ON THE DERIVED SUBGROUPS OF THE FREE NILPOTENT GROUPS OF FINITE RANK

RUSSELL D. BLYTH, PRIMOŽ MORAVEC,
AND ROBERT FITZGERALD MORSE

Abstract. We provide a detailed structure description of the derived subgroups of the free nilpotent groups of finite rank. This description is then applied to computing the nonabelian tensor squares of the free nilpotent groups of finite rank.

1. Introduction

A systematic structure description of the subgroups of the free nilpotent groups is given in S. Moran's paper "A subgroup theorem for free nilpotent groups" [8] that is based on the work of Gol′dina [6]. Moran's result is necessarily general and does not provide a detailed structure description for any specific subgroup of a free nilpotent group, such as its derived subgroup.

The purpose of this paper is to provide a detailed structure description of the derived subgroups of the free nilpotent groups of finite rank. The motivation for this investigation is that the derived subgroup of a free nilpotent group of class $c+1$ and rank n is isomorphic to the nonabelian exterior square of the free nilpotent group of class c and rank n. Moreover, the results presented in this paper give complete structure descriptions of the nonabelian tensor squares of free nilpotent groups of finite rank using a result from [1].

We fix our notation. Let F_n be the free group of rank n with generators $f_1, \ldots, f_n$ and denote the free abelian group of rank n by F_n^{ab}. Let $\mathcal{C}_n$ be a fixed basic sequence of commutators in the free generators of F_n. The weight of the commutator $c_i \in \mathcal{C}_n$ is denoted by w_i. We denote the subsequence of commutators of $\mathcal{C}_n$ whose weight is at most w by $\mathcal{C}_{n,w}$. The number of commutators in $\mathcal{C}_{n,w}$ is denoted by $M(n,w)$. The subset of simple left normed commutators in $\mathcal{C}_n$ of weight at most w is denoted by $\mathcal{S}_{n,w}$. Let $\mathcal{N}_{n,c} = F_n/\gamma_{c+1}(F_n)$ be the free nilpotent group of rank n and class c generated by $g_1, \ldots, g_n$. Denote by $\mathcal{D}_{n,c}$ the derived subgroup of $\mathcal{N}_{n,c}$. The elements of $\mathcal{C}_{n,c}$ map to $\mathcal{N}_{n,c}$ via the

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natural homomorphism $F_n \to F_n/\gamma_{c+1}(F_n) = \mathcal{N}_{n,c}$. With slight abuse of notation we identify elements of $\mathcal{C}_{n,c}$ as the same as their images in $\mathcal{N}_{n,c}$.

The group $N_{m,w}$, which will be used several times in this paper, is defined as follows. We first fix a free nilpotent group $\mathcal{K}$. Given $m \geq 2$ and $w \geq 3$, let

$$\mathcal{K} = \begin{cases} \mathcal{N}_{s,\lfloor w/2 \rfloor - 1} & \text{if } m = 2 \\ \mathcal{N}_{s,\lfloor w/2 \rfloor} & \text{if } m \geq 3 \end{cases}$$

where $s = |\mathcal{S}_{m,w-2} \setminus \mathcal{S}_{w,1}|$. Let $k_1, \ldots, k_s$ be the free generating set for $\mathcal{K}$. Fix a bijection β from the set $\{1, \ldots, s\}$ to $\mathcal{S}_{m,w-2} \setminus \mathcal{S}_{m,1}$. Associate a weight ω_{k_i} to each generator k_i of $\mathcal{K}$ via β by setting ω_{k_i} to be the weight of the simple left normed commutator $\beta(i)$. Then for $m \geq 2$ and $w \geq 3$, define

$$N_{m,w} = \mathcal{K}/R,$$

where $R = \langle [k_i, k_j] \mid \omega_{k_i} + \omega_{k_j} > w \rangle$. The group $N_{m,w}$ is minimally generated by $s = |\mathcal{S}_{m,w-2} \setminus \mathcal{S}_{w,1}|$ generators.

Our main result is the following description of the derived subgroup of a free nilpotent group of finite rank.

Theorem 1. *Let $\mathcal{N}_{n,c}$ be the free nilpotent group of class $c \geq 1$ and rank $n \geq 1$. If $n = 1$ or $c = 1$ then $\mathcal{N}_{n,c}$ is abelian and $\mathcal{D}_{n,c}$ is trivial. If $n > 1$ and $c = 2$ then $\mathcal{D}_{n,c}$ is free abelian of rank $M(n, 2) = \binom{n}{2}$. If $n > 1$ and $c > 2$ then*

$$\mathcal{D}_{n,c} \cong N_{n,c} \times F_f^{\text{ab}},$$

where $f = |\mathcal{S}_{n,c} \setminus \mathcal{S}_{n,c-2}|$.

In Section 2 we prove Theorem 1 and provide formulas for $|\mathcal{S}_{n,c} \setminus \mathcal{S}_{n,c-2}|$ and $|\mathcal{S}_{n,c} \setminus \mathcal{S}_{n,1}|$. These formulas allow us to restate Theorem 1 giving an explicit rank of the free abelian factor of $\mathcal{D}_{n,c}$ and for the number of minimal generators for $N_{n,c}$ (Theorem 8).

The nonabelian tensor square $G \otimes G$ of a group G was introduced by Brown and Loday [3] following the ideas of Dennis [4] and Miller [7] and is of topological significance. In Section 3 we give a brief exposition of the nonabelian tensor square of a group and use the formulas found in Section 2 to give a full description of $\mathcal{N}_{n,c} \otimes \mathcal{N}_{n,c}$. We apply Theorem 1 to Corollary 10 to obtain a general structure description for this nonabelian tensor square.

Corollary 2. *Let $G = \mathcal{N}_{n,c}$ be the free nilpotent group of class c and rank $n > 2$. Then*

$$G \otimes G \cong N_{n,c+1} \times F_g^{\text{ab}},$$

where $g = |\mathcal{S}_{n,c} \setminus \mathcal{S}_{n,c-2}| + \binom{n+1}{2}$.

2. The Derived Subgroup of a Free Nilpotent Group

In this section we first prove Theorem 1 and then use a result of Gaglione and Spellman [5] to find a formula for the number $|\mathcal{S}_{m,w} \setminus \mathcal{S}_{m,1}|$ of simple left normed commutators in $\mathcal{C}_{m,w}$ of weight at least 2. From this formula we derive explicit expressions for s, the minimal number of generators for $N_{n,w}$, and for $|\mathcal{S}_{m,w} \setminus \mathcal{S}_{m,w-1}|$.

The proof of Theorem 1 uses the following results proved in [8].

Theorem 3 ([8], Theorem 1.5)**.** *Every abelian subgroup of a free nilpotent group is free abelian.*

Theorem 4 ([8], Theorem 3.1)**.** *Let B be a subgroup of a free nilpotent group $\mathcal{N}_{n,c}$ of class $c \geq 1$ and rank $n \geq 1$. Then B is generated by a set of c subgroups*

$$B_1, B_2, \ldots, B_c,$$

where

(i) for $k = 1, 3, \ldots, c$, the subgroup B_k is a free nilpotent group of class $\lfloor \frac{c}{k} \rfloor$;
(ii) for $n = 2$, the subgroup B_2 is infinite cyclic, otherwise B_2 is nilpotent of class $\lfloor \frac{c}{2} \rfloor$;
(iii) for $i + j \leq c$, the subgroup $[B_i, B_j]$ is contained in the subgroup $\langle B_{i+j}, \ldots, B_c \rangle$;
(iv) for $i + j > c$, the subgroup $[B_i, B_j]$ is trivial; and
(v) for $k = 1, 2, \ldots, c-1$, the quotient group

$$\langle B_k, B_{k+1}, \ldots, B_c \rangle / \langle B_{k+1}, \ldots, B_c \rangle$$

is a free abelian group freely generated by the images of the free generators of B_k in the quotient group.

It can be possible for a subgroup B of $\mathcal{N}_{n,c}$ that one or more of the B_i in Theorem 4 might have rank 0. In such cases we treat these subgroups as trivial.

Applying Theorem 4 to the subgroup $\mathcal{D}_{n,c}$ of $\mathcal{N}_{n,c}$ the subgroups $B_1, \ldots, B_c$ are constructed as follows. Set B_1 to be a group of rank 0, as there are no basic commutators of weight 1 in $\mathcal{D}_{n,c}$, and set

$$B_k = \langle \mathcal{C}_{n,k} \setminus \mathcal{C}_{n,k-1} \rangle$$

for $k = 2, \ldots, c$. It follows that $\mathcal{D}_{n,c}$ is generated by $\mathcal{C}_{n,c} \setminus \mathcal{C}_{n,1}$. This fact can also be obtained by the Hall Basis Theorem.

We now prove Theorem 1.

Proof of Theorem 1. Let $\mathcal{N}_{n,c}$ be a free nilpotent group of class $c \geq 1$ and rank $n \geq 1$. If $n = 1$ or $c = 1$ then $\mathcal{N}_{n,c}$ is abelian and its derived subgroup $\mathcal{D}_{n,c}$ is trivial.

If $c = 2$ then $\mathcal{D}_{n,c}$ is abelian and hence free abelian by Theorem 3. The $M(n, 2)$ commutators of $\mathcal{C}_n$ of weight 2 are independent and generate $\mathcal{D}_{n,c}$. Hence $\mathcal{D}_{n,c}$ is free abelian of rank $M(n, 2) = \binom{n}{2}$.

Suppose $c > 2$ and $n > 1$. Let $B_1, \ldots, B_c$ be the subgroups of $\mathcal{D}_{n,c}$ constructed above. The abelian subgroups B_c and B_{c-1} are free abelian by Theorem 3 and both are contained in the center of $\mathcal{D}_{n,c}$. The Hall Basis Theorem states that $\mathcal{C}_{n,c} \setminus \mathcal{C}_{n,c-1}$ is an independent generating set for B_c. By property (v) of Theorem 4 the quotient $\langle B_{c-1}, B_c\rangle/\langle B_c\rangle$ is freely generated by the images of $\mathcal{C}_{n,c-1} \setminus \mathcal{C}_{n,c-2}$. However, since B_{c-1} is also central, $\langle B_{c-1}, B_c\rangle = B_{c-1} \times B_c$ with rank $|\mathcal{C}_{n,c} \setminus \mathcal{C}_{n,c-2}|$. Let A be the subgroup of $\mathcal{D}_{n,c}$ generated by the basis elements $\mathcal{S}_{n,c} \setminus \mathcal{S}_{n,c-2}$ of $\langle B_{c-1}, B_c\rangle$.

Let N be the subgroup of $\mathcal{D}_{n,c}$ generated by the set $(\mathcal{C}_{n,c} \setminus \mathcal{C}_{n,1}) \setminus (\mathcal{S}_{n,c} \setminus \mathcal{S}_{n,c-2})$. By Theorem 4 this subgroup is generated by subgroups $B_1, \ldots, B^*_{c-1}, B^*_c$ that satisfy properties (i)-(v). We define this sequence of subgroups as before, replacing the subgroups B_c and B_{c-1} with

$$B^*_c = \langle(\mathcal{C}_{n,c} \setminus \mathcal{C}_{n,c-1}) \setminus (\mathcal{S}_{n,c} \setminus \mathcal{S}_{n,c-1})\rangle \text{ and}$$
$$B^*_{c-1} = \langle(\mathcal{C}_{n,c-1} \setminus \mathcal{C}_{n,2-1}) \setminus (\mathcal{S}_{n,c-1} \setminus \mathcal{S}_{n,c-2})\rangle.$$

It follows from A being central in $\mathcal{D}_{n,c}$ and from the Hall Basis Theorem that any element of $\mathcal{D}_{n,c}$ can be written as xy, where $x \in N$ and $y \in A$. Hence $\mathcal{D}_{n,c} = NA$. The subgroup N is normal in $\mathcal{D}_{n,c}$ since A is central and $\mathcal{D}_{n,c} = NA$.

Any element $g \in N \cap A$ would have to be written both as a product of powers of the generators of N and as a product of powers of simple commutator generators of A. This is only possible if $g = 1$. Hence $N \cap A = 1$. Since both A and N are normal in $\mathcal{D}_{n,c}$, it follows that $\mathcal{D}_{n,c} = N \times A$.

To show that N is isomorphic to $N_{n,c}$ we define a new sequence of subgroups of N. For $i = 2, \ldots, c-2$ we define

$$S_i = \langle\mathcal{S}_{n,i} \setminus \mathcal{S}_{n,i-1}\rangle.$$

Set $N^* = \langle S_1, \ldots, S_{c-2}\rangle$. The generators of the subgroups S_i for $i = 2, \ldots, c-2$ are independent and cannot be expressed as products of powers of the other generators. This holds since the generators are simple left normed commutators and no generator of N^* has weight 1.

Using the bijection β the generators $\mathcal{K}$ are mapped to the generators of N^*. The groups $\mathcal{K}$ and N^* both have the same nilpotency class, either $\lfloor c/2\rfloor$ if $n > 2$ or $\lfloor c/2\rfloor - 1$ if $n = 2$. Hence the mapping of generators induces a homomorphism $\phi : \mathcal{K} \to N^*$. Since the S_i are subgroups of the B_i then $[S_i, S_j]$ is trivial if $i + j > c$. Therefore the commutator $[k_i, k_j]$ is in the kernel of ϕ whenever the sum of the weights of the commutators $\beta(i)$ and $\beta(j)$ is larger than c. Hence $R \subseteq \ker(\phi)$. On the other hand, N^* by construction does not introduce relations other than those found in Theorem 4. Hence $R = \ker(\phi)$, and $N^* \cong \mathcal{K}/R = N_{n,c}$.

We complete the proof by showing that $N^* = N$. If c_i is a generator of N that is not simple left normed commutator then $c_i = [c_q, c_p]$, where $w_q > 1$ and $w_p > 1$. If c_q and c_p are simple commutators then c_i is a product of the generators of N^*. If either c_q or c_p is not a simple commutator then we repeat the process and determine that all generators of N are products of simple commutators that generate N^*. □

The following result from [5] enables us to provide precise values for the rank s of $\mathcal{K}$ and for f in Theorem 1 in terms of n and c.

Proposition 5. *Let m and w be positive integers larger than 1. Then the value of $|\mathcal{S}_{m,w} \setminus \mathcal{S}_{m,w-1}|$, the number of simple left normed commutators of $\mathcal{C}_{m,w}$ of weight exactly w is*

$$(w-1)\binom{m+w-2}{w}.$$

We derive an immediate consequence.

Corollary 6. *Let m and w be positive integers greater than 1. Then the value of $|\mathcal{S}_{m,w} \setminus \mathcal{S}_{m,w-2}|$ is*

$$\binom{m+w-2}{w}\left((w-1)+\frac{w(w-2)}{m+w-2}\right).$$

Proof. By Proposition 5,

$$\begin{aligned}
|\mathcal{S}_{m,w} \setminus \mathcal{S}_{m,w-2}| &= |\mathcal{S}_{m,w-1} \setminus \mathcal{S}_{m,w-2}| + |\mathcal{S}_{m,w} \setminus \mathcal{S}_{m,w-1}| \\
&= (w-2)\binom{m+w-3}{w-1} + (w-1)\binom{m+w-2}{w} \\
&= (w-2)\frac{(m+w-3)!}{(w-1)!(m-2)!} + (w-1)\binom{m+w-2}{w} \\
&= \binom{m+w-2}{w}\left((w-1)+\frac{w(w-2)}{m+w-2}\right).
\end{aligned}$$

□

Using Proposition 5 we may determine a formula for the number $|\mathcal{S}_{n,w} \setminus \mathcal{S}_{n,1}|$ of simple left normed commutators of $\mathcal{C}_{n,c}$ of weight at least 2.

Proposition 7. *Let w be a positive integer greater than* 1. *Let $S_w^* = \mathcal{S}_{n,w} \setminus \mathcal{S}_{n,1}$ be the set of simple left normed commutators of $\mathcal{C}_n$ of weights* $2, \ldots, w$. *Then*

$$|S_w^*| = \sum_{j=2}^{w} |S_{n,j} \setminus S_{n,j-1}| = \frac{(n+w-1)!(wn-w-n)+w!n!}{w!n!}.$$

Proof. The proof is by induction on w for each fixed value of n. For $w = 2$, the right side gives

$$\frac{(n+1)!(n-2)+2n!}{2n!} = \frac{(n+1)(n-2)+2}{2} = \frac{n(n-1)}{2},$$

which is equal to $|S_{n,2} \setminus S_{n,1}| = \binom{n}{2}$.

Suppose that the formula holds for $w = k \geq 2$, that is, that

$$|S_k^*| = \sum_{j=2}^{k} |S_{n,j} \setminus S_{n,j-1}| = \frac{(n+k-1)!(kn-k-n)+k!n!}{k!n!}.$$

Then, by the inductive hypothesis and Proposition 5,

$$\begin{aligned}
|S_{k+1}^*| &= |S_k^*| + |S_{n,k+1} \setminus S_{n,k}| \\
&= \frac{(n+k-1)!(kn-k-n)+k!n!}{k!n!} + k\binom{n+k-1}{k+1} \\
&= \frac{(n+k-1)!(kn-k-n)+k!n!}{k!n!} + k\frac{(n+k-1)!}{(k+1)!(n-2)!} \\
&= \frac{(k+1)((n+k-1)!(kn-k-n)+k!n!)}{(k+1)!n!} + \frac{kn(n-1)(n+k-1)!}{(k+1)!n!} \\
&= \frac{(n+k-1)!(n+k)(nk-k-1)+(k+1)!n!}{(k+1)!n!} \\
&= \frac{(n+k)!(n(k+1)-(k+1)-n)+(k+1)!n!}{(k+1)!n!},
\end{aligned}$$

which shows that the formula holds for $w = k+1$, and thus the formula holds for all integers $w \geq 2$. □

In particular, when $w = c - 2$,

$$\begin{aligned}
|S_{c-2}^*| &= \frac{(n+c-3)!((c-2)n-(c-2)-n)+(c-2)!n!}{(c-2)!n!} \\
&= \frac{(n+c-3)!(cn-3n-c+2)+(c-2)!n!}{(c-2)!n!},
\end{aligned}$$

which is the value of the rank s of $\mathcal{K}$. We thus obtain the following refinement of the statement of our main result.

Theorem 8. *Let $\mathcal{N}_{n,c}$ be the free nilpotent group of class $c \geq 1$ and rank $n > 2$. Then*

$$\mathcal{D}_{n,c} \cong N_{n,c} \times F_f^{\mathrm{ab}},$$

where

$$f = \binom{n+c-2}{c}\left((c-1) + \frac{c(c-2)}{n+c-2}\right)$$

and $N_{n,c}$ has

$$s = \frac{(n+c-3)!(cn-3n-c+2)+(c-2)!n!}{(c-2)!n!}$$

minimal generators.

3. Application

In this section we apply Theorem 1 to describe the structure of the nonabelian tensors square of the free nilpotent groups of finite rank.

Let G be any group. Then the group $G \otimes G$ generated by the symbols $g \otimes h$, where $g, h \in G$, subject to the relations

$$gh \otimes k = ({}^g h \otimes {}^g k)(g \otimes k) \quad \text{and} \quad g \otimes hk = (g \otimes h)({}^h g \otimes {}^h k)$$

for all g, h, and k in G, where ${}^x y = xyx^{-1}$ for $x, y \in G$, is called the nonabelian tensor square of G. Let $\nabla(G)$ be the subgroup of $G \otimes G$ generated by the set $\{g \otimes g \mid g \in G\}$. The group $\nabla(G)$ is a central subgroup of $G \otimes G$ [3]. The factor group $G \otimes G/\nabla(G)$ is called the nonabelian exterior square of G, denoted by $G \wedge G$. For elements g and h in G, the coset $(g \otimes h)\nabla(G)$ is denoted $g \wedge h$. Hence $G \otimes G$ is a central extension of $G \wedge G$ by $\nabla(G)$ and we have the short exact sequence

$$1 \longrightarrow \nabla(G) \xrightarrow{\iota} G \otimes G \xrightarrow{\sigma} G \wedge G \longrightarrow 1.$$

The following theorem from [1] provides the basic structure for the nonabelian tensor square of a free nilpotent group of finite rank.

Theorem 9. *Let $G = \mathcal{N}_{n,c}$ be the free nilpotent group of class c and rank $n > 1$. Then*

$$G \otimes G \cong \nabla(G) \times G \wedge G.$$

In [1] it was shown that $\nabla(\mathcal{N}_{n,c}) \cong F^{\mathrm{ab}}_{\binom{n+1}{2}}$. It follows from Corollary 2 of [2] that $\mathcal{D}_{n,c+1}$ is isomorphic to $\mathcal{N}_{n,c} \wedge \mathcal{N}_{n,c}$. Putting these facts together we obtain Corollary 1.7 of [2], which we now state.

Corollary 10. *Let $G = \mathcal{N}_{n,c}$ be the free nilpotent group of class c and rank $n > 1$. Then*

$$G \otimes G \cong \mathcal{D}_{n,c+1} \times F^{\mathrm{ab}}_{\binom{n+1}{2}}.$$

The following theorem combines our detailed description of $\mathcal{D}_{n,c+1}$ from Theorem 8 with Corollary 10.

Theorem 11. *Let $G = \mathcal{N}_{n,c}$ be the free nilpotent group of class c and rank $n > 2$. Then*

$$G \otimes G \cong N_{n,c+1} \times F_g^{\mathrm{ab}},$$

where

$$g = \binom{n+c-1}{c+1}\left(c + \frac{(c+1)(c-1)}{n+c-1}\right) + \binom{n+1}{2}.$$

and $N_{n,c+1}$ has

$$\frac{(n+c-2)!((c+1)n - 3n - c + 3) + (c-1)!n!}{(c-1)!n!}$$

minimal generators.

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Department of Mathematics and Computer Science, Saint Louis University, St. Louis, MO 63103, USA

E-mail address: blythrd@slu.edu

Fakulteta za matematiko in fiziko, Univerza v Ljubljani, Jadranska 19, 1000 Ljubljana, Slovenia

E-mail address: primoz.moravec@fmf.uni-lj.si

Department of Electrical Engineering and Computer Science, University of Evansville, Evansville IN 47722 USA

E-mail address: rfmorse@evansville.edu

URL: faculty.evansville.edu/rm43