# FINITE GROUPS IN WHICH SOME PROPERTY OF TWO-GENERATOR SUBGROUPS IS TRANSITIVE

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Abstract. Finite groups in which a given property of two-generator subgroups is a transitive relation are investigated. We obtain a description of such groups and prove in particular that every finite soluble-transitive group is soluble. A classification of finite nilpotent-transitive groups is also obtained.

## 1. Introduction

Let $\mathfrak{X}$ be a group theoretical class. A group $G$ is said to be $\mathfrak{X}$-*transitive* (or an $\mathfrak{X}$T-*group*) if for all $x, y, z \in G\backslash\{1\}$ the relations $\langle x, y\rangle \in \mathfrak{X}$ and $\langle y, z\rangle \in \mathfrak{X}$ imply $\langle x, z\rangle \in \mathfrak{X}$. In graph theoretical terms, let $\Gamma_{\mathfrak{X}}(G)$ be the simple graph whose vertices are the nontrivial elements of $G$, and $a$ and $b$ are connected by an edge if and only if $\langle a, b\rangle \in \mathfrak{X}$. Then $G$ is an $\mathfrak{X}$T-group precisely when all the connected components of $\Gamma_{\mathfrak{X}}(G)$ are complete graphs. Several authors have studied $\mathfrak{X}$T-groups for some special classes $\mathfrak{X}$. When $\mathfrak{X}$ is the class of all abelian groups, these groups are also known as commutative-transitive groups or $C$T-groups. Weisner [10] has shown that finite $C$T-groups are either soluble or simple. Finite nonabelian simple $C$T-groups have been classified by Suzuki [6]. These are precisely $\mathrm{PSL}(2, 2^f)$, where $f > 1$. A characterization of finite soluble $C$T-groups has been given by Wu [11] who has also obtained information on locally finite $C$T-groups and polycyclic $C$T-groups. When $\mathfrak{X} = \mathfrak{N}_c$, the class of all groups which are nilpotent of class $\leq c$, similar results have been obtained in [1].

The purpose of this note is to obtain a description of finite $\mathfrak{X}$T-groups for the group theoretical classes $\mathfrak{X}$ having the following properties:

($\star$) $\mathfrak{X}$ is subgroup closed, it contains all finite abelian groups and is bigenetic in the class of all finite groups.

Here a class $\mathfrak{X}$ is said to be *bigenetic* (a terminology due to Lennox [4]) in the class of all finite groups when a finite group $G$ is in $\mathfrak{X}$ if and only if all its two-generator subgroups are. Examples of classes satisfying ($\star$) are the class of all abelian groups, all nilpotent groups, all supersoluble groups and all soluble groups. First we show that if $\mathfrak{X}$ is a class satisfying ($\star$), then every finite $\mathfrak{X}$T-group

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which does not belong to $\mathfrak{X}$ is either a Frobenius group with kernel and complement belonging to $\mathfrak{X}$, or it has no normal $\mathfrak{X}$-subgroups, i.e., it is $\mathfrak{X}$-*semisimple* as defined in [5]. We also show that in several cases, e.g., in the soluble or supersoluble case, the second possibility does not occur. As a consequence we obtain that a finite group is soluble if and only if it is soluble-transitive. In the case when $\mathfrak{X} = \mathfrak{N}$, the class of all nilpotent groups, there exist simple $\mathfrak{N}$T-groups. We obtain a complete classification of finite $\mathfrak{N}$T-groups which generalizes some results of [11].

## 2. Results

Given a group theoretical class $\mathfrak{X}$, let $R_{\mathfrak{X}}(G)$ be the product of all normal $\mathfrak{X}$-subgroups of $G$ (the $\mathfrak{X}$-*radical* of $G$). In general $R_{\mathfrak{X}}(G)$ does not belong to $\mathfrak{X}$. Our first result shows that this is however true within the class of all finite $\mathfrak{X}$-transitive groups when $\mathfrak{X}$ satisfies the properties $(\star)$.

**Lemma 2.1.** *Let $\mathfrak{X}$ be a class of groups satisfying $(\star)$ and let $G$ be a finite $\mathfrak{X}$T-group. Then $R_{\mathfrak{X}}(G)$ is an $\mathfrak{X}$-group.*

*Proof.* Let $M$ and $N$ be normal $\mathfrak{X}$-subgroups of $G$. It suffices to show that $MN$ also belongs to $\mathfrak{X}$. Suppose first that $M \cap N \neq 1$ and let $x \in M \cap N \backslash \{1\}$. First note that for any $m \in M\backslash\{1\}$ and $n \in N\backslash\{1\}$ we have that $\langle m, x\rangle$ and $\langle x, n\rangle$ belong to $\mathfrak{X}$. As $G$ is an $\mathfrak{X}$T-group, we conclude that $\langle m, n\rangle$ is an $\mathfrak{X}$-group. Now let $m_1, m_2 \in M\backslash\{1\}$ and $n \in N\backslash\{1\}$. We may suppose that $m_1 n \neq 1$. Then $\langle m_1 n, m_1\rangle = \langle m_1, n\rangle$ is in $\mathfrak{X}$ and $\langle m_1, m_2\rangle$ is in $\mathfrak{X}$. Thus it follows that $\langle m_1 n, m_2\rangle$ also belongs to $\mathfrak{X}$. Similarly we can prove that $\langle m n_1, n_2\rangle$ is in $\mathfrak{X}$ for every $m \in M\backslash\{1\}$ and $n_1, n_2 \in N\backslash\{1\}$. Now take $m_1, m_2 \in M\backslash\{1\}$ and $n_1, n_2 \in N\backslash\{1\}$ and suppose that $m_1 n_1 \neq 1$, $m_2 n_2 \neq 1$. Then $\langle m_1 n_1, m_2\rangle \in \mathfrak{X}$, $\langle m_2, m_2 n_2\rangle \in \mathfrak{X}$, whence $\langle m_1 n_1, m_2 n_2\rangle$ belongs to $\mathfrak{X}$. This shows that every two-generator subgroup of $MN$ belongs to $\mathfrak{X}$. Since $\mathfrak{X}$ is bigenetic in the class of all finite groups, we get that $MN$ is an $\mathfrak{X}$-group, as required.

Suppose now that $M \cap N = 1$. Then $[M, N] = 1$. As above it suffices to prove that every two-generator subgroup of $MN$ is in $\mathfrak{X}$. At first let $m_1, m_2 \in M\backslash\{1\}$ and $n \in N\backslash\{1\}$. Then the groups $\langle m_1 n, n\rangle = \langle m_1, n\rangle$ and $\langle n, m_2\rangle$ are abelian, hence they belong to $\mathfrak{X}$. By the transitivity we have that $\langle m_1 n, m_2\rangle$ belongs to $\mathfrak{X}$. Similar argument shows that $\langle m n_1, n_2\rangle \in \mathfrak{X}$ for every $m \in M\backslash\{1\}$ and $n_1, n_2 \in N\backslash\{1\}$. From this it follows that if $m_1, m_2 \in M\backslash\{1\}$ and $n_1, n_2 \in N\backslash\{1\}$, then $\langle m_1 n_1, m_2\rangle$ and $\langle m_2, m_2 n_2\rangle$ are in $\mathfrak{X}$, hence $\langle m_1 n_1, m_2 n_2\rangle$ is also in $\mathfrak{X}$. This concludes the proof. □

**Theorem 2.2.** *Let $\mathfrak{X}$ be a class of groups satisfying $(\star)$. Let $G$ be a finite $\mathfrak{X}$T-group. Then one of the following holds.*

(i) *$G$ belongs to $\mathfrak{X}$.*
(ii) *$G$ is $\mathfrak{X}$-semisimple.*
(iii) *$G$ is a Frobenius group with kernel and complement both belonging to $\mathfrak{X}$.*

*Proof.* Let $R$ be the $\mathfrak{X}$-radical of $G$. By Lemma 2.1, $R$ belongs to $\mathfrak{X}$. If $R = G$, then $G$ belongs to $\mathfrak{X}$. If $R = 1$, then $G$ is $\mathfrak{X}$-semisimple. So from now on we assume that $1 \neq R \neq G$.

Let $y \in R\backslash\{1\}$ and suppose that there exists $a \in C_G(y)\backslash R$. Then $\langle a, y\rangle$ is abelian, hence it belongs to $\mathfrak{X}$. As $R$ is an $\mathfrak{X}$-group and $G$ is an $\mathfrak{X}$T-group, we have that $\langle a, h\rangle$ is in $\mathfrak{X}$ for every $h \in R$. By conjugation we get that $\langle a^x, h\rangle \in \mathfrak{X}$ for every $x \in G$ and $h \in R$. Since $G$ is an $\mathfrak{X}$T-group, we get that

$$\langle a^x, a^z\rangle \in \mathfrak{X} \tag{1}$$

for every $x, z \in G$. We claim that $\langle u, v\rangle \in \mathfrak{X}$ for every $u, v \in a^G$. To prove this, we first introduce some notation. For $u \in a^G$ let $r$ be the smallest integer such that $u$ can be written as $a^{\pm g_1} \cdots a^{\pm g_r}$ for some $g_1, \ldots, g_r \in G$. Then we say that $u$ is of weight $r$ and denote $\mathrm{wt}(u) = r$. The proof of our claim goes by induction on $\mathrm{wt}(u)+\mathrm{wt}(v)$. If $\mathrm{wt}(u)+\mathrm{wt}(v) \leq 2$, then the claim follows from (1). Suppose that the claim holds true for all $u, v \in a^G$ with $\mathrm{wt}(u) + \mathrm{wt}(v) \leq l$. Let now $u, v \in a^G$ be such that $\mathrm{wt}(u) + \mathrm{wt}(v) = l + 1$. Without loss of generality we may assume that $\mathrm{wt}(u) > 1$ and $v \neq 1$. Then we can write $u = u'a^{\pm g}$ for some $g \in G$ and $u' \in a^G\backslash\{1\}$ with $\mathrm{wt}(u') = \mathrm{wt}(u)-1$. We have that $\langle u, a^g\rangle = \langle u', a^g\rangle$ belongs to $\mathfrak{X}$ by the induction assumption. For the same reason we have that $\langle a^g, v\rangle \in \mathfrak{X}$. As $G$ is an $\mathfrak{X}$T-group, we conclude that $\langle u, v\rangle$ belongs to $\mathfrak{X}$. This proves that every two-generator subgroup of $a^G$ belongs to $\mathfrak{X}$. As $\mathfrak{X}$ is bigenetic in the class of all finite groups, we get $a^G \in \mathfrak{X}$, hence $a \in R$, a contradiction. By Satz 8.5 in [2] we have that $G$ is a Frobenius group and $R$ is its kernel. In particular, it follows from here that $R$ is nilpotent. Let $H$ be its complement. Then $H$ is an $\mathfrak{X}$T-group with nontrivial center. It follows from here that every two-generator subgroup of $H$ belongs to $\mathfrak{X}$, hence $H \in \mathfrak{X}$. □

A characterization of $\mathfrak{X}$-semisimple $\mathfrak{X}$T-groups is usually not easy and depends heavily on a choice of the class $\mathfrak{X}$; see [1, 6, 11]. In the case of Frobenius groups we provide a general characterization of $\mathfrak{X}$T-groups. At first we prove the following technical result.

**Lemma 2.3.** *Let $\mathfrak{X}$ satisfy $(\star)$. Let $G$ be a finite $\mathfrak{X}$T-group and $H$ an $\mathfrak{X}$-subgroup of $G$. Then*

$$C_G^{\mathfrak{X}}(H) = \{x \in G : \langle x, h\rangle \in \mathfrak{X} \text{ for some } h \in H\backslash\{1\}\}$$

*is an $\mathfrak{X}$-subgroup of $G$ containing $H$.*

*Proof.* Clearly $C_G^{\mathfrak{X}}(H)$ contains $H$. Let $x, y \in C_G^{\mathfrak{X}}(H)\backslash\{1\}$. Then there exist $h, k \in H\backslash\{1\}$ such that $\langle x, h\rangle \in \mathfrak{X}$ and $\langle y, k\rangle \in \mathfrak{X}$. Since $\langle h, k\rangle \in \mathfrak{X}$, we get that $\langle x, y\rangle$ also belongs to $\mathfrak{X}$. If $xy \neq 1$, then $\langle xy, y\rangle = \langle x, y\rangle$ belongs to $\mathfrak{X}$, hence also $\langle xy, k\rangle \in \mathfrak{X}$. Thus $xy \in C_G^{\mathfrak{X}}(H)$. Note also that every two-generator subgroup of $C_G^{\mathfrak{X}}(H)$ is in $\mathfrak{X}$, hence $C_G^{\mathfrak{X}}(H)$ also belongs to $\mathfrak{X}$. □

**Proposition 2.4.** *Let $\mathfrak{X}$ be a group theoretical class satisfying $(\star)$. Let $G$ be a Frobenius group with kernel $F$ and complement $H$. Then $G$ is an $\mathfrak{X}$T-group if and only if $C_G^{\mathfrak{X}}(F)$ and $C_G^{\mathfrak{X}}(H)$ are $\mathfrak{X}$-groups.*

*Proof.* Let $\mathfrak{X}$ and $G$ be as above. If $G$ is an $\mathfrak{X}$T-group, then it follows from Theorem 2.2 that $F$ and $H$ belong to $\mathfrak{X}$. Consequently $C_G^{\mathfrak{X}}(F)$ and $C_G^{\mathfrak{X}}(H)$ are also $\mathfrak{X}$-groups by Lemma 2.3. Conversely, suppose that $C_G^{\mathfrak{X}}(F)$ and $C_G^{\mathfrak{X}}(H)$ are $\mathfrak{X}$-groups. Let $x, y, z \in G\backslash\{1\}$ and suppose that $\langle x, y\rangle \in \mathfrak{X}$ and $\langle y, z\rangle \in \mathfrak{X}$. Assume first that $y \in F$. Then $x, z \in C_G^{\mathfrak{X}}(F)$ and consequently $\langle x, z\rangle \in \mathfrak{X}$. If $y \notin F$, then $y \in H^g$ for some $g \in G$. But then $x, z \in C_G^{\mathfrak{X}}(H^g) = (C_G^{\mathfrak{X}}(H))^g$, whence $\langle x, z\rangle$ belongs to $\mathfrak{X}$. Thus $G$ is an $\mathfrak{X}$T-group. □

When $\mathfrak{X}$ is the class of all abelian groups, then all three possibilities of Theorem 2.2 can occur [6, 11]. In some cases, however, we can exclude the existence of $\mathfrak{X}$-semisimple $\mathfrak{X}$T-groups.

**Theorem 2.5.** *Let $\mathfrak{X}$ be a class of groups satisfying $(\star)$, and suppose that $\mathfrak{X}$ contains all finite dihedral groups and that every finite $\mathfrak{X}$-group is soluble. If $G$ is a finite $\mathfrak{X}$T-group which is not in $\mathfrak{X}$, then $G$ is a Frobenius group with complement belonging to $\mathfrak{X}$. In particular, $G$ is soluble.*

Before proving this result we mention here the well known Thompson's classification of minimal simple groups, i.e., finite nonabelian simple groups all whose proper subgroups are soluble. It turns out [9] that every such group is isomorphic to one of the following groups.

(i) $\mathrm{PSL}(2, p)$, where $p$ is a prime, $p > 3$ and $p^2 - 1 \neq 0 \mod 5$.
(ii) $\mathrm{PSL}(2, 2^f)$, where $f$ is a prime.
(iii) $\mathrm{PSL}(2, 3^f)$, where $f$ is an odd prime.
(iv) $\mathrm{PSL}(3, 3)$.
(v) $\mathrm{Sz}(q)$, where $q = 2^{2n+1}$ and $2n + 1$ is a prime.

If $G = \mathrm{PSL}(2, F)$ where $F$ is a Galois field of odd characteristic and $|F| > 5$, then $G$ can be generated by an involution and an element of even order. This can be easily seen as follows. Let $q = |F|$. By Dickson's theorem [2], $G$ contains elements $a$ and $b$ with $|a| = (q-1)/2$ and $|b| = (q+1)/2$. Note that precisely one of $|a|$, $|b|$ is even, without loss of generality we may assume that this is true for $|a|$. Then $N_G(a) = D_{q-1}$ and this is the only maximal subgroup of $G$ containing $a$; this follows from the proof of Dickson's theorem [2]. So if we choose any involution $u$ from $G\backslash N_G(a)$, we have $\langle a, u\rangle = G$, as required. A similar result holds true for $\mathrm{PSL}(3, 3)$ and $\mathrm{Sz}(q)$. In the first case note that $\mathrm{PSL}(3, 3)$ can be generated by the canonical projections of matrices

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 1 & 2 & 1 \end{pmatrix},$$

which are of orders 2 and 8 in $\mathrm{PSL}(3,3)$, respectively. For the Suzuki groups $\mathrm{Sz}(q)$ it follows from [8] that they can always be generated by an involution and an element of order 4. We summarize this in the following lemma.

**Lemma 2.6.** *Let $G$ be one of the following groups: $\mathrm{PSL}(2,F)$ where $F$ is a Galois field of odd characteristic and $|F| > 5$, $\mathrm{PSL}(3,3)$ or $\mathrm{Sz}(q)$. Then $G$ can be generated by an involution and an element of even order.*

Note that for the groups $\mathrm{PSL}(2,2^f)$ the conclusion of the above lemma does not hold. In this case we have the following result that can be verified by straightforward calculation.

**Lemma 2.7.** *Let $G = \mathrm{PSL}(2,2^f)$, $f > 1$. Denote by $\zeta$ a generator of $\mathrm{GF}(2^f)$ and let $a$, $b$ and $c$ be the elements of $G$ which are projections of*

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \zeta \\ \zeta^{-1} & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & \zeta \\ \zeta^{-1} & \zeta \end{pmatrix},$$

*respectively. Then $\langle a,b\rangle$ and $\langle b,c\rangle$ are dihedral groups and $\langle a,c\rangle = G$.*

*Proof of Theorem 2.5.* We may suppose that $G$ does not belong to $\mathfrak{X}$, hence $R_{\mathfrak{X}}(G) \neq G$. If we prove that $G$ is soluble, then $R_{\mathfrak{X}}(G) \neq 1$ and our claim follows from Theorem 2.2. So suppose that there exist finite insoluble $\mathfrak{X}$T-groups, and let $G$ be a counterexample of minimal order. Then every proper subgroup of $G$ is soluble. By Theorem 2.2 we have that $R_{\mathfrak{X}}(G) = 1$. Let $R$ be the soluble radical of $G$. Since $\mathfrak{X}$ contains all finite abelian groups, we have that $R = 1$. It is now easy to see that $G$ has to be simple. By Thompson's classification of minimal simple groups [9], $G$ is isomorphic to one of the groups in the above mentioned list. By Lemma 2.7, $G$ is not isomorphic to any of $\mathrm{PSL}(2,2^f)$, where $f$ is a prime. If $G$ is one of the groups of Lemma 2.6, then $G = \langle a,b\rangle$, where $|a| = 2$ and $|b| = 2k$, $k > 1$. We have that $\langle a, b^k\rangle$ is a dihedral group and $\langle b^k, b\rangle$ is a cyclic group, hence $G$ is in $\mathfrak{X}$ by the $\mathfrak{X}$T-property, a contradiction. This concludes the proof. ☐

Using Theorem 2.5, we obtain a rather surprising characterization of finite soluble groups.

**Corollary 2.8.** *Every finite soluble-transitive group is soluble.*

Note that the class of all supersoluble groups also satisfies all the assumptions of Theorem 2.5. Thus we have the following.

**Corollary 2.9.** *Let $G$ be a finite supersoluble-transitive group. If $G$ is not supersoluble, then $G$ is a Frobenius group with supersoluble complement. In particular, $G$ is always soluble.*

In view of Corollary 2.8 we may ask if every finite supersoluble-transitive group is supersoluble. This is not true however, as the group $A_4$ shows. It is also not difficult to find an example of a Frobenius group with supersoluble complement which is not supersoluble-transitive. This example also shows that Proposition

2.4 is in a certain sense best possible. Indeed, it is not possible to replace $C_G^{\mathfrak{X}}(F)$ and $C_G^{\mathfrak{X}}(H)$ by $F$ and $H$, respectively.

*Example* 2.10. Let $A = \langle x\rangle \oplus \langle y\rangle$ be an elementary group of order 9 and let $\alpha$ be the automorphism of $A$ given by the matrix

$$\begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}.$$

Then $\langle\alpha\rangle$ acts fixed-point-freely on $A$. Let $G = A \rtimes \langle\alpha\rangle$. This is a group of order 36 which is not supersoluble-transitive. To see this, note that $\langle\alpha^2, (\alpha y)^2\rangle$ is a dihedral group, $\langle(\alpha y)^2, \alpha y\rangle$ is cyclic, whereas $\langle\alpha^2, \alpha y\rangle = G$ is not supersoluble. Denoting by $\mathfrak{S}$ the class of all supersoluble groups, note that $C_G^{\mathfrak{S}}(\langle\alpha\rangle)$ has 20 elements and it is thus not a subgroup of $G$. On the other hand, $C_G^{\mathfrak{S}}(A)$ is a subgroup of index 2 in $G$.

Theorem 2.5 cannot be applied in the case of $\mathfrak{N}$T-groups, where $\mathfrak{N}$ denotes the class of all nilpotent groups. Thus it is to be expected that there exist finite insoluble $\mathfrak{N}$T-groups. This is confirmed by the following characterization of finite $\mathfrak{N}$T-groups which is essentially contained in [1]. We include a proof for the sake of completeness.

**Theorem 2.11.** *Let $G$ be a finite $\mathfrak{N}$T-group. Then one of the following holds.*

(i) *$G$ is nilpotent.*
(ii) *$G$ is a Frobenius group with nilpotent complement.*
(iii) *$G \cong \mathrm{PSL}(2, 2^f)$ for some $f > 1$.*
(iv) *$G \cong \mathrm{Sz}(q)$ with $q = 2^{2n+1} > 2$.*

*Conversely, every finite group under* (i)–(iv) *is an $\mathfrak{N}$T-group.*

*Proof.* If $G$ is soluble and not nilpotent, then the Fitting subgroup $F$ of $G$ is a proper nontrivial subgroup of $G$. By Theorem 2.2, $G$ is a Frobenius group with nilpotent complement. So suppose that $G$ is not soluble. It is easy to see that in every finite $\mathfrak{N}$T-group $G$ the centralizers of nontrivial elements are nilpotent, i.e., $G$ is an CN-group. By a result of Suzuki [7, Part I, Theorem 4], the centralizer of any involution in $G$ is a 2-group. Let $P$ and $Q$ be any Sylow $p$-subgroups of $G$ and suppose that $P \cap Q \neq 1$. Since $P$ and $Q$ are nilpotent and $G$ is an $\mathfrak{N}$T-group, we conclude that $\langle P, Q\rangle$ is nilpotent. This shows that the Sylow subgroups of $G$ are independent. Combining Theorem 1 in Part I and Theorem 3 in Part II of [7], we conclude that $G$ has to be simple. Additionally, it follows from [8] that $G$ is isomorphic either to $\mathrm{PSL}(2, 2^f)$, where $f > 1$, or to $\mathrm{Sz}(q)$ with $q = 2^{2n+1} > 2$.

Let $G$ be a finite Frobenius group with the kernel $N$ and a complement $H$ and suppose that $H$ nilpotent. Let $x, y, z \in G\backslash\{1\}$ and let the groups $\langle x, y\rangle$ and $\langle y, z\rangle$ be nilpotent. Let $c$ be the nilpotency class of $\langle x, y\rangle$. First suppose that $x \in N$ and $y \notin N$. Then there exists an integer $c$ such that $[x, {}_c y] = 1$, which implies $[x, {}_{c-1} y] = 1$, since $H$ acts fixed-point-freely on $N$. By the same argument we get $x = 1$, which is not possible. This shows that if $x \in N$ then $y \in N$ and similarly also $z \in N$. But in this case $\langle x, z\rangle$ is clearly nilpotent, since $N$ is nilpotent. Thus we may assume that $x, y, z \notin N$. Let $x \in H^g$ and $y \in H^k$ for some $g, k \in G$ and suppose $H^g \neq H^k$. We clearly have $C_G(x) \leq H^g$ and $C_G(y) \leq H^k$. Let $\omega$ be any commutator of weight $c$ with entries in $\{x, y\}$. Then $\omega \in C_G(x) \cap C_G(y) = 1$ implies that $\langle x, y\rangle$ is nilpotent of class $\leq c-1$, a contradiction. Hence we conclude that $\langle x, y\rangle \leq H^g$ and similarly also $\langle y, z\rangle \leq H^g$. Therefore we have $\langle x, z\rangle \leq H^g$. But $H^g$ is nilpotent, hence the group $\langle x, z\rangle$ is also nilpotent. This shows that the groups under (ii) are $\mathfrak{N}$T-groups.

It remains to prove that the groups under (iii) and (iv) are $\mathfrak{N}$T-groups. If $G = \mathrm{PSL}(2, 2^f)$, $f > 1$, then every centralizer of a nontrivial element of $G$ is abelian by [6]. It follows from here that $G$ is an $\mathfrak{N}$T-group. Now let $G = \mathrm{Sz}(q)$ where $q = 2^{2n+1} > 2$. By Theorem 3.10 (c) in [3], $G$ has a nontrivial partition $(G_i)_{i\in I}$, where for every $i \in I$ the group $G_i$ is nilpotent and contains the centralizers of all of its nontrivial elements. Let $x, y, z \in G\backslash\{1\}$ and suppose that the groups $\langle x, y\rangle$ and $\langle y, z\rangle$ are nilpotent. Let $a$ and $b$ be nontrivial elements in $Z(\langle x, y\rangle)$ and $Z(\langle y, z\rangle)$, respectively, and suppose that $a \in G_i$ and $b \in G_j$ for some $i, j \in I$. Then $y \in C_G(a) \cap C_G(b) \leq G_i \cap G_j$, hence $i = j$. But now we get $x, z \in G_i$ and since $G_i$ is nilpotent, the same is true for the group $\langle x, z\rangle$. Hence $G$ is an $\mathfrak{N}$T-group. □

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