SOME COMMUTATOR GROUP LAWS EQUIVALENT TO THE COMMUTATIVE LAW

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Abstract

Let G be a group, n an integer greater than 2 and $x, y \in G$. Let c_n be a commutator of weight n with entries from the set $\{x, x^{-1}, y, y^{-1}\}$. The question, originally raised by N. D. Gupta, is: Under what conditions is a group satisfying the law $c_n = [x, y]$ abelian? Some sufficient conditions for an affirmative answer are the finiteness or the solvability of the group. It is known that in the case $n = 3$ the answer is positive for an arbitrary group. We prove that every group satisfying the law $c_4 = [x, y]$ is abelian. We establish some group laws of the form $c_n = [x, y]$ which imply the commutative law without any other restriction. We also prove that every 4-Engel group satisfying the law $c_n = [x, y]$ is abelian.

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1 Introduction

Let G be a group satisfying the law $c_n = [x, y]$, where c_n is a commutator of weight $n > 2$ with entries drawn from the set $\{x, x^{-1}, y, y^{-1}\}$. N. D. Gupta [1] showed that every solvable or finite group with such a law is abelian. He also investigated a number of examples for $n = 3$ and $n = 4$ and showed that for those cases the restrictions on the structure of G are unnecessary. L.-C. Kappe and M. J. Tomkinson [4] examined the case $n = 3$. They proved that all the laws for $n = 3$ imply the commutative law. They also raised the following question: Do similar laws with the commutators of bigger weight imply that G is abelian? In case the answer is no, what is the lowest value of n such that the law $c_n = [x, y]$ is not equivalent to the commutative law? In this paper we contribute to the solution of this question by proving the following result:

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Theorem 1.1. *Every group satisfying the law $c_4 = [x, y]$ is abelian.*

In addition, we establish some group laws of the form $c_n = [x, y]$ which imply the commutative law without any restriction on the structure of the group, cf. Theorem 2.7. These results may be of independent interest. Another result of the kind is Theorem 2.4, which states that every 4-Engel group satisfying the law $c_n = [x, y]$ is abelian. It is based on a recent result of Traustason [7] which describes the structure of the subgroup of the 4-Engel group generated by an element and its conjugate. Some auxiliary results give a large family of group laws which imply that a group satisfies the 3-Engel law, cf. Theorem 3.5 and Theorem 3.8. One could possibly go further in generalizing these results.

We prove Theorem 1.1 in the following way: We begin by exploring some general properties of the group laws $c_n = [x, y]$. These results, together with Theorems 3.5 and 3.8, cover over 70% of the possibilities for the law $c_4 = [x, y]$. The remaining laws are handled individually. It is interesting to note that most of these cases can be reduced in a few steps to the conditions considered in Theorem 3.5 or 3.8.

We remind the reader that a commutator of two elements x, y of the group G is defined by $[x, y] := x^{-1}y^{-1}xy$. We also write $x^y := y^{-1}xy$. The commutator of weight $n > 2$ is defined inductively as $[x_1, \dots, x_n] := [[x_1, \dots, x_{n-1}], x_n]$. We use the notation $[x, ny] := [x, y, \dots, y]$, where y appears n times on the right. A group is said to be an n-Engel group if it satisfies the law $[x, ny] = 1$. The standard commutator identities, see [4] and [6], will be used throughout the paper without further reference.

Since techniques similar to those in [4] are used to deal with the laws of the form $c_n = [x, y]$, we will now outline them. Let $\mathcal{F}_G$ be the near-ring of all functions from G to G described in [6]. For $\alpha, \beta \in \mathcal{F}_G$ we define $\alpha\beta$ and $\alpha + \beta$ by $x^{\alpha\beta} := (x^\alpha)^\beta$ and $x^{\alpha+\beta} := x^\alpha x^\beta$, respectively. The additive unit is the map 0, defined by $x^0 := 1$, and the additive inverse of α is denoted by $-\alpha$, where $x^{-\alpha} := (x^\alpha)^{-1}$. Let $\mathcal{N}_G$ be the sub near-ring of the near-ring $\mathcal{F}_G$ generated by the inner automorphisms of the group G. Let $x, y \in G$ and set $a := [x, y]$. We define the equivalence relation $\sim_a$ on $\mathcal{N}_G$ by $g \sim_a h$ if and only if $a^g = a^h$. We set $\mathcal{N}_a := \mathcal{N}_G/_{\sim_a}$. One should notice that the equivalence relation $\sim_a$ is not compatible with the multiplication from the left, for if $h \sim_a k$, it is not necessary that the equivalence classes $[fh]_a$ and $[fk]_a$ coincide. On the other hand, it is compatible with addition and with multiplication from the right, and the left distributive law also holds. The right distributive law holds when multiplying on the right by an element of the group G. From now on we identify the equivalence class $[f]_a$ with f.

Every law of the form $c_n = [x, y]$ can be transcribed into $a^{f(x,y)} = a^{g(x,y)}$, where $f(x, y)$ and $g(x, y)$ are elements of $\mathcal{N}_G$, so the problem of reducing this law to the commutative law is in fact the question whether the equation $f(x, y) = g(x, y)$ in $\mathcal{N}_a$ implies the identity $1 = 0$. We suppress writing the base a and write only the exponents, so we often say that a group satisfies the law $f(x,y) = g(x,y)$ instead of $a^{f(x,y)} = a^{g(x,y)}$. When we replace (x,y) by (u,v) in the $\mathcal{N}_a$-equation $f(x,y) = g(x,y)$, we mean the replacement of (x,y) by (u,v) in the group identity $[x,y]^{f(x,y)} = [x,y]^{g(x,y)}$, so we obtain $[u,v]^{f(u,v)} = [u,v]^{g(u,v)}$.

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2 General properties

For our purpose it is obviously enough to consider the case when G is a 2-generator group. We summarize some of the known results from [1] and [4] in the following lemma:

Lemma 2.1 ([1], [4]).
(a) A solvable or a finite group satisfying the law $c_n = [x,y]$ is abelian;
(b) A group satisfying the law of the form $c_3 = [x,y]$ is abelian.

Since every 2-generator 3-Engel group is nilpotent of class ≤ 4 by [3], it follows that:

Lemma 2.2 ([1]). *A 3-Engel group satisfying the law $c_n = [x,y]$ is abelian.*

We prove that every 4-Engel group G with the law $c_n = [x,y]$ is abelian. The following lemma is required in the proof:

Lemma 2.3 ([7]). *Let G be a 4-Engel group and $u,v \in G$. Then the group $\langle u, u^v \rangle$ is metabelian and nilpotent of class at most 4.*

Theorem 2.4. *Let G be a 4-Engel group with the law $c_n = [x,y]$. Then G is an abelian group.*

Proof. Let $u,v \in G$. It follows from Lemma 2.3 and Lemma 2.1 that the group $\langle u, u^v \rangle$ is abelian. This is equivalent to the fact that G is a 2-Engel group. Thus, by the celebrated result of F. W. Levi [5] (see also [6; Theorem 12.3.6]), G is nilpotent of class at most 3, and hence is abelian by Lemma 2.1. $\square$

The next two lemmas are frequently used:

Lemma 2.5. *The 3-Engel law is equivalent to the law $[x,y]^{y^{-1}}[x,y]^y = [x,y]^2$.*

Proof. The law $[x,y,y,y] = 1$ can be rewritten in $\mathcal{N}_a$ as $-y + 1 - y + y^2 = 0$. Replacing (x,y) by (x^{-1}, y^{-1}), we get

$$-x^{-1}y^{-2} + x^{-1}y^{-1} - x^{-1}y^{-2} + x^{-1}y^{-3} = 0. \tag{1}$$

Multiplying (1) by y^2x from the right, we get $-1 + x^{-1}yx - 1 + y^{-1} = 0$, which is equivalent to $y^{-1} + y = 2$. $\square$

Lemma 2.6 (Higman [2]). *If $G = \langle u, v \rangle$ satisfies the law $[x, y, [x, y^{-1}]] = 1$, then it is metabelian.*

We start with some general laws, which settle 32 out of the 128 cases for the law $c_4 = [x, y]$:

Theorem 2.7. *Let n be a positive integer greater than 1 and $k_1, \ldots, k_n \in \{-1, 1\}$. The following group laws are equivalent to the commutative law:*
(a) $[x^{-1}, y^{k_1}, \ldots, y^{k_n}] = [x, y]$;
(b) $[y^{-1}, x^{k_1}, \ldots, x^{k_n}] = [x, y]$;
(c) $[y^{k_1}, x^{-1}, y^{k_2}, \ldots, y^{k_n}] = [x, y]$;
(d) $[x^{k_1}, y^{-1}, x^{k_2}, \ldots, x^{k_n}] = [x, y]$.

Proof. Replacing x by $x^{-1}y^{-1}$ in (a) leads to

$$\begin{aligned} [yx, y^{k_1}, \ldots, y^{k_n}] &= [x^{-1}y^{-1}, y]; \\ [x, y^{k_1}, \ldots, y^{k_n}] &= [x^{-1}, y]^{y^{-1}}. \end{aligned}$$

Thus we get $[x^{-1}, y] = [x^{-1}, y]^{y^{-1}}$ or, equivalently, $[x, y]^y = [x, y]$. But this is equivalent to $[x, y, y] = 1$, hence the group is a 2-Engel group and therefore abelian. The proofs of (b), (c) and (d) are very similar and are therefore omitted. $\square$

Theorem 2.8. *Let $n \geq 2$ and let G be a group satisfying one of the laws $[y, nx] = [x, y]$ or $[x, ny^{-1}] = [x, y]$. Then G is abelian.*

Proof. Replacing y by $[y, x]$ in the law $[y, nx] = [x, y]$, we get $[y, (n+1)x] = [x, [y, x]]$ which implies $[x, y, x] = [x, [y, x]]$. This is equivalent to $[x, y, [x^{-1}, y]] = 1$, hence G is metabelian by Lemma 2.6 and thus abelian by Lemma 2.1. The proof of the remaining law can be found in [1]. $\square$

Lemma 2.9. *Let $n \geq 1$ and $k_1, \ldots, k_n \in \{-1, 1\}$. The following pairs of group laws are equivalent:*
(a) $[x, y, y^{k_1}, \ldots, y^{k_n}] = [x, y]$ *and* $[x, y, x^{-k_1}, \ldots, x^{-k_n}] = [x, y]$;
(b) $[y, x, x^{k_1}, \ldots, x^{k_n}] = [x, y]$ *and* $[y, x, y^{-k_1}, \ldots, y^{-k_n}] = [x, y]$.

Proof. Let G be a group satisfying the law $[x, y, y^{k_1}, \ldots, y^{k_n}] = [x, y]$. Replacing y by y^{-1}, we get $[x, y^{-1}, y^{-k_1}, \ldots, y^{-k_n}] = [x, y^{-1}]$. Conjugating this identity by y, we obtain $[y, x, y^{-k_1}, \ldots, y^{-k_n}] = [y, x]$ which is equivalent to $[x, y, x^{-k_1}, \ldots, y^{-k_n}] = [x, y]$. The proof of (b) is similar. $\square$

Corollary 2.10. *Let $n \geq 1$. If a group G satisfies the law $[y, x, ny^{-1}] = [x, y]$, then it is abelian.*

Corollary 2.11. *If a group G satisfies the law $[y, x^{-1}, x^{-1}, x] = [x, y]$, then it is abelian.*

Proof. Let us consider a group G which satisfies the law $[y, x^{-1}, x^{-1}, x] = [x, y]$. This law can be rewritten in $\mathcal{N}_a$ as $-x^{-2} + x^{-1} - 1 + x^{-1} = 1$. Interchanging x and y, we get $y^{-1} - 1 + y^{-1} - y^{-2} = 1$, which is the corresponding equation for $[y, x, y^{-1}, y^{-1}] = [x, y]$. This implies the commutative law by Corollary 2.10. □

Lemma 2.12. *Let $n \geq 2$ and $k_1, \dots, k_n \in \{-1, 1\}$.*
(a) The law $[x^{k_1}, y^{k_2}, \dots, y^{k_n}, x] = [x, y]$ implies $[x, y^{k_2}, \dots, y^{k_n}, y^{\pm 1}] = 1$;
(b) The law $[y^{k_1}, x^{k_2}, \dots, x^{k_n}, y] = [x, y]$ implies $[y, x^{k_2}, \dots, x^{k_n}, x^{\pm 1}] = 1$.

Proof. We only prove (a) since the proof of (b) is similar. In the case where $k_1 = 1$, we replace x by yx and we get $[x, y^{k_2}, \dots, y^{k_n}, x] \cdot [x, y^{k_2}, \dots, y^{k_n}, y]^x = [x, y]$. It follows that $[x, y^{k_2}, \dots, y^{k_n}, y] = 1$. If we instead of that replace x by $y^{-1}x$, we also get $[x, y^{k_2}, \dots, y^{k_n}, y^{-1}] = 1$. In the case where $k_1 = -1$, we replace x by x^{-1} and apply the above procedure. □

This result, together with Lemma 2.2, yields:

Corollary 2.13. *The group laws of the form $[x^k, y^{\pm 1}, y^{\pm 1}, x] = [x, y]$ or $[y^k, x^{\pm 1}, x^{\pm 1}, y] = [x, y]$, where $k \in \{-1, 1\}$, imply the commutative law.*

The following result is rather technical and will be used in the next section:

Lemma 2.14. *Let $n \geq 1$ and $k_1, \dots, k_n \in \{-1, 1\}$.*
(a) The law $[y^{-1}, x^{k_1}, \dots, x^{k_n}, y^{-1}] = [x, y]$ implies $[y^{-1}, x^{k_1}, \dots, x^{k_n}, x^{\pm 1}]^{y^{-1}} = [x, y, x^{\mp 1}]$;
(b) The law $[x^{-1}, y^{k_1}, \dots, y^{k_n}, x^{-1}] = [x, y]$ implies $[x^{-1}, y^{k_1}, \dots, y^{k_n}, y^{\pm 1}]^{x^{-1}} = [x, y, y^{\mp 1}]$;
(c) The law $[x, y^{k_1}, \dots, y^{k_n}, x^{-1}] = [x, y]$ implies $[x, y^{k_1}, \dots, y^{k_n}, y^{\pm 1}] = [y, x, y^{\pm 1}]$.

Proof. Replacing y by $y^{-1}x$ in the law $[y^{-1}, x^{k_1}, \dots, x^{k_n}, y^{-1}] = [x, y]$ leads to

$$[y, x^{k_1}, \dots, x^{k_n}, y] \cdot [y, x^{k_1}, \dots, x^{k_n}, x^{-1}]^y = [x, y^{-1}]^x,$$

which implies $[y, x^{k_1}, \dots, x^{k_n}, x^{-1}]^y = [x, y^{-1}, x]$. The rest of the proof is very similar, so we omit the details. □

3 The laws $c_4 = [x, y]$

In this section we consider the laws of the form $[u, v, w, z] = [x, y]$, where u, v, w, z are the elements of the set $\{x, x^{-1}, y, y^{-1}\}$. When $u = v^{\pm 1}$, these laws clearly imply that the group is abelian, so, in fact, we have 128 non-trivial possibilities for the commutator $[u, v, w, z]$. For a number of cases N. D. Gupta [1] showed that they imply the commutative law.

Since there are so many cases of the law $[u, v, w, z] = [x, y]$ to handle, it is convenient to set up a table of 32 lines and 4 columns, where each of the lines represents the commutator $[u, v, w]$ and the columns indicate the possible values for z. The entries of this table note which theorem or proposition takes care of the case - see Table 1:

	x	x^{-1}	y	y^{-1}
1. $[x, y, y]$	2.13	3.13	3.3	3.1
2. $[x, y, y^{-1}]$	3.9	3.9	3.1	3.3
3. $[x, y^{-1}, y]$	3.1	3.1	3.1	3.1
4. $[x, y^{-1}, y^{-1}]$	2.13	3.13	3.3	2.8
5. $[x, y, x]$	3.4	3.1	3.9	3.9
6. $[x, y, x^{-1}]$	3.1	3.4	3.9	3.9
7. $[x, y^{-1}, x]$	2.7	2.7	3.9	3.20
8. $[x, y^{-1}, x^{-1}]$	2.7	2.7	3.9	3.18
9. $[x^{-1}, y, y]$	2.13	3.19	2.7	2.7
10. $[x^{-1}, y, y^{-1}]$	3.9	3.18	2.7	2.7
11. $[x^{-1}, y^{-1}, y]$	3.1	3.9	2.7	2.7
12. $[x^{-1}, y^{-1}, y^{-1}]$	2.13	3.2	2.7	2.7
13. $[x^{-1}, y, x]$	3.1	3.1	3.1	3.1
14. $[x^{-1}, y, x^{-1}]$	3.4	3.30	3.9	3.10
15. $[x^{-1}, y^{-1}, x]$	2.7	2.7	3.1	3.9
16. $[x^{-1}, y^{-1}, x^{-1}]$	2.7	2.7	3.9	3.16
17. $[y, x, x]$	2.8	3.29	2.13	3.15
18. $[y, x, x^{-1}]$	3.26	3.28	3.9	3.9
19. $[y, x^{-1}, x]$	3.26	3.25	3.9	3.9
20. $[y, x^{-1}, x^{-1}]$	2.11	3.31	2.13	3.11
21. $[y, x, y]$	3.9	3.9	3.28	3.26
22. $[y, x, y^{-1}]$	3.9	3.9	3.29	2.10
23. $[y, x^{-1}, y]$	3.9	3.22	2.7	2.7
24. $[y, x^{-1}, y^{-1}]$	3.9	3.21	2.7	2.7
25. $[y^{-1}, x, x]$	2.7	2.7	2.13	3.23
26. $[y^{-1}, x, x^{-1}]$	2.7	2.7	3.9	3.14
27. $[y^{-1}, x^{-1}, x]$	2.7	2.7	3.9	3.9
28. $[y^{-1}, x^{-1}, x^{-1}]$	2.7	2.7	2.13	3.24
29. $[y^{-1}, x, y]$	3.9	3.9	3.29	3.25
30. $[y^{-1}, x, y^{-1}]$	3.9	3.10	3.28	3.27
31. $[y^{-1}, x^{-1}, y]$	3.9	3.9	2.7	2.7
32. $[y^{-1}, x^{-1}, y^{-1}]$	3.9	3.17	2.7	2.7

Table 1: List of the laws $c_4 = [x, y]$.

There are 14 group laws of the form $c_4 = [x, y]$ which almost directly imply the commutative law, since their corresponding equations in $\mathcal{N}_a$ are equivalent to the equations for some laws of the form $c_3 = [x, y]$. The reader should consult [4] for the details.

Proposition 3.1. *Let $x, y \in G$ and $z \in \{x^{\pm 1}, y^{\pm 1}\}$. If G satisfies the law $c_4 = [x, y]$, where c_4 is one of the commutators of the form $[x, y, z, z^{-1}]$, $[x, y^{-1}, y, z]$, $[x^{-1}, y, x, z]$, $[x^{-1}, y^{-1}, x, y]$ or $[x^{-1}, y^{-1}, y, x]$, then it is an abelian group.*

If we consider Lemma 2.12 and Theorem 2.4, we obtain 8 more laws of the form $c_5 = [x, y]$ which imply the commutative law; these laws are $[x^k, y^{\pm 1}, y^{\pm 1}, y^{\pm 1}, x] = [x, y]$ and $[y^k, x^{\pm 1}, x^{\pm 1}, x^{\pm 1}, y] = [x, y]$, where $k \in \{-1, 1\}$. This yields:

Proposition 3.2. *A group G, which satisfies $[x^{-1}, y^{-1}, y^{-1}, x^{-1}] = [x, y]$, is abelian.*

Proof. By Lemma 2.14 G also satisfies the law

$$[x^{-1}, y^{-1}, y^{-1}, y^{-1}]^{x^{-1}} = [x, y, y]. \tag{1}$$

In addition, $[x, y, y, x] = [x^{-1}, y^{-1}]$ also holds. Using (1), we obtain

$$[[x^{-1}, y^{-1}, y^{-1}, y^{-1}]^{x^{-1}}, x] = [x^{-1}, y^{-1}]. \tag{2}$$

Conjugating (2) by x, we get $[x^{-1}, y^{-1}, y^{-1}, y^{-1}, x] = [y^{-1}, x]$. This yields $[y^{-1}, x, x, x, y] = [x, y]$, hence G is an abelian group. □

The following group laws are already dealt with in [1]:

Proposition 3.3 ([1]). *The group laws $[x, y, y, y] = [x, y]$, $[x, y^{-1}, y^{-1}, y] = [x, y]$ and $[x, y, y^{-1}, y^{-1}] = [x, y]$ are equivalent to the commutative law.*

Corollary 3.4. *The following group laws imply that a group is abelian:*
(a) $[x, y, x, x] = [x, y]$;
(b) $[x^{-1}, y, x^{-1}, x] = [x, y]$;
(c) $[x, y, x^{-1}, x^{-1}] = [x, y]$.

Proof. The law $[x, y, x, x] = [x, y]$ is equivalent to the law $[x, y, y^{-1}, y^{-1}] = [x, y]$ by Lemma 2.9. This law has in $\mathcal{N}_a$ the equation $-y^{-1} + 1 - y^{-1} + y^{-2} = 1$. Replacing (x, y) by (y, x) gives $x^{-2} - x^{-1} + 1 - x^{-1} = 1$. This is the corresponding $\mathcal{N}_a$-equation for the law (b). Similarly, the equivalence between the laws $[x, y, x^{-1}, x^{-1}] = [x, y]$ and $[x, y, y, y] = [x, y]$ can be proved. □

Let us consider for a while the group laws which have in $\mathcal{N}_a$ the equation of the form $-y^{-1} + xy^{-1} = f(x)$, where $f(x)$ is an expression in $\mathcal{N}_a$ not containing y. Replacing y by xy in this equation gives $-y^{-1}x^{-1} + xy^{-1}x^{-1} = f(x)$. Comparing both equations and multiplying by y from the right give $-x^{-1} + [x^{-1}, y] = -1 + x$. Since $a^{[x^{-1}, y]} = a^{x^{-1} + 1 - x^{-1}}$ in G, we have in $\mathcal{N}_a$ that $x^{-1} + x = 2$ which is equivalent to $y + y^{-1} = 2$. Lemma 2.5 implies that G is a 3-Engel group. Using a similar procedure, we get:

Theorem 3.5. *The group laws with the following equations in $\mathcal{N}_a$ imply that a group satisfies the 3-Engel law:*

$$\begin{aligned} \pm y \mp xy &= f(x); \\ \pm y \mp x^{-1}y &= f(x); \\ \pm y^{-1} \mp x^{-1}y^{-1} &= f(x); \\ \pm y^{-1} \mp y^{-1}x &= f(x); \\ -y^{-1} + xy^{-1} &= f(x). \end{aligned}$$

Similar manipulations as above give:

Lemma 3.6. *The laws, which have in $\mathcal{N}_a$ the equation of the form $y^{-1} - xy^{-1} = f(x)$, imply $x^2 = 1 + x - 2 + x$.*

Interchanging x and y in the above yields:

Corollary 3.7. *The law with the $\mathcal{N}_a$-equation of the form $-yx^{-1} + x^{-1} = f(y)$ implies $y^2 = y - 2 + y + 1$.*

Let us examine now the group law with the equation in $\mathcal{N}_a$ of the form $-x^n y + x^{n+1} y = f(x)$ where n is a nonnegative integer. Replacing y by $x^{-1}y$ gives $-x^{n-1}y + x^n y = f(x)$. Using induction leads to $-y + xy = f(x)$, which implies the 3-Engel law. A similar argument shows that $x^{-n}y - x^{-n-1}y = f(x)$ implies that G is a 3-Engel group; we only have to replace y by xy. Using Theorem 3.5 as the basis of induction, we obtain:

Theorem 3.8. *The laws with the following $\mathcal{N}_a$-equations imply the 3-Engel law:*

$$\begin{aligned} \pm x^{kn} y \mp x^{k(n+1)} y &= f(x), \\ \pm y^{-1} x^{kn} \mp y^{-1} x^{k(n+1)} &= f(x), \end{aligned}$$

where $k \in \{-1, 1\}$ and n is a nonnegative integer.

By a straightforward calculation and using Lemma 2.2, we get the following:

Corollary 3.9. *Let G be a group satisfying the law $c_4 = [x, y]$, where c_4 is one of the following commutators: $[z, w, z^{k_1}, w^{k_2}]$, $[z, w, w^{-1}, z^{\pm 1}]$, $[z, w^{-1}, z^{\pm 1}, w]$, $[z^{-1}, w^{\pm 1}, z^{-1}, w]$, $[y^{k_1}, x^{-1}, x, y^{k_2}]$, $[y^{-1}, x^{k_1}, y, x^{k_2}]$, $[x^{-1}, y^{\pm 1}, y^{\mp 1}, x^{\pm 1}]$, $[y^{-1}, x, x^{-1}, y]$ or $[x^{-1}, y^{-1}, x, y^{-1}]$, where $z \in \{x, y\}$, $w \in \{x, y\} \backslash \{z\}$ and $k_1, k_2 \in \{-1, 1\}$. Then G is an abelian group.*

Proposition 3.10. *If a group G satisfies $[x^{-1}, y, x^{-1}, y^{-1}] = [x, y]$ or $[y^{-1}, x, y^{-1}, x^{-1}] = [x, y]$, then it is an abelian group.*

Proof. The law $[y^{-1}, x, y^{-1}, x^{-1}] = [x, y]$ can be rewritten in $\mathcal{N}_a$ as $-y^{-1}x^{-1} + y^{-2}x^{-1} = -y^{-1} + y^{-2} + 1$. Replacing y by y^{-1} in this equation, we get $yx^{-1} - x^{-1} = y^{-1} + y - 1$. By Theorem 3.5, G is a 3-Engel group, and hence is abelian by Lemma 2.2. The proof for the law $[x^{-1}, y, x^{-1}, y^{-1}] = [x, y]$, which has in $\mathcal{N}_a$ the equation $x^{-1}y^{-1} - x^{-2}y^{-1} = x^{-1} - x^{-2} + 1$, is similar. $\square$

Proposition 3.11. *A group G satisfying $[y, x^{-1}, x^{-1}, y^{-1}] = [x, y]$ is abelian.*

Proof. Rewriting this law in $\mathcal{N}_a$ gives

$$-x^{-1}y^{-1} + x^{-2}y^{-1} = -x^{-1} + x^{-2} + 1. \tag{1}$$

If we substitute x^{-1} for x in (1), we get $y^{-1} - xy^{-1} = 1 - x - x^{-1}$. By Lemma 3.6,

$$-1 + x - x^2 = 1 - x - 1 \tag{2}$$

also holds. Replacing y by yx^{-1} in (1), we get $-x^{-1}y^{-1} + x^{-2}y^{-1} = -x^{-2} + x^{-3} + x^{-1}$. Using (1), we get $x^2 - x^3 - x = -1 + x - x^2$. Replacing x by x^{-1} in this equation, we get $x - x^2 - 1 = x^2 - x^3 - x$. Comparing both the equations, we have $x - x^2 - 1 = -1 + x - x^2$. Using (2), we obtain $x^2 - x + 1 - x = 0$ which is the corresponding equation for the 3-Engel law $[y, x, x, x] = 1$. Thus G is abelian by Lemma 2.2. □

Lemma 3.12. *Let G be a group which satisfies two laws that can be rewritten in $\mathcal{N}_a$ as*

$$y + 1 + y = 1 + y + 1; \tag{1}$$

$$y^2 = y - 2 + y + 1. \tag{2}$$

Then G is a 2-Engel group.

Proof. Combining (1) and (2) leads to

$$y^2 = y - 1 + y + 1 - y. \tag{3}$$

Replacing x by x^{-1} in (2) gives $-x^{-1}y^2x = 1 - y + 2 - y - 2$. Since the identity $-x^{-1}y^2x = 1 + y - y^2 - y - 1$ always holds, we get $y^2 = -y + 1 + y - 2 + 2y$. Observing (1) in this equation yields

$$y^2 = 1 + y - 3 + 2y. \tag{4}$$

Next we replace (x, y) by (y, ya) in (1) to obtain

$$-y - y^2 + y + 1 - y = -1 - y^2 + (y + 1 - y - y^2 + y). \tag{5}$$

Applying (3) to (5) yields $-y - y^2 + y + 1 - y = -1 - y^2 + 1$. Using (1) and (3) leads to $y^2 = -y + 1 + y - 1 + y$. Another application of (1) yields $y^2 = 1 + y - 2 + y$. Observing (4), we get $y = 1$. This proves the lemma. □

Corollary 3.13. *A group G satisfying $[x, y^{\pm 1}, y^{\pm 1}, x^{-1}] = [x, y]$ is abelian.*

Proof. The group law $[x,y^{-1},y^{-1},x^{-1}]=[x,y]$ can be rewritten in $\mathcal{N}_a$ as

$$y^{-2}-y^{-1}+y^{-1}x^{-1}-y^{-2}x^{-1}=1. \tag{1}$$

Replacing y by y^{-1} in (1), we get $-yx^{-1}+x^{-1}=y^{-1}-y+1$. By Corollary 3.7, this yields

$$y^2=y-2+y+1. \tag{2}$$

Replacing (x,y) by (xy^{-1},y) in (1), we get $y^{-3}-y^{-2}+y^{-1}x^{-1}-y^{-2}x^{-1}=y^{-1}$. Substituting this into (1), we obtain $y^2-y+y^3=y-1+y^2$. Replacing y by y^{-1} in this equation and using (2), we get

$$y^2-1=-y+y^2. \tag{3}$$

Replacing y by y^{-1} in (3) gives $y^2=1+y-1$ or, using (3), $y^2=-y+1+y$. If we compare both the equations, we get $y+1+y=1+y+1$; the commutative law follows from Lemma 3.12 and Lemma 2.1.

The law $[x,y,y,x^{-1}]=[x,y]$ has in $\mathcal{N}_a$ the corresponding equation $-x^{-1}+yx^{-1}=-1+y+1$. As before, this leads to $y^2=y-2+y+1$. Also, by Lemma 2.14, G satisfies the law $[x,y,y,y^{-1}]=[y,x,y^{-1}]$. This may be rewritten in $\mathcal{N}_a$ as $y^2=y-1+y+1-y$. The equation $1+y+1=y+1+y$ is now easily obtained, so G is abelian by Lemma 3.12 and Lemma 2.1. □

Corollary 3.14. *A group G satisfying $[y^{-1},x,x^{-1},y^{-1}]=[x,y]$ is abelian.*

Proof. The law $[y^{-1},x,x^{-1},y^{-1}]=[x,y]$ can be rewritten in $\mathcal{N}_a$ as

$$-y^{-1}x^{-1}+y^{-1}-y^{-2}+x^{-1}y^{-2}=1. \tag{1}$$

Replacing (x,y) by (x^{-1},y^{-1}) in (1) and using (1), multiplied by y on the right, lead to $x^{-1}-x^{-1}y-x^{-1}=1+y^{-1}-1$ or, equivalently, $x^{-1}y^{-1}-x^{-1}-x^{-1}y^{-1}=y^{-1}+y^{-2}-y^{-1}$. Substituting x by $y^{-1}x$ in the preceding equation and then multiplying it by y^{-1} on the right lead to

$$y^2+y-y^2=y+1-y. \tag{2}$$

Replacing x by x^{-1} in (2) and multiplying by x from the right yield

$$-y^2+y+y^2=-2y+1+2y. \tag{3}$$

Substituting y^{-1} for y in (2), we obtain

$$y^2=y-1+y+1-y. \tag{4}$$

Comparing (2) and (4) gives $y^3=y^2+1-y$. Changing x to a in this equation yields $y=1-y-1+y+(-y^2+y+y^2)$, since $-y^3+y^4=y-y^2$ and $-y^3+y^2=y-1$. Observing (3), we get $y+1+y=1+y+1$. In addition, if we use (4), we get $y^2=y-2+y+1$. Therefore G is an abelian group by Lemma 3.12 and 2.1. □

Proposition 3.15. *A group G satisfying $[y,x,x,y^{-1}]=[x,y]$ is abelian.*

Proof. The law $[y,x,x,y^{-1}]=[x,y]$ can be rewritten in $\mathcal{N}_a$ as

$$x-1+y^{-1}-xy^{-1}=1. \tag{1}$$

But then

$$x^2=1+x-2+x \tag{2}$$

also holds. Changing y to y^{-1} in (1) gives $-y^{-1}+y^{-1}x-y^{-1}=-2+x+1$. Further replacement of y by xy in the preeceding equation and then comparing give

$$x^2=2x-1+x-1. \tag{3}$$

Replacing x by x^{-1} in (3), we get $x^2=2x-1-x^2+x$. Using (3) yields

$$x^2=x+1-x. \tag{4}$$

Comparing (3) and (4), we obtain

$$x-1+x=1-x+1. \tag{5}$$

Using (2) in (5), we get $x^2=1+x-1-x+1-x+1$ or, considering (4),

$$-1+x+1-x-1+x=x-1-x+1. \tag{6}$$

Replacing (x,y) by (a,x) in (5) gives, after some straightforward calculation, the equation $-1+x+1-x-1+x=x-1-x+1+x-1$. Comparing this with (6), we get $x=1$, hence G is 2-Engel and thus abelian. □

Proposition 3.16. *A group G satisfying $[x^{-1},y^{-1},x^{-1},y^{-1}]=[x,y]$ is abelian.*

Proof. The law $[x^{-1},y^{-1},x^{-1},y^{-1}]=[x,y]$ can be rewritten in $\mathcal{N}_a$ as

$$-y^{-1}x^{-2}+x^{-1}y^{-1}-x^{-1}y^{-2}+y^{-1}x^{-2}y^{-1}=1. \tag{1}$$

Replacing (x,y) by (x^{-1},xy) in (1) and multiplying by xy on the right lead to

$$x+y^{-1}x^{-1}=1-y+y^{-1}. \tag{2}$$

Replacing x by yx in (2) and using (2) lead to $1+y^{-1}x^{-2}=y+x^{-1}y^{-2}x^{-1}$ or, equivalently, $-x^{-1}y^{-2}+y^{-1}x^{-2}y^{-1}=1-x$. Substituting this into (1), we get $-y^{-1}x+y^{-1}=x^2-x^3-x^2$. Thus G is a 3-Engel group by Theorem 3.5, and hence abelian. □

The proof for the next law is similar:

Proposition 3.17. *A group satisfying the law $[y^{-1},x^{-1},y^{-1},x^{-1}]=[x,y]$ is abelian.*

Proof. This law has in $\mathcal{N}_a$ the equation

$$x^{-1}y^{-2} - x^{-1}y^{-1} + y^{-1}x^{-2} - x^{-1}y^{-2}x^{-1} = 1. \tag{1}$$

Replacing (x,y) by (yx, y^{-1}), we get

$$y^{-1}x^{-1} + y = x^{-1} + x + 1. \tag{2}$$

If we replace (x,y) by (x,xy) in (2) and then observe (2), we obtain $x^{-1}y^{-2} + 1 = y^{-1}x^{-2}y^{-1} + x$. This is equivalent to $y^{-1}x^{-2} - x^{-1}y^{-2}x^{-1} = -1 + y$ (to see this replace (x,y) by (y,x)). Using (1), we get $x^{-1}y - x^{-1} = -y^2 + y^3 - y^2$. After interchanging x and y and inversion, the commutative law easily follows from Theorem 3.5 and Lemma 2.2. □

Proposition 3.18. *A group G satisfying $[x, y^{-1}, x^{-1}, y^{-1}] = [x,y]$ or $[x^{-1}, y, y^{-1}, x^{-1}] = [x,y]$ is abelian.*

Proof. We rewrite the first law in $\mathcal{N}_a$ as

$$y^{-1}x^{-1} - y^{-1} + y^{-2} - x^{-1}y^{-2} = 1. \tag{1}$$

Replacing x by x^{-1} in (1), we get $-y^{-2} + x^{-1}y^{-2} - x^{-1}y^{-1} + y^{-1} = x^{-1}$. Similarly, if we replace y by $x^{-1}y^{-1}$ in (1), we obtain $-1 + x - y^{-1} + y^{-1}x^{-1} = x^{-1}y^{-2}$. Comparing both equations lead to

$$y^{-2} = -1 + x - x^{-1}. \tag{2}$$

Replacing y by xy in (2) and observing (2) give $y^{-2} = y^{-1}x^{-1}y^{-1}x^{-1}$. Replacing (x,y) by (xy^{-1}, x) in this equation, we get, after the multiplication by xy^{-1} from the right, $x^{-1}y^{-2} = x^{-3}$. Observing this and (2) together with (1) yields $y^{-1}x^{-1} - y^{-1} = 1 + x^{-3} + x^{-1} - x + 1$. This concludes the proof, since G is a 3-Engel group by Theorem 3.5.

By interchanging x and y in the preceding argument, we obtain that the remaining law also implies the commutative law. □

Proposition 3.19. *A group G satisfying $[x^{-1}, y, y, x^{-1}] = [x,y]$ is abelian.*

Proof. In $\mathcal{N}_a$, the law $[x^{-1}, y, y, x^{-1}] = [x,y]$ can be rewritten as

$$x^{-1}y - x^{-1} + x^{-2} - x^{-1}yx^{-1} = 1. \tag{1}$$

Replacing y by y^{-1} in (1) gives $-x^{-1}y^{-1} + x^{-1} - 1 + x^{-1}y^{-2}xy = -1 + x + 1$. Since $x^{-1}y^{-2}xy = -1 + y^{-1} + 1$ always holds, we get

$$-x^{-1}y^{-1} + x^{-1} - 2 + y^{-1} = -1 + x \tag{2}$$

or, equivalently, $y-1+y^{-1}x^{-1}=x^{-1}-2+y^{-1}$. If we replace (x,y) by (y^{-1},x^{-1}) in (1) and observe (1), we get $x+x^{-1}y=x^{-2}yx+y^{-1}$. Replacing y by xy in this equation leads to $x+y=1+y-1+y^{-1}x^{-1}$. Using the equation obtained from (2), we get $y-y^{-1}=-x+1+x^{-1}-2$. Replacing y by $x^{-1}y$ in this equation, we get $x^{-1}y-y^{-1}x=-x+1+x^{-1}-2$. Comparing both the equations leads to $-y^{-1}xy^{-1}+y^{-2}=-x^{-1}+1$. This is equivalent to $x^{-2}-x^{-1}yx^{-1}=1-y^{-1}$. Using this in (1), we obtain $x^{-1}-x^{-1}y=1-y^{-1}-1$. Interchanging x and y leads to $-y^{-1}+y^{-1}x=-1+x^{-1}+1$ which implies that the group is abelian by Theorem 3.5 and Lemma 2.2. □

Proposition 3.20. *A group G satisfying $[x,y^{-1},x,y^{-1}]=[x,y]$ is abelian.*

Proof. This law can be rewrtitten in $\mathcal{N}_a$ as

$$y^{-1}x-y^{-1}+y^{-2}-y^{-1}xy^{-1}=1. \tag{1}$$

Replacing y by $y^{-1}x$, we obtain $[x,y,x,x^{-1}]=[x,y]^{y^{-2}-y^{-1}xy^{-1}}$. Rewriting this in $\mathcal{N}_a$, we get $-x+1-x^{-1}=y^{-2}-y^{-1}xy^{-1}-1$. Using (1) leads to $y^{-1}-y^{-1}x=-x+1-x^{-1}$, so G is 3-Engel by Theorem 3.5 and is thus abelian. □

A similar approach can be used to show that the next group law implies the commutative law.

Proposition 3.21. *A group G satisfying $[y,x^{-1},y^{-1},x^{-1}]=[x,y]$ is abelian.*

Proof. This law can be rewritten in $\mathcal{N}_a$ as $-x^{-1}y^{-1}+x^{-1}-x^{-2}+y^{-1}x^{-2}=1$. Replacing y by y^{-1} in this equation, we get

$$x^{-2}-y^{-1}x^{-2}=y^{-1}+x^{-1}-y^{-1}x^{-1}. \tag{1}$$

Clearly $[y,x,y^{-1},x]=[x^{-1},y]$ also holds. If we replace x by yx, we obtain $[y,x,y^{-1},y]=[x,y]^{x^{-2}-y^{-1}x^{-2}}$. Rewriting this in $\mathcal{N}_a$, we get $-y^{-1}+x^{-2}-y^{-1}x^{-2}=-1+y-1$. Using (1) and interchanging x and y lead to $-y^{-1}+x^{-1}y^{-1}=1-x+1$, hence G is abelian by Theorem 3.5 and Lemma 2.2. □

Proposition 3.22. *A group G satisfying $[y,x^{-1},y,x^{-1}]=[x,y]$ is abelian.*

Proof. This law can be rewritten in $\mathcal{N}_a$ as

$$-x^{-1}y+x^{-1}-x^{-2}+x^{-1}yx^{-1}=1. \tag{1}$$

Replacing x by x^{-1} in (1), we get $x^{-1}+y-1+x=yx$. If we observe that (1) is equivalent to $x^{-1}+y-1+x=x^{-1}y$, we obtain $x^{-1}y=yx$ or, equivalently, $y=x^{-1}yx^{-1}$. Replacing y by $x^{-1}y$ in this equation, we get $x^{-2}=y^{-1}+1-y^{-1}$. Applying these two equations to (1), we get $-x^{-1}y+x^{-1}=1-y+y^{-1}+1-y^{-1}$; similarly as in the last proposition, the commutative law follows from Theorem 3.5 and Lemma 2.2. □

Proposition 3.23. *A group G satisfying $[y^{-1}, x, x, y^{-1}] = [x, y]$ is abelian.*

Proof. This law can be rewritten in $\mathcal{N}_a$ as

$$-y^{-1}x + y^{-1} - y^{-2} + y^{-1}xy^{-1} = 1. \tag{1}$$

Furthermore, G satisfies the law $[y, x, x, x]^y = [x, y^{-1}, x^{-1}]$, by Lemma 2.14. Replacing y by xy in this equation, we get $[y, x, x, x]^x = [y^{-1}, x^{-1}, x^{-1}]^{y^{-1}}$ which can be rewritten in $\mathcal{N}_a$ as $x^2 - x + x^2 - x^3 = x^{-1}y^{-2} - y^{-1}x^{-2}y^{-1}$. Replacing x by x^{-1} in this equation gives $-y^{-2} + y^{-1}xy^{-1} = -x^{-3} + x^{-2} - x^{-3} + x^{-4}$. Observing (1), we get $-y^{-1} + y^{-1}x = -x^{-3} + x^{-2} - x^{-3} + x^{-4} - 1$. Theorem 3.5 and Lemma 2.2 imply that G is abelian. □

Proposition 3.24. *A group G satisfying $[y^{-1}, x^{-1}, x^{-1}, y^{-1}] = [x, y]$ is abelian.*

Proof. The law $[y^{-1}, x^{-1}, x^{-1}, y^{-1}] = [x, y]$ has in $\mathcal{N}_a$ the equation

$$y^{-1}x^{-2} - x^{-1}y^{-1} + x^{-1}y^{-2} - y^{-1}x^{-2}y^{-1} = 1. \tag{1}$$

Using Lemma 2.14, we observe that G also satisfies the law $[y, x^{-1}, x^{-1}, x]^y = [x, y^{-1}, x^{-1}]$. Replacing y by xy in this equation, we get $[x, y, x^{-1}, x]^y = [y^{-1}, x^{-1}, x^{-1}]$. This can be rewriten in $\mathcal{N}_a$ as $-x^{-1} + 1 - x = x^{-1}y^{-2} - y^{-1}x^{-2}y^{-1} - 1$. Using (1), we obtain $y^{-1}x - y^{-1} = -x + x^2 - x^3$, hence G is a 3-Engel group and is therefore abelian. □

Proposition 3.25. *A group G satisfying $[y, x^{-1}, x, x^{-1}] = [x, y]$ or $[y^{-1}, x, y, y^{-1}] = [x, y]$ is abelian.*

Proof. The law $[y, x^{-1}, x, x^{-1}] = [x, y]$ can be rewritten in $\mathcal{N}_a$ as $x^{-1} - x^{-2} + x^{-1} = 2$. Replacing (x, y) by (x^{-1}, y) and (x^{-1}, y^{-1}) in (1), we obtain $x - x^2 + x = 2$ and $-x^2 + 2x = 2$. Comparing both the equations, we get $x + 1 = 1 + x$ which implies that G is metabelian by Lemma 2.6, and is thus abelian by Lemma 2.1. The law $[y^{-1}, x, y, y^{-1}] = [x, y]$ is equivalent to $[y, x^{-1}, x, x^{-1}] = [x, y]$; to see this replace (x, y) by (y, x). □

Proposition 3.26. *A group G satisfying $[y, x^{\pm 1}, x^{\mp 1}, x] = [x, y]$ or $[y, x, y, y^{-1}] = [x, y]$ is abelian.*

Proof. Rewriting the first two laws in $\mathcal{N}_a$, we get

$$x^{-1} - 1 + x = 2. \tag{1}$$

Replacing (x, y) by (x, a) in (1) and using (1) yield $1 - x^2 + x = -x + 1 - x + 1$. By multiplying this equation by x^{-1} from the right and using (1), we obtain $1 + x^{-1} = x^{-1} + 1$. This implies that G is metabelian by Lemma 2.6, and hence is abelian. The law $[y, x, y, y^{-1}] = [x, y]$ is equivalent to $[y, x, x^{-1}, x] = [x, y]$ by Lemma 2.9. □

Proposition 3.27. *A group G satisfying $[y^{-1}, x, y^{-1}, y^{-1}] = [x, y]$ is abelian.*

Proof. The law $[y^{-1}, x, y^{-1}, y^{-1}] = [x, y]$ can be rewritten in $\mathcal{N}_a$ as

$$y^3 = -y + y^2 - y + 1. \tag{1}$$

Replacing y by y^{-1} in (1) gives

$$y^3 = 1 + y^2 - y + y^2. \tag{2}$$

Substituting a for x in (1) and using (2) yield $y^4 = 1 + y^3 - y^2 + y - 1 + y$. Using $y^4 = -y^2 + y^3 - y^2 + y$ and $y^3 - y^2 = 1 + y^2 - y$ gives

$$2 + y^2 - 1 = -y^2 + 1 + y^2 - y. \tag{3}$$

Replacing (x, y) by (a, y) in (2) and using (1) yield $y^3 - y^2 + y = -1 + y^2 - y + 1$. Using (2) leads to $2 + y^2 - 1 = y^2 - y$. Observing (3) gives $y^2 = 1$. Substituting this into (3), we obtain $y = -1$ which implies that G is metabelian by Lemma 2.6, and hence is abelian. □

Proposition 3.28. *A group G satisfying $[y^{\pm 1}, x, y^{\pm 1}, y] = [x, y]$ or $[y, x, x^{-1}, x^{-1}] = [x, y]$ is abelian.*

Proof. The law $[y^{-1}, x, y^{-1}, y] = [x, y]$ can be rewritten in $\mathcal{N}_a$ as

$$-1 + y - y^2 + y = y^2. \tag{1}$$

Replacing y by y^{-1} in (1), we get

$$y - 1 + y - y^2 = 1. \tag{2}$$

This is the corresponding equation for the law $[y, x, y, y] = [x, y]$. Replacing (x, y) by (a, y) in (1) and observing (2), we get $-y + 1 - y + y^2 - y^3 + y = -y^2$. Multiplying (2) by y from the right, we obtain $-y + y^2 - y^3 = -y^2 + y$. The last two equations imply $-y^2 + 2y = -1 + y - y^2$. Using (2), we obtain $-y + y^2 = y - 1 + y$. Observing (2) once again, we get $y = -1$. Lemma 2.6 implies that G is metabelian, and hence is abelian by Lemma 2.1. The remaining law is equivalent to $[y, x, y, y] = [x, y]$ by Lemma 2.9. □

Proposition 3.29. *Let G be a group satisfying $[y^{\pm 1}, x, y^{\mp 1}, y] = [x, y]$ or $[y, x, x, x^{-1}] = [x, y]$. Then G is an abelian group.*

Proof. The first two laws can be rewritten in $\mathcal{N}_a$ as

$$y^{-1} = 2 - y + 1. \tag{1}$$

Replacing (x, y) by (y, ya) in (1) and using (1) and $y^2 = y - 1 + 2y$, we obtain

$$1 - y - 1 + y - 1 - y = -1 - y + 1 - y - 1 + y. \tag{2}$$

Replacing (x, y) by (y, a) in (1) leads to $1 - y - 1 + y - 1 - y = -y + 1 - y - 1$ which is equivalent to $-y - 1 + y - 1 = -1 - y + 1 - y - 1 + y$. Using these two equations in (2) gives $2y = 2$. Substituting this into (1), we get $y^2 + y = 1$, which is equivalent to $x + x^2 = 1$; this is the corresponding equation for the law $[y, x^{-1}, x^{-1}] = [x, y]$, so G is an abelian group by Lemma 2.1. The law $[y, x, x, x^{-1}] = [x, y]$ is equivalent to the first two laws by Lemma 2.9. □

Proposition 3.30. *A group G satisfying $[x^{-1}, y, x^{-1}, x^{-1}] = [x, y]$ is abelian.*

Proof. This group law can be rewritten in $\mathcal{N}_a$ as

$$x^3 = x - x^2 + x - 1. \tag{1}$$

Replacing x by x^{-1} in (1) leads to $x^3 = x^2 - x + x^2 - 1$. Using (1), we get

$$1 - x + 1 = x - 1 + x. \tag{2}$$

Replacing y by a in (2) and observing (1) give $3 = 0$. If we replace x by x^2 in this equation, we get

$$x + 1 + x + 1 + x + 1 = 0. \tag{3}$$

Replacing (x, y) by (a, x) in (1) and using (2) lead to $x + 1 + x + 1 + x + 1 - x + 1 = 0$. Finally, by observing (3), we get $x = 1$. It follows that G is a 2-Engel group and hence is abelian by Lemma 2.1. □

There is only one more case left to prove Theorem 1.1, namely the law $[y, x^{-1}, x^{-1}, x^{-1}] = [x, y]$. This one seems to be the most difficult case to handle.

Proposition 3.31. *A group G satisfying $[y, x^{-1}, x^{-1}, x^{-1}] = [x, y]$ is abelian.*

Proof. The law $[y, x^{-1}, x^{-1}, x^{-1}] = [x, y]$ can be rewritten in $\mathcal{N}_a$ as

$$x^3 = -x + x^2 - x + 1. \tag{1}$$

Replacing (x, y) by (x^{-1}, y) in (1) gives

$$x^3 = 1 + x^2 - x + x^2. \tag{2}$$

Combining (1) and (2) yields

$$x^4 = -x^2 + 1 + x^2. \tag{3}$$

Replacing y by a in (2), we get $-x^4+x^3=(-x+1-x^3)+(x^2-x+x^2-x^3)+x^2$. Using (1) and (2), we obtain $x^3=x^4-x^2+x-1+x^2$ or, by observing (3),

$$x^3=-x^2+1+x-1+x^2. \tag{4}$$

Replacing x by x^{-1} in (4) leads to

$$x-x^3+x^2+x^3-x=1. \tag{5}$$

If we replace y by a in (1), we get $-x^4+x^3=-x+x^2-x^3+(x^2-x+x^2)-x+1$. Observing (2) together with (5) yields $x^3=x^4-x+x^2-x^3+(-1+x^2)+x^3-x$. Using (3), we get $x^2=(x^3-1+x-x^2+x)-x^3+x^2-1$. Considering (1), this yields

$$x^3=x^2-1-x^2. \tag{6}$$

Replacing x by x^{-1} in (6) and observing (6) lead to

$$x^2+1-x^2=x+1-x. \tag{7}$$

Substituting x^{-1} for x in (7) gives

$$-1+x^2+1=-x+x^2+x. \tag{8}$$

If we replace y by a in (7), we obtain $x^2-x+(1-x^2+x^3)=(x^3-x^2+1-x)+x^2$. This, together with (4) and (6), yields $2x^2-x=1+2x^2$. Replacing x by x^{-1} in this equation gives

$$x^2=-2-x+2. \tag{9}$$

Replacing (x,y) by (a,x) in (1) and using (9) and (8) yield $x^2=-1+x-1-x+1$. Finally, multiplying this equation by x^2 from the right and observing (3) and (2) give $x^2+1=x$. This is the corresponding equation for the law $[x^{-1},y,x^{-1}]=[x,y]$, so G is abelian by Lemma 2.1. □

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