On the autocommutator subgroup and absolute center of a group

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Abstract. We show that if the quotient of a group by its absolute center is locally finite of exponent n, then the exponent of its autocommutator subgroup is n-bounded, that is, bounded by a function depending only on n. If the group itself is locally finite, then its exponent is n-bounded as well. Under some extra assumptions, the exponent of its automorphism group is n-bounded. We determine the absolute center and autocommutator subgroup for a large class of (infinite) abelian groups.

1. Introduction

A classical result of Schur states that if the central quotient $G/Z(G)$ of a group G is finite, then the commutator subgroup G' is finite as well, see [15, (10.1.4)]. Mann [11] generalised this and proved that G' is locally finite of n-bounded exponent if $G/Z(G)$ is locally finite of exponent n. In 1994, Hegarty [8] introduced the *autocommutator subgroup*

$$G'^{\star} = \langle g^{-1}g^{\alpha} \mid g \in G, \alpha \in \mathrm{Aut}(G)\rangle$$

of G and its *absolute center*

$$Z^{\star}(G) = \{g \in G \mid g^{\alpha} = g \text{ for all } \alpha \in \mathrm{Aut}(G)\},$$

where he used the notation $L(G) = Z^{\star}(G)$ and $G^{\star} = G'^{\star}$. Hegarty proved that if $G/Z^{\star}(G)$ is finite, then also $G'^{\star}$ and the automorphism group $\mathrm{Aut}(G)$ are finite.

One of the goals of this paper is to extend Mann's result to G-groups as considered by Ellis [4]. A group M is a *G-group* if G is a group acting on M via a homomorphism $\varphi\colon G \to \mathrm{Aut}(M)$ with image containing the inner automorphisms $\mathrm{Inn}(M)$. Accordingly, the *G-center* of M is defined as

$$Z_G(M) = \{m \in M \mid m^g = m \text{ for all } g \in G\}$$

and its *G-commutator subgroup* is

$$[M, G] = \langle m^{-1}m^g \mid m \in M, g \in G\rangle.$$

We prove that if $M/Z_G(M)$ is locally finite of exponent n, then $[M, G]$ is locally finite of n-bounded exponent. If $\mathrm{Aut}(M/Z_G(M))$ is nilpotent of class c and G acts faithfully, then this allows us to prove that the exponent of G is bounded in terms of n and c. In the special case when M is locally finite and $G = \mathrm{Aut}(M)$, we show that the exponent of M is n-bounded.

Further results on the exponents of G, $[M, G]$, and $\mathrm{Aut}(M)$ are summarised in Section 2. As an application, we consider the relative Schur multiplier as defined by Loday [10]. We show that if G is a group with locally finite normal subgroup N of exponent n, then the exponent of the relative Schur multiplier of the pair (G, N) is n-bounded. This improves a recent result of [13], where the exponent is $\exp(G)$-bounded.

Having established a connection between the exponents of $M/Z_G(M)$, $[M, G]$, and M, respectively, it is of interest to investigate the structure of $Z_G(M)$ and $[M, G]$. We consider the special case of $G = \mathrm{Aut}(M)$ and M abelian. A recent result of Chiş et al. [2] proves that every finite abelian group M occurs as the autocommutator subgroup

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of a group. Here we generalise this result to a certain class of infinite abelian groups, see Section 3.1. In Section 3.2 we prove that $Z^\star(M)$ has at most two elements if $M = D \times R$ is abelian where D is divisible and R is reduced having no 2-elements of infinite height. We note that in this paper a p-element means a *non-trivial* element of p-power order.

In the last section we define the upper G-central series of a G-group M, and prove a partial converse of a stability result of Hall [7].

2. Exponents

Let M be a G-group. In this section we consider some consequences of the assumption that $M/Z_G(M)$ has finite exponent. Our starting point is a recent result of Mann [11] who proved that if $G/Z(G)$ is locally finite of exponent n, then G' is locally finite of n-bounded exponent. His proof can be generalised as follows.

Theorem 2.1. *Suppose that M is a G-group. If $M/Z_G(M)$ is locally finite of exponent n, then $[M, G]$ is locally finite of n-bounded exponent.*

Proof. Let $x, y \in M$ and put $H = \langle x, y\rangle$. The group $HZ_G(M)/Z_G(M)$ is a finite 2-generator group of exponent dividing n. Its order k divides the order $b(n)$ of the largest finite 2-generator group of exponent n, which exists by the solution of the restricted Burnside problem [16]. Since $H \cap Z_G(M) \leq Z(H)$, the transfer map from H to $H \cap Z_G(M)$ is given by $m \mapsto m^k$. Hence, $m \mapsto m^{b(n)}$ is a homomorphism on M, and $(m^{-1}m^g)^{b(n)} = 1$ for all $m \in M$ and $g \in G$, that is, the exponent of $[M, G]$ divides $b(n)$. Since $[M, G]/([M, G] \cap Z_G(M))$ is locally finite and $[M, G] \cap Z_G(M)$ is abelian of finite exponent, $[M, G]$ is locally finite as well. □

The bound for $\exp[M, G]$ obtained in the proof of Theorem 2.1 is very crude. In some cases, better bounds can be found.

Proposition 2.2. *Let M be a G-group with $M/Z_G(M)$ of exponent n.*

a) *If M is a finite p-group of class less than p, then* $\exp[M, G]$ *divides n.*

b) *If $n \in \{2, 3, 4, 6\}$, then* $\exp[M, G]$ *divides n^2.*

Proof. a) Since M has class less than p, it is an immediate consequence of [9, Satz III.10.2] that M is regular, that is, for every $u, v \in M$ and every integer $i \geq 1$ there exists $w \in [M, M]$ with $(uv)^{p^i} = u^{p^i}v^{p^i}w^{p^i}$, see also [9, Satz III.10.8g)]. In addition, this property implies that $[M, M]^n = [M^n, M]$ as proved in [9, Satz III.10.8c)]. Since $M^n \leq Z_G(M)$, we have $[M, M]^n = 1$, thus $m \mapsto m^n$ is an endomorphism of M. Hence, if $g \in G$ and $m \in M$, then $(m^{-1}m^g)^n = (m^n)^{-1}(m^n)^g = 1$ as $m^n \in Z_G(M)$.

b) Clearly, the exponent of $M/Z(M)$ divides n, and a group M with this property is called n-central. It is proved in [12] that if M is an n-central group with $n \in \{2, 3, 4, 6\}$, then $m \mapsto m^{n^2}$ is an endomorphism of M, see [12, Theorems A, 1.1, and 1.2] for the precise reference. This implies that $(m^{-1}m^g)^{n^2} = 1$ for all $m \in M$ and $g \in G$. □

The next result shows that, under certain assumptions, there is a finite quotient A of M with induced G-action such that $\exp[M, G] = \exp[A, G]$.

Proposition 2.3. *Let M be a finitely generated G-group with $M/Z_G(M)$ finite of exponent n. Then M embeds into a direct product $A \times B$, where A and B are G-groups, A is finite and $A/Z_G(A)$ has exponent dividing n, and B is torsion-free abelian of finite rank. Moreover, G acts trivially on B and* $\exp[M, G] = \exp[A, G]$.

Proof. We only have to consider infinite M. In this case, $Z_G(M)$ is infinite and, as a subgroup of finite index in a finitely generated group, $Z_G(M)$ is finitely generated as well, see for example [15, (1.6.11)]. By assumption, $M/Z(M)$ is finite and so is M' by Schur's theorem. This implies that the elements in M of finite order form a characteristic subgroup T. By Zorn's lemma we can choose a maximal torsion-free subgroup S of $Z_G(M)$. Since $M/Z_G(M)$ is finite and $Z_G(M)/S$ is a finitely generated abelian torsion group, the quotient $A = M/S$ is finite with induced G-action. Clearly, $\exp A/Z_G(A)$ divides n. Since M' is finite, $M' \leq T$, and $B = M/T$ is torsion-free abelian. It follows from Theorem 2.1 that $[M,G]$ has finite exponent, that is, $[M,G] \leq T$, and G acts trivially on B. Finally, since $S \cap T = 1$, there is an embedding of M into $A \times B$. Hence, $[M,G]$ embeds into $[A,G]$. Since A is a quotient of M, the assertion follows. □

We now consider the exponent of G where M is a G-group and G acts faithfully. Note that $\exp G$ may not be finite even though M has finite exponent. For example, consider the direct product $M = \prod_{z\in\mathbb{Z}} C_n$ of cyclic groups of order n and $G = \mathrm{Aut}(M)$; we note that all direct products in this paper are restricted direct products. This shows that some extra assumptions are required to bound the exponent of G. Let $\mathrm{Var}_G(M) = \{g \in G \mid m^{-1}m^g \in Z_G(M) \text{ for all } m \in M\}$, cf. [8].

Proposition 2.4. *Let M be a G-group with G acting faithfully and $M/Z_G(M)$ locally finite of exponent n. Then the following hold:*

a) *The exponent of* $\mathrm{Var}_G(M)$ *is n-bounded.*

b) *If* $\mathrm{Aut}(M/Z_G(M))$ *is nilpotent of class c, then* $\exp G$ *is (c,n)-bounded.*

Proof. a) By Theorem 2.1, the exponent of $[M,G]$ is bounded by $k = k(n)$. If $g \in \mathrm{Var}_G(M)$ and $m \in M$, then, by induction, $m^{g^i} = m^g(m^{-1}m^g)^{i-1}$. Now the assertion follows from $m^{g^{k+1}} = m^g$ for all $m \in M$.

b) Note that $\mathrm{Var}_G(M)$ is the kernel of the induced G-action on $M/Z_G(M)$, hence the exponent of G does not exceed $\exp \mathrm{Var}_G(M) \cdot \exp \mathrm{Aut}(M/Z_G(M))$. By a), the assertion follows if we prove that the exponent of $\mathrm{Aut}(M/Z_G(M))$ is (c,n)-bounded. Denote $N = M/Z_G(M)$ and $A = \{\alpha \in \mathrm{Aut}(N) \mid x^{-1}x^\alpha \in Z(N)\}$. From the proof of [14, Theorem 4.1] we conclude that $Z(A)/C_{Z(A)}([N,A])$ is isomorphic to a subgroup of power automorphisms of $[N,A]$. If θ is a such a power automorphism and $x \in [N,A]$, then $x^\theta = x^{t_x}$ for some integer t_x coprime to the order of x. Hence, $x^{\theta^{\phi(|x|)}} = x$, where ϕ is Euler's totient function. The exponent of $[N,A]$ divides n and, hence, $\phi(|x|)$ is a divisor of $\phi(n)$. This shows that $\theta^{\phi(n)} = 1$, and the exponent of $Z(A)/C_{Z(A)}([N,A])$ divides $\phi(n)$. It follows directly from the proof of [14, Theorem 4.1] that $\exp C_{Z(A)}([N,A])$ is n-bounded. We conclude that $\exp Z(A)$ is n-bounded, thus $\exp Z(\mathrm{Aut}(N))$ is n-bounded. Since $\mathrm{Aut}(N)$ is nilpotent of class c, the exponent of $\mathrm{Aut}(N)$ divides $(\exp Z(A))^c$, see [15, (5.2.22)], thus is (c,n)-bounded. This completes the proof. □

In the remainder of this section we consider the special case when $G = \mathrm{Aut}(M)$. We start with a preliminary lemma.

Lemma 2.5. *Let M be a locally finite group such that $M/Z^\star(M)$ is a π-group. If M contains a unique involution, then M is a $(\pi \cup \{2\})$-group, and a π-group otherwise.*

Proof. Assume that $Z^\star(M)$ contains a p-element, where $p \notin \pi$. Then every p-element of M lies in $Z^\star(M)$, and the p-component S of $Z^\star(M)$ is characteristic in M. Since $M = C_M(S)$ and S is a normal maximal p-subgroup of M, the groups M and S satisfy the conditions of [3, Theorem 2], which shows that S is a direct factor of M. Thus, every automorphism of S lifts to an automorphism of M, and $S \leq Z^\star(M)$ implies $\mathrm{Aut}(S) = 1$. Since $s \mapsto s^{-1}$ is an automorphism of S, we have $\exp S = 2$. But every abelian group of finite exponent is a direct product of cyclic groups, see [5, Theorem 17.2], which implies that $S = C_2$. Thus, $p = 2$ and M has a unique involution. □

Theorem 2.6. *Let M be a locally finite group. If $M/Z^\star(M)$ has exponent n, then $\exp M$ is n-bounded.*

Proof. Let π be the set of prime divisors of n, and let $t = t(n)$ be their product. From Lemma 2.5 it follows that M is a $(\pi \cup \{2\})$-group. The exponent of $M/Z(M)$ divides n, and, using the notation of the proof of Theorem 2.1, the map $M \to Z(M)$, $m \mapsto m^{b(n)}$, is an endomorphism of M. By the solution of the Restricted Burnside problem, $b = b(n)$ is a π-number and n divides b, see [16]. Clearly, the map $m \mapsto m^{2bt+1}$ is an endomorphism as well. In fact, it is an automorphism, since $2bt + 1$ is not a π-number. Therefore, $m^{2bt} = 1$ for all $m \in Z^\star(M)$, and the exponent of M divides $2bnt$. □

Corollary 2.7. *Let $M = \prod_i P_i$ be a direct product of p_i-groups P_i for pairwise distinct primes p_i such that no P_i is locally cyclic. Suppose that $\mathrm{Aut}(M)$ is nilpotent of class c. If $\exp M/Z^\star(M) = n$, then the exponent of $\mathrm{Aut}(M)$ is (c, n)-bounded.*

Proof. Since $\mathrm{Aut}(M)$ is nilpotent, the group M is nilpotent as well, which implies that M is locally finite. Theorem 2.6 shows that $\exp M$ is n-bounded. We can assume that $i = 1$, that is, M is a p-group which is not locally cyclic. It follows from $\exp M/Z^\star(M) = n$ that the inner automorphism group of G has finite exponent. Now [6, Theorem 2] can be applied, which proves that $\mathrm{Aut}(M)/Z(\mathrm{Aut}(M))$ has finite exponent, e say. Since $\mathrm{Aut}(M)$ is nilpotent of class c, its exponent divides e^c, see [15, (5.2.22)]. In particular, $\mathrm{Aut}(M)$ has finite exponent and we can apply [6, Proposition 9b)], which proves the assertion. □

2.1. Relative Schur multipliers. As an application of the above results we consider the following. Let G be a group with normal subgroup N. The *relative Schur multiplier* of the pair (G, N) is defined as the relative homology group $\mathrm{H}_2(G, N; \mathbb{Z})$, see Loday [10]. By [4], it can also be described as $\mathrm{M}(G, N) = \ker(N \wedge G \to [N, G])$, where $N \wedge G$ is the non-abelian exterior product as defined in [1]. For finite G, it is proved in [13] that the exponent of $\mathrm{M}(G, N)$ can be bounded in terms of $\exp G$. The following is an improvement of this result.

Proposition 2.8. *Let G be a group and N a locally finite normal subgroup of G. If the exponent of N is n, then the exponent of $\mathrm{M}(G, N)$ is n-bounded.*

Proof. Our proof relies on the notion of a *relative central extension* of the pair (G, N), see [10]. It consists of a group homomorphism $\partial\colon M \to G$ with image equal to N, an action of G upon M such that ∂ is a G-homomorphism, $\ker \partial \leq Z_G(M)$, and $u^{\partial(v)} = u^v$ for all $u, v \in M$. Let $\partial\colon M \to G$ and $\partial'\colon M' \to G$ be relative central extensions of the pair (G, N). A morphism between these is a G-homomorphism $\phi\colon M \to M'$ satisfying $\partial'\phi = \partial$. Thus all relative central extension of the pair (G, N) form a category $\mathcal{RCE}(G, N)$. At first we show that $\mathcal{RCE}(G, N)$ admits a projective object. Let F be the free group on the set $N \times G$ with G-action defined by $(k, g')^g = (k, g'g)$. The map $(k, g) \mapsto k^g$ induces a homomorphism $\bar\partial\colon F \to G$. Now define

$$M = F/\langle x^{\bar\partial(y)} x^{-y},\ a^{-1}a^g \text{ for all } x, y \in F,\ a \in \ker \bar\partial,\ g \in G\rangle^F.$$

It follows that $\bar\partial$ induces a homomorphism $\partial\colon M \to G$, and it is straightforward but technical to verify that the latter is a projective object in $\mathcal{RCE}(G, N)$.

Note that $N \cong M/\ker\partial$, and there is an epimorphism $M/\ker\partial \to M/Z_G(M)$, that is, $\exp M/Z_G(M)$ divides n. Theorem 2.1 shows that $\exp[M,G]$ is n-bounded. It is proved in [4, Theorem 7] that $N \wedge G$ is isomorphic to $[G,M]$. Thus, the group $\mathrm{M}(G,N)$ embeds into $[M,G]$, which proves the assertion. □

3. The absolute center and autocommutator subgroup of an abelian group

The aim of this section is to investigate the absolute center and the autocommutator subgroup of an abelian group. It is proved in [5, Theorem 21.3] that every abelian group G is a direct product $G = D \times R$ where D is the unique maximal divisible subgroup of G and R is *reduced*, that is, R has no non-trivial divisible subgroup. If p is a prime, then the p-component of R is denoted by R_p. The *(p-)height* of an element $g \in R_p$ is p^x if x is maximal with the property that $g = k^{p^x}$ for some $k \in R_p$. If there is no bound on x, then g has *infinite height*. We say a group has no elements of infinite height if every non-trivial element has finite height.

Reduced groups with no elements of infinite height feature nice properties, which is the main reason why we restrict attention to abelian groups $G = D \times R$ where R satisfies certain conditions on the height of some of its elements. We often make use of the following lemma about elements of finite height; it is proved in [5, Corollaries 27.2 & 27.9].

Lemma 3.1. *Let G be an abelian group.*

a) *Let $g \in G$ be of order p and finite height p^k. If $h \in G$ with $h^{p^k} = g$, then $\langle h\rangle$ is a direct factor of G.*
b) *An element $g \in G$ of prime-power order belongs to a finite direct summand of G if and only if the group $\langle g\rangle$ contains no elements of infinite height.*

3.1. The autocommutator subgroup. Chiş et al. [2] proved that every finite abelian group is the autocommutator subgroup of a finite group. We now generalise this result to the class of abelian groups $G = D \times R$ where D is divisible and R is torsion and reduced having no 2-elements of infinite height.

First, we need some more notation. An element $k \in R_p$ is *p-minimal* if it has height 1 and $k^{|k|/p}$ has height $|k|/p$. Note that every element of height 1 and order p is p-minimal. If $g \in R_p$ has order p^m and $g^{p^{m-1}}$ has height p^y, then $g^{p^{m-1}} = k^{p^y}$ for some $k \in R_p$. It is easy to see that k is p-minimal.

For an abelian group A and a positive integer m let A^m denote the subgroup of m-th powers. The next lemma summarises some preliminary results.

Lemma 3.2. *Let $G = D \times R$ be an abelian group such that D is divisible and R is reduced having no 2-elements of infinite height.*

a) *$G'^\star = D \times R'^\star$ and $R^2 \le R'^\star$; in particular, $R_p \le R'^\star$ for all $p > 2$.*
b) *The p-component R_p is generated by the p-minimal elements. If R_p has unbounded exponent, then the order of p-minimal elements is unbounded.*
c) *An element $k \in R$ is 2-minimal if and only if $\langle k\rangle$ is a direct factor of R.*
d) *Let $h, k \in R$ be 2-minimal. If $|k| > |h|$, then $h \in R'^\star$. If $|k| = |h|$ and $\langle h\rangle \cap \langle k\rangle = 1$, then $h, k \in R'^\star$.*

Proof. a) Note that $D \le G$ is characteristic and it follows readily that the restriction homomorphism $\varphi\colon \mathrm{Aut}(G) \to \mathrm{Aut}(D) \times \mathrm{Aut}(R)$ is surjective where we identify R with G/D. This shows that $D'^\star \times R'^\star \le G'^\star$. Inversion is an automorphism of D and, hence, $D^2 \leq D'^{\star}$. But $D^2 = D$ since D is divisible, that is, $D = D'^{\star}$. The kernel of φ is isomorphic to $\mathrm{Hom}(R, D)$ where a homomorphism $\psi\colon R \to D$ corresponds to the automorphism $(d, r) \mapsto (dr^{\psi}, r)$ of $D \times R$. Hence, if $\alpha \in \ker\varphi$ and $r \in R$, then $r^{-1}r^{\alpha} \in D$, which proves that $G'^{\star} = D \times R'^{\star}$. Clearly, $R^2 \leq R'^{\star}$.

b) If $g \in R_p$ has height p^x, then $g = k^{p^x}$ for some $k \in R_p$ of height 1. It remains to show that every element k of height 1 is a product of p-minimal elements. We use induction on the order $|k| = p^m$. If $m = 1$, then k is p-minimal. Now let $m > 1$ and assume that k is not p-minimal and the height of $k^{p^{m-1}}$ is p^y with $y \geq m$. Then $k^{p^{m-1}} = w^{p^y} = (w^{p^{y-m+1}})^{p^{m-1}}$ for some p-minimal $w \in R_p$. We now write $k = w^{p^{y-m+1}}c$ for some $c \in R_p$ with $|c| \leq p^{m-1}$; note that $c \neq 1$ as k has height 1. Since $w^{p^{y-m+1}}$ and k have heights $\geq p$ and 1, respectively, the height of c is 1. By induction, c is a product of p-minimal elements.

c) This follows from Lemma 3.1a).

d) By c), we have $R = \langle h\rangle \times U = \langle k\rangle \times V$, and $k = h^i u$ for some $u \in U$ and $i \in \mathbb{Z}$. First, let $|h| = 2^x$ and $|k| = 2^y$ with $y > x$. Since $k^{2^{y-1}} = u^{2^{y-1}}$, the element u is 2-minimal as well, and $\langle h\rangle \times \langle u\rangle$ is a direct factor of R. The homomorphism defined by $u \mapsto uh$ lifts to an automorphism α of R, and $h = u^{-1}u^{\alpha} \in R'^{\star}$.

Now let $|h| = |k| = 2^x$. If i is even, then $k^{2^{x-1}} = u^{2^{x-1}}$. As above, $\langle h\rangle \times \langle u\rangle$ is a direct factor of R, and $h, u \in R'^{\star}$. Hence, let i be odd and consider $w = h^{-1}k$ with $w^{2^{x-1}} = u^{2^{x-1}} \neq 1$. If $u^{2^{x-1}}$ has height 2^{x-1}, then u is p-minimal, and $\langle h\rangle \times \langle u\rangle$ is a direct factor of R. This implies that $h, u \in R'^{\star}$. If the height of $u^{2^{x-1}}$ is larger than 2^{x-1}, then $u^{2^{x-1}} = v^{2^y}$ for some $y \geq x$ and p-minimal v. Write $v = h^i r$ with $r \in U$ and note that $v^{2^y} = r^{2^y}$, that is, r is p-minimal. Now $\langle h\rangle \times \langle r\rangle$ is a direct factor of R, and $h \in R'^{\star}$. A similar argument yields $k \in R'^{\star}$. □

Theorem 3.3. *Let $G = D \times R$ be an abelian group such that D is divisible and R is reduced and torsion. If R has no 2-elements of infinite height, then $G = K'^{\star}$ for some abelian group K.*

Proof. We construct $K = D \times S$ for some reduced group S having no elements of infinite height such that $R = S'^{\star}$, cf. Lemma 3.2. Denote by $R = \prod_p R_p$ the decomposition of R into its p-components. First, we assume that R_2 has finite exponent 2^m. It is proved in [5, Theorem 17.2] that every abelian group of bounded exponent is a direct product of cyclic groups. Hence, we can decompose $R_2 = \prod_{n \leq m}\prod_{i \in \mathcal{I}_n} C_{2^n}$ for some index sets $\mathcal{I}_n$. If $|\mathcal{I}_m| > 1$, then Lemma 3.2 implies that $R = R'^{\star}$, and we can choose $S = R$, that is, $K = G$. If $|\mathcal{I}_m| = 1$, then define $T = C_{2^{m+1}} \times \prod_{n<m}\prod_{i \in \mathcal{I}_n} C_{2^n}$ and $S = T \times \prod_{p>2} R_p$. We embed C_{2^m} into $C_{2^{m+1}}$ and consider R as a subgroup of S. Then R has index 2 in S, and $S'^{\star} \leq R$: If $\alpha \in \mathrm{Aut}(S)$ and $s \in S \setminus R$, then $s^{\alpha} \notin R$ and $s^{\alpha}R = sR$, that is, $[s, \alpha] \in R$. Now Lemma 3.2 implies that $R = K'^{\star}$ with $K = D \times S$. If R_2 has unbounded exponent, then $R = R'^{\star}$ follows directly from Lemma 3.2, and we can choose $K = G$. □

The following corollary deals with a special case where R is not torsion. We denote by C_∞ the cyclic group of infinite order.

Corollary 3.4. *Let $G = D \times R$ be an abelian group such that D is divisible and R is reduced having no 2-elements of infinite height. Assume that $R = T \times F$ where T is the torsion subgroup of R and $F = \prod_{i \in \mathcal{I}} C_\infty$ for some non-empty index set $\mathcal{I}$. Then*

$$G'^{\star} = \begin{cases} G & \text{if } |\mathcal{I}| > 1, \\ D \times T \times F^2 & \text{if } |\mathcal{I}| = 1. \end{cases}$$

Proof. By Lemma 3.2, we can assume that $D = 1$. Let z be a generator of a direct factor of F and let $r \in T$ be a torsion element. Then $z \mapsto rz$ lifts to an automorphism α of R with $r = [z, \alpha] \in G'^{\star}$, that is, $T \leq G'^{\star}$. If $|\mathcal{I}| > 1$ and $y, z \in F$ are generators of two different factors of F, then $y \mapsto yz$ lifts to an automorphism of G, which implies that $F \leq G'^{\star}$ as well. If $F = C_\infty$, then $T \times F^2$ is a characteristic subgroup of R of index 2. The proof of Theorem 3.3 shows that $G'^{\star} \leq T \times F^2$, which implies the assertion. □

3.2. The absolute center. Since inversion is an automorphism of every abelian group G, the absolute center $Z^{\star}(G)$ has exponent dividing 2. The aim of this section is to prove the following result.

Theorem 3.5. *Let $G = D \times R$ be an abelian group such that D is divisible and R is reduced having no 2-elements of infinite height.*

a) *Suppose that D has 2-elements. If D has a unique involution z, then $Z^{\star}(G) = \langle z \rangle$, and $Z^{\star}(G) = 1$ otherwise.*

b) *Suppose that D has no 2-elements. If R has a unique involution z of maximal height, then $Z^{\star}(G) = \langle z \rangle$, and $Z^{\star}(G) = 1$ otherwise.*

Proof. a) First, we assume that $R = 1$. The structure of divisible groups is determined in [5, Theorem 23.1], and we can assume that $D = \prod_p \prod_{i \in \mathcal{I}_p} C_{p^\infty} \times \prod_{j \in \mathcal{J}} \mathbb{Q}$, where p runs over all primes, $\mathcal{I}_p$ and $\mathcal{J}$ are index sets, and C_{p^∞} is a quasicyclic p-group. Recall that $Z^{\star}(D)$ is a 2-group contained in $H = \prod_{i \in \mathcal{I}_2} C_{2^\infty}$. Every automorphism of H extends to an automorphism of D, and we have $Z^{\star}(D) = Z^{\star}(H)$. Suppose that $|\mathcal{I}_2| > 1$ and choose a well-ordering on $\mathcal{I}_2$ with smallest element i_0. For $i \in \mathcal{I}_2$ let i^+ be its unique successor if exists, and $i^+ = i_0$ otherwise. Let $\varphi_i \colon C_{2^\infty}^{(i)} \to C_{2^\infty}^{(i^+)}$ be an isomorphism between the i-th and i^+-th direct factor of H. We use this notation to define an automorphism α_i of H which maps $g \in C_{2^\infty}^{(i)}$ to $g g^{\varphi_i}$, and fixes all other elements of H. Since α_i lifts to an automorphism of D, we have $Z^{\star}(D) = 1$. Thus, it remains to consider $|\mathcal{I}_2| = 1$, that is, $H = C_{2^\infty}$. Clearly, the map $H \to H$, $h \mapsto h^3$, is an automorphism of H with fixed point set $\{1, z\}$, where $z \in H$ is the unique element in H of order 2.

Now we consider the general case and, first, show that $Z^{\star}(G) \leq D$. Since $D \leq G$ is characteristic, the restriction homomorphism $\mathrm{Aut}(G) \to \mathrm{Aut}(D) \times \mathrm{Aut}(R)$ is surjective and has kernel isomorphic to $\mathrm{Hom}(R, D)$, see the proof of Lemma 3.2a). Since D contains elements of 2-power order, it can be read of the above decomposition of D that D has a direct factor C_{2^∞}. Thus, if R is a $2'$-group, then we can assume that $R = 1$ and the assertion follows from the above argument. Now let $g \in R$ be a 2-element. By our assumptions, g has finite height and Lemma 3.1b) shows that $R = C \times \tilde{R}$ for some cyclic 2-group C containing g. Let α be the automorphism of G defined by a homomorphism $R \to D$ which has kernel $\tilde{R}$ and embeds C into a quasicyclic factor $C_{2^\infty} \leq D$. Then α moves g, which implies that $Z^{\star}(G) \leq Z^{\star}(D)$. Hence, $Z^{\star}(D) = Z^{\star}(G)$, which proves the assertion.

b) Using the structure of $\mathrm{Aut}(G)$ as determined in the proof of Theorem 3.5a), we have $Z^{\star}(G) = Z^{\star}(R)$. Assume that $Z^{\star}(R) \neq 1$ and choose $a \in Z^{\star}(R)$ of minimal height, 2^n say. Let $b \in R$ with $b \neq a$ be an involution of height 2^m with $m \geq n$, and choose $g, h \in R$ with $a = g^{2^n}$ and $b = h^{2^m}$. If no such b exists, then the assertion follows immediately. By Lemma 3.1a), we have $R = \langle g \rangle \times U = \langle h \rangle \times V$, and $h = g^i u$ for some integer i and $u \in U$. If $m > n$, or $m = n$ and $a^i = 1$, then $b = h^{2^m} = u^{2^m}$, and Lemma 3.1a) shows that $\langle g \rangle \times \langle u \rangle$ is a direct factor of R. Clearly, there is an automorphism of R which maps g to $g u^{2^{m-n}}$, which contradicts $a \in Z^{\star}(R)$. Hence, we have $m = n$ and $b = h^{2^m} = a u^{2^m}$.

Now consider $k = g^{-1}h$. Then $c = k^{2^m} = u^{2^m}$ is an involution of height $l \geq m$. If $l = m$, then $\langle g \rangle \times \langle u \rangle$ is a direct factor of R and the same argument as above yields a contradiction. If $l > m$, then $c = f^{2^l}$ for some $f = g^j v$ with $v \in U$. Again, as shown above, this gives a contradiction. It follows that $Z^\star(R) = \{1, a\}$, where $a \in R$ is the unique involution of maximal height. □

4. Stability series

Let M be a G-group. A subgroup series $M = M_1 > M_2 > \cdots$ is a *G-central series* of M if G acts trivially on each section M_i/M_{i+1}. Hall [7, Theorem 1] proved that if $M = M_1 > \cdots > M_{n+1} = 1$ is a G-central series with G acting faithfully, then G is nilpotent of class at most n. In this section we provide a partial converse of this result.

First, we introduce some notation. The *upper G-central series* of M

$$1 = \zeta_{G,0}(M) \leq \zeta_{G,1}(M) \leq \cdots$$

is defined by $\zeta_{G,1}(M) = Z_G(M)$ and $\zeta_{G,i+1}(M)/\zeta_{G,i}(M) = Z_G(M/\zeta_{G,i}(M))$. If n is minimal with $\zeta_{G,n}(M) = M$, then M is *G-nilpotent* of *G-class* n. Note that M is G-nilpotent if and only if it admits a G-central series.

Being G-nilpotent can be quite restrictive: Let M be an abelian G-group with $G = \mathrm{Aut}(M)$ and G-class n. Since inversion is an automorphism of M, it can be proved by induction that the exponent of $\zeta_{G,i}(M)$ divides 2^i. This shows that M is a torsion group with exponent dividing 2^n.

Proposition 4.1. *Let M be a finite nilpotent G-group with G acting faithfully.*

a) *Suppose that $G = \mathrm{Aut}(M)$ is nilpotent. Then M is G-nilpotent if and only if its Sylow p-subgroups are non-cyclic for all $p \geq 3$.*

b) *If G acts trivially on M/M', then G is nilpotent and M is G-nilpotent.*

Proof. a) We can assume that M is a p-group. If M is cyclic and $p \geq 3$, then $Z_G(M) = 1$, and M is not G-nilpotent. If M is a cyclic 2-group and $\zeta_{G,i}(M) < M$, then $Z_G(M/\zeta_{G,i}(M))$ is the unique subgroup of $M/\zeta_{G,i}(M)$ of order 2, and M is G-nilpotent. Now let M be a non-cyclic p-group. Since M has finite exponent and $\mathrm{Aut}(M)$ is nilpotent, we can apply [14, Corollary 4.5], which shows that G is a p-group. Assume, for a contradiction, that M is not G-nilpotent. Then there exists $i \geq 0$ such that G acts fixed point freely on $K = M/\zeta_{G,i}(M)$. Let H be the image of $G \to \mathrm{Aut}(K)$ and consider $W = H \ltimes K$. Since H acts fixed point freely, W is a Frobenius group. The structure of Frobenius groups is well-known and it is proved in [9, Satz V.8.3] that the order of $|H|$ divides $|K| - 1$. Since H and K are p-groups, this yields a contradiction.

b) Assume that M is a p-group which is not G-nilpotent. As in a), there exists i such that $W = H \ltimes K$ is a Frobenius group where $K = M/\zeta_{G,i}(M)$ and H the image of $G \to \mathrm{Aut}(K)$. Recall that $|H|$ divides $|K|-1$, see [9, Satz V.8.3], hence H is a p'-group. Thus, $\mathrm{Inn}(M)$ acts trivially on K, that is, K is abelian and $M' \leq \zeta_{G,i}(M)$. But this implies $H = 1$, a contradiction. Thus, M is G-nilpotent, and Hall's result [7, Theorem 1] completes the proof. □

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