectively. Suppose $m > n$ and let m be even. Then $[A, {}_mC] = 1$ implies $y^{2^m}(x-1) = 1$ and $y^{-2^m}x = -1$. Similarly, from $[A, {}_{m+1}C] = 1$ we obtain $y^{-2^{m+1}}x = -1$ and $y^{2^{m+1}}(x-1) = 1$. This implies $y^{2^m} = 1$ and hence $x = -1$, which contradicts the choice of x. $\square$

References

[1] S. Bachmuth and H.Y. Mochizuki, *Third Engel Groups and the Macdonald-Neumann Conjecture*, Bull. Austral. Math. Soc. **5** (1971), 379–386.

[2] C. Delizia and C. Nicotera, *On the Commutative-Transitive Kernel of Locally Finite Groups*, Algebra Colloq. **10** (2003), no. 4, 567–570.

[3] C. Delizia and C. Nicotera, *On the Commutative-Transitive Kernel of Certain Infinite Groups*, JP Journal of Algebra, Number Theory and Applications, to appear.

[4] B. Fine, A. Gaglione, G. Rosenberger and D. Spellman, *The Commutative Transitive Kernel*, Algebra Colloq. **4** (1997), 141–152.

[5] B. Huppert, *Endliche Gruppen*, Springer-Verlag, Berlin, 1967.

[6] B. Huppert and N. Blackburn, *Finite Groups III*, Springer-Verlag, New York, 1982.

[7] O.H. Kegel and B.A.F. Wehrfritz, *Locally Finite Groups*, North-Holland Publishing Company, 1973.

[8] P. Lescot, *A Note on CA-groups*, Comm. Algebra **18** (1990), no. 3, 833–838.
[9] D.J.S. Robinson, *Finiteness Conditions and Generalized Soluble Groups*, Part 2, Springer-Verlag, Berlin, 1972.
[10] D.J.S. Robinson, *A Course in the Theory of Groups*, Springer-Verlag, New York, 1993.
[11] M. Suzuki, *The Nonexistence of Certain Type of Simple Groups of Odd Order*, Proc. Amer. Math. Soc. **8** (1957), 686–695.
[12] M. Suzuki, *Finite Groups with Nilpotent Centralizers*, Trans. Amer. Math. Soc. **99** (1961), no. 3, 425–470.
[13] M. Suzuki, *On a Class of Doubly Transitive Groups*, Ann. of Math. **75** (1962), 105–145.
[14] L. Weisner, *Groups in Which the Normaliser of Every Element Except Identity is Abelian*, Bull. Amer. Math. Soc. **31** (1925), 413–416.
[15] Y.F. Wu, *Groups in Which Commutativity Is a Transitive Relation*, J. Algebra **207** (1998), 165–181.
[16] H. Zassenhaus, *Über endliche Fastkörper*, Abh. Math. Sem. Univ. Hamburg **11** (1936), 187–220.

Dipartimento di Matematica e Informatica, Università di Salerno, Via Ponte don Melillo, 84084 - Fisciano (SA), Italy
E-mail address: cdelizia@unisa.it

Fakulteta za Matematiko in Fiziko, Univerza v Ljubljani, Jadranska 19, 1000 Ljubljana, Slovenia
E-mail address: primoz.moravec@fmf.uni-lj.si

Dipartimento di Matematica e Informatica, Università di Salerno, Via Ponte don Melillo, 84084 - Fisciano (SA), Italy
E-mail address: cnicotera@unisa.it