UNRAMIFIED BRAUER GROUPS OF FINITE AND INFINITE GROUPS

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In loving memory of my father

ABSTRACT. The Bogomolov multiplier is a group theoretical invariant isomorphic to the unramified Brauer group of a given quotient space. We derive a homological version of the Bogomolov multiplier, prove a Hopf-type formula, find a five term exact sequence corresponding to this invariant, and describe the role of the Bogomolov multiplier in the theory of central extensions. A new description of the Bogomolov multiplier of a nilpotent group of class two is obtained. We define the Bogomolov multiplier within K-theory and show that proving its triviality is equivalent to solving a long-standing problem posed by Bass. An algorithm for computing the Bogomolov multiplier is developed.

1. Introduction

In this paper we develop a homological version of a group theoretical invariant that has served as one of the main tools in studying the problem of stable rationality of quotient spaces. Let G be a finite group and V a faithful representation of G over $\mathbb{C}$. Then there is a natural action of G upon the field of rational functions $\mathbb{C}(V)$. A problem posed by Emmy Noether [25] asks as to whether the field of G-invariant functions $\mathbb{C}(V)^G$ is purely transcendental over $\mathbb{C}$, i.e., whether the quotient space V/G is *rational*. A question related to the above mentioned is whether V/G is *stably rational*, that is, whether there exist independent variables $x_1, \ldots, x_r$ such that $\mathbb{C}(V)^G(x_1, \ldots, x_r)$ becomes a pure transcendental extension of $\mathbb{C}$. This problem has close connection with Lüroth's problem [27] and the inverse Galois problem [32, 28]. By Hilbert's Theorem 90 stable rationality of V/G does not depend upon the choice of V, but only on the group G. Saltman [28] found examples of groups G such that V/G is not stably rational over $\mathbb{C}$. His main method was application of the unramified cohomology group $\mathrm{H}^2_{\mathrm{nr}}(\mathbb{C}(V)^G, \mathbb{Q}/\mathbb{Z})$ as an obstruction. A version of this invariant had been used before by Artin and Mumford [1] who constructed unirational varieties over $\mathbb{C}$ that were not rational. Bogomolov [3] further explored this cohomology group. He proved that $\mathrm{H}^2_{\mathrm{nr}}(\mathbb{C}(V)^G, \mathbb{Q}/\mathbb{Z})$ is canonically isomorphic to a certain subgroup $\mathrm{B}_0(G)$ (defined in Section 3) of the *Schur multiplier* $\mathrm{H}^2(G, \mathbb{Q}/\mathbb{Z})$ of G. Kunyavskiĭ [18] coined the term the *Bogomolov multiplier* of G for the group $\mathrm{B}_0(G)$. Bogomolov used the above description to find new examples of groups with $\mathrm{H}^2_{\mathrm{nr}}(\mathbb{C}(V)^G, \mathbb{Q}/\mathbb{Z}) \neq 0$. Subsequently, Bogomolov, Maciel and Petrov [4] showed that $\mathrm{B}_0(G) = 0$ when G is a finite simple group of Lie type A_ℓ, whereas Kunyavskiĭ [18] recently proved that $\mathrm{B}_0(G) = 0$ for every quasisimple or almost simple group G. Bogomolov's conjecture that V/G is stably rational over $\mathbb{C}$ for every finite simple group G, nevertheless, still remains open.

We first observe that if G is a finite group, then $\mathrm{B}_0(G)$ is canonically isomorphic to $\mathrm{Hom}(\tilde{\mathrm{B}}_0(G), \mathbb{Q}/\mathbb{Z})$, where the group $\tilde{\mathrm{B}}_0(G)$ can be described as a section of the

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nonabelian exterior square $G \wedge G$ of the group G. The latter appears implicitly in Miller's work [21], and was further developed by Brown and Loday [6]. Let $\gamma_2(G)$ be the derived subgroup of G, and denote the kernel of the commutator homomorphism $G \wedge G \to \gamma_2(G)$ by $\mathrm{M}(G)$. Miller [21] proved that there is a natural isomorphism between $\mathrm{M}(G)$ and $\mathrm{H}_2(G,\mathbb{Z})$. Using this description, we prove that $\tilde{\mathrm{B}}_0(G) = \mathrm{M}(G)/\mathrm{M}_0(G)$, where $\mathrm{M}_0(G)$ is the subgroup of $\mathrm{M}(G)$ generated by all $x \wedge y$ such that $x, y \in G$ commute. In the finite case, $\tilde{\mathrm{B}}_0(G)$ is thus (non-canonically) isomorphic to $\mathrm{B}_0(G)$. The functor $\tilde{\mathrm{B}}_0$ can be studied within the category of all groups, and this is the main goal of the paper. In the first part we prove a Hopf-type formula for $\tilde{\mathrm{B}}_0(G)$ by showing that if G is given by a free presentation $G = F/R$, then

$$\tilde{\mathrm{B}}_0(G) \cong \frac{\gamma_2(F) \cap R}{\langle \mathrm{K}(F) \cap R \rangle},$$

where $\mathrm{K}(F)$ denotes the set of commutators in F. A special case of this was implicitly used before by Bogomolov [3], and Bogomolov, Maciel and Petrov [4]. With the help of the above formula we derive a five term exact sequence

$$\tilde{\mathrm{B}}_0(G) \longrightarrow \tilde{\mathrm{B}}_0(G/N) \longrightarrow \frac{N}{\langle \mathrm{K}(G) \cap N] \rangle} \longrightarrow G^{\mathrm{ab}} \longrightarrow (G/N)^{\mathrm{ab}} \longrightarrow 0,$$

where G is any group and N a normal subgroup of G. This is a direct analogue of the well-known five term homological sequence. By applying Kunyavskiĭ's work and the above sequence we obtain the following group theoretical result: If G is a finite group and S its solvable radical, then $S \cap \gamma_2(G) = \langle S \cap \mathrm{K}(G) \rangle$. Furthermore, we compute $\tilde{\mathrm{B}}_0(G)$ when G is a finite group that is a split extension. This corresponds to a well-known result of Tahara [33] who computed the Schur multiplier of semidirect product of groups (see also [17]). In particular, we obtain a closed formula for $\mathrm{B}_0(G)$ when G is a Frobenius group.

In his paper [3], Bogomolov extended the definition of $\mathrm{B}_0(G)$ to cover all algebraic groups G, cf. Section 3. This can be further extended in a natural way to cover all infinite groups. We prove here that if G is any group, then $\mathrm{B}_0(G)$ is canonically isomorphic to $\mathrm{Hom}(\tilde{\mathrm{B}}_{0\mathcal{F}}(G), \mathbb{Q}/\mathbb{Z})$, where $\tilde{\mathrm{B}}_{0\mathcal{F}}(G)$ is the quotient of the subgroup $\mathrm{M}_{\mathcal{F}}(G)$ of $\mathrm{H}_2(G,\mathbb{Z})$ generated by all images of corestriction maps $\mathrm{cor}_G^H : \mathrm{H}_2(H,\mathbb{Z}) \to \mathrm{H}_2(G,\mathbb{Z})$, where H runs through all finite subgroups of G, by the subgroup $\mathrm{M}_{0\mathcal{F}}(G)$ generated by all $\mathrm{im}\,\mathrm{cor}_G^A$, where A runs through all finite abelian subgroups of G. As a consequence we show that if G is a locally finite group, then $\tilde{\mathrm{B}}_0(G) \cong \tilde{\mathrm{B}}_{0\mathcal{F}}(G)$. This in particular applies to periodic linear groups. On the other hand, $\tilde{\mathrm{B}}_0(G)$ and $\tilde{\mathrm{B}}_{0\mathcal{F}}(G)$ may fail to be isomorphic in general.

One of the goals of the paper is to exhibit the role of $\tilde{\mathrm{B}}_0(G)$ in studying certain types of central extensions of G. This is motivated by the classical theory of Schur multipliers which are the cornerstones of the extension theory of groups. We define $G \curlywedge G$ to be the quotient of $G \wedge G$ by $\mathrm{M}_0(G)$. Then it is clear that the sequence $\tilde{\mathrm{B}}_0(G) \rightarrowtail G \curlywedge G \twoheadrightarrow \gamma_2(G)$ is exact, therefore $\tilde{\mathrm{B}}_0(G)$ can be thought of as the obstruction to $G \curlywedge G$ being isomorphic to $\gamma_2(G)$. This corresponds to a result of Miller [21] who demonstrated that the nonabelian exterior square $G \wedge G$ of a group G fits into the short exact sequence $\mathrm{M}(G) \rightarrowtail G \wedge G \twoheadrightarrow \gamma_2(G)$. This construction enables us to prove that if G is a finite group, then for every stem extension (E, π, A) producing $\tilde{\mathrm{B}}_0(G)$ we have that $\gamma_2(E)$ and $G \curlywedge G$ are of the same order. Furthermore, there exists a stem extension of this kind such that $\gamma_2(E)$ is actually isomorphic to $G \curlywedge G$. This can be seen as a direct analogue of the well-known fact that if G is finite, then $G \wedge G$ is naturally isomorphic to the derived subgroup of an arbitrary covering group of G. In addition to that, we prove that if G is a perfect group, then $G \curlywedge G$ is universal within the class of central extensions E of G with the property that every commuting pair of elements in G has commuting lifts in E. Again, this corresponds to the fact that if G is a perfect group, then $G \wedge G$ is the universal central extension of G.

The first known examples of finite groups G with $\mathrm{B}_0(G) \neq 0$ were found among p-groups of class 2 [3, 28]. Bogomolov obtained a description of $\mathrm{B}_0(G)$ when G is a p-group of class 2 with G^{ab} elementary abelian. Here we obtain a description of $\tilde{\mathrm{B}}_0(G)$ for any group G that is nilpotent of class 2. More precisely, we show that $\tilde{\mathrm{B}}_0(G) \cong \ker(\mathrm{H}_2(G^{\mathrm{ab}}, \mathbb{Z}) \to \gamma_2(G))/\ker(\mathrm{H}_2(G^{\mathrm{ab}}, \mathbb{Z}) \to G \curlywedge G)$. In the case when G is a p-group of class 2 with G^{ab} elementary abelian, this can be further refined using the Blackburn-Evens theory [2].

The functor $\tilde{\mathrm{B}}_0$ has applications in K-theory. For a unital ring Λ define $\tilde{\mathrm{B}}_0\,\Lambda = \tilde{\mathrm{B}}_0(\mathrm{E}(\Lambda))$ where $\mathrm{E}(\Lambda)$ is the subgroup of $\mathrm{GL}(\Lambda)$ generated by elementary matrices. We prove that $\tilde{\mathrm{B}}_0\,\Lambda$ is naturally isomorphic to $\mathrm{K}_2\,\Lambda/\langle \mathrm{K}(\mathrm{St}(\Lambda) \cap \mathrm{K}_2\,\Lambda\rangle$, where $\mathrm{St}(\Lambda)$ is the Steinberg group. This is related to a conjecture posed by Bass [10, Problem 3] that $\mathrm{K}_2\,\Lambda$ is always generated by the so-called Milnor elements. We show that this problem has a positive solution for a ring Λ if and only if $\tilde{\mathrm{B}}_0\,\Lambda$ is trivial. The latter is for instance true for commutative semilocal rings. A possible approach towards solving Bass' problem could be based on the result that $\tilde{\mathrm{B}}_0\,\Lambda$ is naturally isomorphic to $\tilde{\mathrm{B}}_0(\mathrm{GL}(\Lambda))$.

In general it is hard to compute $\mathrm{B}_0(G)$, due to its cohomological description. Chu, Hu, Kang, and Kunyavskiĭ [7] recently completed calculations of $\mathrm{B}_0(G)$ for all groups of order ≤ 64. The homological nature of $\tilde{\mathrm{B}}_0$, on the other hand, allows machine computation of $\tilde{\mathrm{B}}_0(G)$ for polycyclic groups G. There is an efficient algorithm developed recently by Eick and Nickel [11] for computing $G \wedge G$ in case G is polycyclic. Based on that we develop and implement an algorithm for computing $\tilde{\mathrm{B}}_0(G)$ for finite solvable groups G. We use this algorithm to determine the Bogomolov multiplier of all solvable groups of order ≤ 729, apart from the orders 512, 576 and 640. Our computations in particular show that there exist three groups of order 243 with nontrivial unramified Brauer group. This contradicts a result of Bogomolov [3] claiming that if G is a finite p-group of order at most p^5, then $\mathrm{B}_0(G) = 0$.

2. Preliminaries and notations

In this section we fix some notations used throughout the paper. Let G be a group and $x, y \in G$. We use the notation ${}^x y = xyx^{-1}$ for conjugation from the left. The commutator $[x, y]$ of elements x and y is defined by $[x, y] = xyx^{-1}y^{-1} = {}^x y y^{-1}$. If H and K are subgroups of G, then we define $[H, K] = \langle [h, k] \mid h \in H, k \in K\rangle$. The commutator subgroup $\gamma_2(G)$ of G is defined to be the group $[G, G]$. The *set* $\{[x, y] \mid x, y \in G\}$ of all commutators of G is denoted by $\mathrm{K}(G)$.

We recall the definition and basic properties of the nonabelian exterior product of groups. The reader is referred to [6, 21] for more thorough accounts on the theory and its generalizations. Let G be a group and M and N normal subgroups of G. We form the group $M \wedge N$, generated by the symbols $m \wedge n$, where $m \in M$ and $n \in N$, subject to the following relations:

$$mm' \wedge n = ({}^m m' \wedge {}^m n)(m \wedge n), \tag{2.0.1}$$

$$m \wedge nn' = (m \wedge n)({}^n m \wedge {}^n n'), \tag{2.0.2}$$

$$x \wedge x = 1, \tag{2.0.3}$$

for all $m, m' \in M$, $n, n' \in N$ and $x \in M \cap N$.

Let L be a group. A function $\phi : M \times N \to L$ is called a *crossed pairing* if for all $m, m' \in M$, $n, n' \in N$, $\phi(mm', n) = \phi({}^m m', {}^m n)\phi(m, n)$, $\phi(m, nn') =$

$\phi(m,n)\phi({}^n m, {}^n n')$, and $\phi(x,x)=1$ for all $x \in M \cap N$. A crossed pairing ϕ determines a unique homomorphism of groups $\phi^* : M \wedge N \to L$ such that $\phi^*(m \wedge n) = \phi(m,n)$ for all $m \in M$, $n \in N$.

The group $G \wedge G$ is said to be the *nonabelian exterior square* of G. By definition, the commutator map $\kappa : G \wedge G \to \gamma_2(G)$, given by $g \wedge h \mapsto [g,h]$, is a well defined homomorphism of groups. Clearly $\mathrm{M}(G) = \ker\kappa$ is central in $G \wedge G$, and G acts trivially via diagonal action on $\mathrm{M}(G)$. Miller [21] proved that there is a natural isomorphism between $\mathrm{M}(G)$ and $\mathrm{H}_2(G,\mathbb{Z})$. A direct consequence of this result is that if a group G is given by a free presentation $G \cong F/R$, then $G \wedge G$ is naturally isomorphic to $\gamma_2(F)/[R,F]$.

The following lemma collects some basic identities that hold in the nonabelian exterior square of a group:

Lemma 2.1 ([6]). *Let G be a group and $x,y,z,w \in G$.*

(a) $x \wedge y = (y \wedge x)^{-1}$.
(b) ${}^{x^{-1}}(x \wedge y) = y \wedge x^{-1}$.
(c) ${}^{[z,w]}(x \wedge y) = (z \wedge w)(x \wedge y)(z \wedge w)^{-1}$.

3. The unramified Brauer group

Let G be a finite group and V a faithful representation of G over $\mathbb{C}$. Bogomolov [3] proved that the unramified Brauer group $\mathrm{H}^2_{\mathrm{nr}}(\mathbb{C}(V)^G, \mathbb{Q}/\mathbb{Z})$ is canonically isomorphic to the group

$$\mathrm{B}_0(G) = \bigcap_{\substack{A \le G,\\ A \text{ abelian}}} \ker \mathrm{res}^G_A, \tag{3.0.1}$$

where $\mathrm{res}^G_A : \mathrm{H}^2(G,\mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^2(A,\mathbb{Q}/\mathbb{Z})$ is the usual cohomological restriction map. Our first aim is to obtain a homological description of $\mathrm{B}_0(G)$. Thus we need a dual of the above construction. Let H be a subgroup of G. Then there is a corestriction map $\mathrm{cor}^H_G : \mathrm{H}_2(H,\mathbb{Z}) \to \mathrm{H}_2(G,\mathbb{Z})$. On the other hand, we have a natural map $\tau^H_G : H \wedge H \to G \wedge G$. Identifying $\mathrm{H}_2(G,\mathbb{Z})$ with $\mathrm{M}(G)$ and $\mathrm{H}_2(H,\mathbb{Z})$ with $\mathrm{M}(H)$, we can write $\mathrm{cor}^H_G = \tau^H_G|_{\mathrm{M}(H)}$. Thus we have the following commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{H}_2(H,\mathbb{Z}) \arrow[r] \arrow[d, "\mathrm{cor}^H_G"'] & H \wedge H \arrow[r] \arrow[d, "\tau^H_G"'] & \gamma_2(H) \arrow[r] \arrow[d] & 1 \\ 0 \arrow[r] & \mathrm{H}_2(G,\mathbb{Z}) \arrow[r] & G \wedge G \arrow[r] & \gamma_2(G) \arrow[r] & 1 \end{tikzcd} \tag{3.0.2}$$

Now define

$$\mathrm{M}_0(G) = \langle \mathrm{cor}^A_G \mathrm{M}(A) \mid A \le G, A \text{ abelian} \rangle.$$

This group can be described as a subgroup of $G \wedge G$ in the following way.

Lemma 3.1. *Let G be a group. Then*

$$\mathrm{M}_0(G) = \langle x \wedge y \mid x,y \in G, [x,y]=1 \rangle.$$

Proof. Denote $N = \langle x \wedge y \mid x,y \in G, [x,y]=1\rangle$. Suppose that $x,y \in G$ commute. Then $A = \langle x,y \rangle$ is an abelian subgroup of G, hence $\mathrm{cor}^A_G \mathrm{M}(A) \le \mathrm{M}_0(G)$. In particular, $x \wedge y \in \mathrm{M}_0(G)$.

Conversely, let A be an abelian subgroup of G. Let $w \in \mathrm{cor}^A_G \mathrm{M}(A)$. Then w can be written as

$$w = \prod_{i=1}^r (a_i \wedge b_i),$$

where $a_i, b_i \in A$. Since $[a_i, b_i] = 1$ for all $i = 1, \ldots, r$, it follows that $w \in N$. This concludes the proof. $\square$

For a group G denote

$$\tilde{\mathrm{B}}_0(G) = \mathrm{M}(G)/\,\mathrm{M}_0(G).$$

With this notation we have the following result.

Theorem 3.2. *Let G be a finite group. Then $\mathrm{B}_0(G)$ is naturally isomorphic to $\mathrm{Hom}(\tilde{\mathrm{B}}_0(G), \mathbb{Q}/\mathbb{Z})$, and thus $\mathrm{B}_0(G) \cong \tilde{\mathrm{B}}_0(G)$ (non-canonically).*

Proof. At first we describe the natural isomorphism between the Schur multiplier $\mathrm{H}^2(G, \mathbb{Q}/\mathbb{Z})$ and $\mathrm{Hom}(\mathrm{H}_2(G, \mathbb{Z}), \mathbb{Q}/\mathbb{Z})$ in terms of the nonabelian exterior square of G. Choose $\gamma \in \mathrm{H}^2(G, \mathbb{Q}/\mathbb{Z})$ and let

$$0 \longrightarrow \mathbb{Q}/\mathbb{Z} \xrightarrow{\ i\ } G_\gamma \xrightarrow{\ \pi\ } G \longrightarrow 1$$

be the central extension associated to γ. Define a map $G \times G \to \gamma_2(G_\gamma)$ by the rule $(x, y) \mapsto [\bar{x}, \bar{y}]$, where $\bar{x}$ and $\bar{y}$ are preimages in G_γ under π of x and y, respectively. This map is well defined. Furthermore, it is a crossed pairing, hence it induces a homomorphism $\lambda_\gamma : G \wedge G \to \gamma_2(G_\gamma)$ given by $\lambda_\gamma(x \wedge y) = [\bar{x}, \bar{y}]$ for $x, y \in G$. It is clear that if $c \in \mathrm{M}(G)$, then $\lambda_\gamma(c) \in i(\mathbb{Q}/\mathbb{Z})$, therefore the restriction of λ_γ to $\mathrm{M}(G)$ (still denoted by λ_γ) belongs to $\mathrm{Hom}(\mathrm{M}(G), \mathbb{Q}/\mathbb{Z})$. The map $\Theta : \mathrm{H}^2(G, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}(\mathrm{M}(G), \mathbb{Q}/\mathbb{Z})$ given by $\gamma \mapsto \lambda_\gamma$ is a homomorphism of groups.

Conversely, let $\varphi \in \mathrm{Hom}(\mathrm{M}(G), \mathbb{Q}/\mathbb{Z})$. Let H be a covering group of G. In other words, we have a central extension

$$0 \longrightarrow Z \xrightarrow{\ j\ } H \xrightarrow{\ \rho\ } G \longrightarrow 1$$

with $jZ \leq \gamma_2(H)$ and $Z \cong \mathrm{M}(G)$. Every finite group has at least one covering group by a result of Schur, cf. [15, Hauptsatz V.23.5]. By [6] we have that $\gamma_2(H)$ is canonically isomorphic to $G \wedge G$. Upon identifying $\gamma_2(H)$ with $G \wedge G$, we may assume without loss of generality that $\mathrm{M}(G)$ is a subgroup of H. Choose a section $\mu : G \to H$ of ρ and define a map $f : G \times G \to H$ by $f(x, y) = \mu(x)\mu(y)\mu(xy)^{-1}$ for $x, y \in G$. It is straightforward to verify f maps $G \times G$ into $\mathrm{M}(G)$, and that $\varphi f \in Z^2(G, \mathbb{Q}/\mathbb{Z})$. The cohomology class of φf does not depend upon the choice of μ. We therefore have a map (the so-called transgression map)

$$\mathrm{tra} : \mathrm{Hom}(\mathrm{M}(G), \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^2(G, \mathbb{Q}/\mathbb{Z})$$

given by $\mathrm{tra}(\varphi) = [\varphi f]$. This is easily seen to be a homomorphism, and Θ is its inverse.

Now choose $\gamma \in \mathrm{B}_0(G)$ and let the map $\Theta : \mathrm{H}^2(G, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}(\mathrm{M}(G), \mathbb{Q}/\mathbb{Z})$ be defined as above. Denote $\lambda_\gamma = \Theta(\gamma)$. Let $x, y \in G$ and suppose that $[x, y] = 1$. Then $A = \langle x, y \rangle$ is an abelian subgroup of G, therefore $\mathrm{res}^G_A(\gamma) = 0$. This implies that $\lambda_\gamma(x \wedge y) = [\bar{x}, \bar{y}] = 1$. Therefore Θ induces a homomorphism $\tilde{\Theta} : \mathrm{B}_0(G) \to \mathrm{Hom}(\mathrm{M}(G)/\,\mathrm{M}_0(G), \mathbb{Q}/\mathbb{Z})$.

Let $\varphi \in \mathrm{Hom}(\mathrm{M}(G)/\,\mathrm{M}_0(G), \mathbb{Q}/\mathbb{Z})$. Then φ can be lifted to a homomorphism $\bar{\varphi} : \mathrm{M}(G) \to \mathbb{Q}/\mathbb{Z}$. Put $\gamma = \mathrm{tra}(\bar{\varphi})$. Suppose that

$$0 \longrightarrow \mathbb{Q}/\mathbb{Z} \xrightarrow{\ i\ } G_\gamma \xrightarrow{\ \pi\ } G \longrightarrow 1$$

is a central extension associated to γ. Choose an arbitrary bicyclic subgroup $A = \langle a, b \rangle$ of G. Then we have a central extension

$$0 \longrightarrow \mathbb{Q}/\mathbb{Z} \xrightarrow{\ i\ } A_\gamma \xrightarrow{\ \pi|_{A_\gamma}\ } A \longrightarrow 1$$

that corresponds to $\operatorname{res}_A^G(\gamma)$. Since $[a,b]=1$, we have that $a\wedge b\in \mathrm{M}_0(G)\le \ker\bar\varphi$, therefore $[\bar a,\bar b]=1$ in A_γ. It follows that A_γ is abelian, thus $\gamma\in \mathrm{B}_0(G)$. Hence the transgression map induces a homomorphism $\widetilde{\operatorname{tra}}:\operatorname{Hom}(\mathrm{M}(G)/\mathrm{M}_0(G),\mathbb{Q}/\mathbb{Z})\to \mathrm{B}_0(G)$ whose inverse is $\tilde\Theta$. □

The definition of $\mathrm{B}_0(G)$ can be extended to infinite groups as follows [3]. Let G be a group. Define

$$K_G=\{\gamma\in \mathrm{H}^2(G,\mathbb{Q}/\mathbb{Z})\mid \operatorname{res}_H^G\gamma=0 \text{ for every finite } H\le G\}.$$

Let $\mathrm{B}_0(G)$ be the subgroup of $\mathrm{H}^2(G,\mathbb{Q}/\mathbb{Z})/K_G$ consisting of all $\gamma+K_G$ with the property that $\operatorname{res}_A^G\gamma=0$ for every finite abelian subgroup A of G. It is clear that if G is a finite group, then this definition of $\mathrm{B}_0(G)$ coincides with the one given by (3.0.1). Bogomolov [3, Theorem 3.1] showed that if G is an algebraic group, then $\mathrm{B}_0(G)$ is isomorphic to $\mathrm{H}^2_{\mathrm{nr}}(\mathbb{C}(V)^G,\mathbb{Q}/\mathbb{Z})$, where V is any generically free representation of G.

In order to obtain a homological description of $\mathrm{B}_0(G)$ for infinite groups, we denote

$$\mathrm{M}_{\mathcal{F}}(G)=\langle \operatorname{cor}_G^H \mathrm{M}(H)\mid H\le G, |H|<\infty\rangle$$

and

$$\mathrm{M}_{0\mathcal{F}}(G)=\langle \operatorname{cor}_G^A \mathrm{M}(A)\mid A\le G, |A|<\infty, A \text{ abelian}\rangle.$$

Note that a similar argument as that of Lemma 3.1 shows that $\mathrm{M}_{0\mathcal{F}}(G)=\langle x\wedge y\mid [x,y]=1, |x|<\infty, |y|<\infty\rangle$. Now define $\tilde{\mathrm{B}}_{0\mathcal{F}}(G)=\mathrm{M}_{\mathcal{F}}(G)/\mathrm{M}_{0\mathcal{F}}(G)$. Then we have:

Theorem 3.3. *Let G be a group. Then the group $\mathrm{B}_0(G)$ is naturally isomorphic to $\operatorname{Hom}(\tilde{\mathrm{B}}_{0\mathcal{F}}(G),\mathbb{Q}/\mathbb{Z})$.*

Proof. We have a natural isomorphism $\mathrm{H}^2(G,\mathbb{Q}/\mathbb{Z})\cong \operatorname{Hom}(\mathrm{M}(G),\mathbb{Q}/\mathbb{Z})$. For $\gamma\in \mathrm{H}^2(G,\mathbb{Q}/\mathbb{Z})$ denote by λ_γ the corresponding element of $\operatorname{Hom}(\mathrm{M}(G),\mathbb{Q}/\mathbb{Z})$. By our definition we have that $\gamma\in K_G$ if and only if $\mathrm{M}_{\mathcal{F}}(G)\le\ker\lambda_\gamma$. Therefore $\mathrm{H}^2(G,\mathbb{Q}/\mathbb{Z})/K_G$ is naturally isomorphic to $\operatorname{Hom}(\mathrm{M}_{\mathcal{F}}(G),\mathbb{Q}/\mathbb{Z})$. Adapting the argument of the proof of Theorem 3.2, we obtain the required result. □

In general, $\tilde{\mathrm{B}}_0(G)$ and $\tilde{\mathrm{B}}_{0\mathcal{F}}(G)$ may be quite different. For example, if G is a one-relator group with torsion, then G has a presentation $G=\langle X\mid s^m\rangle$, where s is not a proper power in the free group over X. By a result of Newman [24], every finite subgroup of G is conjugate to a subgroup of $\langle s\rangle$, hence $\tilde{\mathrm{B}}_{0\mathcal{F}}(G)=0$. On the other hand, all centralizers of nontrivial elements of G are cyclic [24], hence $\mathrm{M}_0(G)=0$ and therefore $\tilde{\mathrm{B}}_0(G)\cong \mathrm{H}_2(G,\mathbb{Z})$. The latter can be nontrivial, cf. Lyndon [19].

In the case of locally finite groups we have the following:

Corollary 3.4. *Let G be a locally finite group. Then $\tilde{\mathrm{B}}_0(G)\cong\tilde{\mathrm{B}}_{0\mathcal{F}}(G)$.*

Proof. Every group G is a direct limit of its finitely generated subgroups $\{G_\lambda\mid \lambda\in\Lambda\}$. If G is locally finite, then the groups G_λ are all finite. Since $\mathrm{M}(G)\cong \varinjlim \mathrm{M}(G_\lambda)$, we conclude that $\mathrm{M}(G)=\mathrm{M}_{\mathcal{F}}(G)$. Since G is periodic, we also have that $\mathrm{M}_{0\mathcal{F}}(G)=\mathrm{M}_0(G)$, hence the result. □

Corollary 3.4 applies, for example, to periodic linear groups. On the other hand, there exist finitely generated periodic groups G (even of finite exponent) such that $\tilde{\mathrm{B}}_{0\mathcal{F}}(G)=0$, yet $\tilde{\mathrm{B}}_0(G)$ is nontrivial.

Example 3.5. Suppose $m>1$ and let $n>2^{48}$ be odd. Let F be a free group of rank m. Denote $B(m,n)=F/F^n$, the *free Burnside group* of rank m and exponent n. Ivanov [16] showed that all centralizers of nontrivial elements of $B(m,n)$ are cyclic, and that every finite subgroup of $B(m,n)$ is cyclic. From here it follows that $\tilde{\mathrm{B}}_{0\mathcal{F}}(B(m,n)) = 0$ and $\tilde{\mathrm{B}}_0(B(m,n)) \cong \mathrm{H}_2(B(m,n),\mathbb{Z})$. The latter group is free abelian of countable rank, cf. [26, Corollary 31.2].

In the rest of the paper we mainly consider the properties of $\tilde{\mathrm{B}}_0(G)$. Obviously $\tilde{\mathrm{B}}_0$ is a covariant functor from **Gr** to **Ab**. It is well known that the homology functor commutes with direct limits. It turns out that $\tilde{\mathrm{B}}_0$ enjoys the same property:

Proposition 3.6. *The functor* $\tilde{\mathrm{B}}_0$ *commutes with direct limits. More precisely, if* $\{G_\lambda, \alpha_\lambda^\mu \mid \lambda \le \mu \in \Lambda\}$ *is a direct system of groups and* G *its direct limit, then* $\tilde{\mathrm{B}}_0(G)$ *is the direct limit of* $\{\tilde{\mathrm{B}}_0(G_\lambda), \tilde{\mathrm{B}}_0(\alpha_\lambda^\mu) \mid \lambda \le \mu \in \Lambda\}$.

Proof. For every $\lambda \in \Lambda$ we have

$$0 \longrightarrow \mathrm{M}_0(G_\lambda) \longrightarrow \mathrm{M}(G_\lambda) \longrightarrow \tilde{\mathrm{B}}_0(G_\lambda) \longrightarrow 0,$$

hence the diagram

$$\begin{tikzcd}
0 \arrow[r] & \varinjlim \mathrm{M}_0(G_\lambda) \arrow[r] \arrow[d, "\alpha'"'] & \varinjlim \mathrm{M}(G_\lambda) \arrow[r] \arrow[d, "\alpha"'] & \varinjlim \tilde{\mathrm{B}}_0(G_\lambda) \arrow[r] \arrow[d, "\tilde{\alpha}"'] & 0 \\
0 \arrow[r] & \mathrm{M}_0(G) \arrow[r] & \mathrm{M}(G) \arrow[r] & \tilde{\mathrm{B}}_0(G) \arrow[r] & 0
\end{tikzcd}$$

is commutative with exact rows. Here α is the natural isomorphism, and α' is its restriction. Clearly α' is an isomorphism, hence so is $\tilde{\alpha}$. □

Proposition 3.7. *Let* G_1 *and* G_2 *be groups. Then* $\tilde{\mathrm{B}}_0(G_1 * G_2) \cong \tilde{\mathrm{B}}_0(G_1) \times \tilde{\mathrm{B}}_0(G_2)$ *and* $\tilde{\mathrm{B}}_{0\mathcal{F}}(G_1 * G_2) \cong \tilde{\mathrm{B}}_{0\mathcal{F}}(G_1) \times \tilde{\mathrm{B}}_{0\mathcal{F}}(G_2)$.

Proof. Let $G = G_1 * G_2$ and let $\iota_1 : G_1 \to G$ and $\iota_2 : G_2 \to G$ be the canonical injections. Then the induced maps $\iota_i^* : \mathrm{M}(G_i) \to \mathrm{M}(G)$ are injective ($i = 1, 2$), $\iota_1^* \mathrm{M}(G_1) \cap \iota_2^* \mathrm{M}(G_2) = 1$ and $\mathrm{M}(G) = \iota_1^* \mathrm{M}(G_1) \times \iota_2^* \mathrm{M}(G_2)$ by [21]. Now let $a, b \in G \setminus \{1\}$ with $[a, b] = 1$. By [20, p. 196] we have the following possibilities. If $a \in {}^h\iota_1(G_1)$, then $b \in C_G(a) \le {}^h\iota_1(G_1)$, hence we can write $a = {}^h\iota_1(x)$ and $b = {}^h\iota_1(y)$ for some commuting elements $x, y \in G_1$. In this case we get $a \wedge b = {}^h(\iota_1(x) \wedge \iota_1(y)) = \iota_1(x) \wedge \iota_1(y)$, as G acts trivially on $\mathrm{M}(G)$. For $a \in {}^h\iota_2(G_2)$, the situation is similar. If neither $a \in {}^h\iota_1(G_1)$ nor $a \in {}^h\iota_2(G_2)$, $C_G(a)$ is infinite cyclic. In this case we clearly have that $a \wedge b = 1$. Therefore we conclude that $\mathrm{M}_0(G) = \iota_1^* \mathrm{M}_0(G_1) \times \iota_2^* \mathrm{M}_0(G_2)$. It follows from here that

$$\tilde{\mathrm{B}}_0(G) \cong \iota_1^\sharp \, \tilde{\mathrm{B}}_0(G_1) \times \iota_2^\sharp \, \tilde{\mathrm{B}}_0(G_2),$$

where $\iota_i^\sharp : \tilde{\mathrm{B}}_0(G_1) \to \tilde{\mathrm{B}}_0(G)$ are the maps induced by ι_i, $i = 1, 2$. From the diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{M}_0(G_i) \arrow[r] \arrow[d, "\iota_i^*"'] & \mathrm{M}(G_i) \arrow[r] \arrow[d, "\iota_i^*"'] & \tilde{\mathrm{B}}_0(G_i) \arrow[r] \arrow[d, "\iota_i^\sharp"'] & 1 \\
1 \arrow[r] & \mathrm{M}_0(G) \arrow[r] & \mathrm{M}(G) \arrow[r] & \tilde{\mathrm{B}}_0(G) \arrow[r] & 1
\end{tikzcd}$$

we see that $\iota_i^\sharp$, $i = 1, 2$, are both injective, therefore $\tilde{\mathrm{B}}_0(G_1 * G_2) \cong \tilde{\mathrm{B}}_0(G_1) \times \tilde{\mathrm{B}}_0(G_2)$.

It remains to prove the corresponding assertion for $\tilde{\mathrm{B}}_{0\mathcal{F}}(G_1 * G_2)$. The above argument shows that $\mathrm{M}_{0\mathcal{F}}(G) = \iota_1^* \mathrm{M}_{0\mathcal{F}}(G_1) \times \iota_2^* \mathrm{M}_{0\mathcal{F}}(G_2)$. By [5, p. 54], every finite subgroup of G is conjugate to a subgroup of G_1 or G_2. Since G acts trivially on $\mathrm{M}(G)$, we therefore conclude that $\mathrm{M}_{\mathcal{F}}(G) = \langle \mathrm{cor}_G^H \mathrm{M}(H) \mid H \le G_1 \text{ or } H \le G_2, |H| < \infty \rangle = \iota_1^* \mathrm{M}_{\mathcal{F}}(G_1) \times \iota_2^* \mathrm{M}_{\mathcal{F}}(G_2)$. From here the result follows along the same lines as above. □

Let the group G be given by a free presentation $G = F/R$, where F is a free group and R a normal subgroup of F. By the well known Hopf formula [5, Theorem II.5.3] we have that $\mathrm{M}(G) \cong (\gamma_2(F) \cap R)/[R, F]$. The isomorphism is induced by the canonical isomorphism $G \wedge G \to \gamma_2(F)/[R, F]$ given by $xR \wedge yR \mapsto [x, y][R, F]$. Under this map, $\mathrm{M}_0(G)$ can be identified with the subgroup of $F/[F, R]$ generated by all the commutators in $F/[F, R]$ that belong to the Schur multiplier of G. In other words, we have that $\mathrm{M}_0(G) \cong \langle \mathrm{K}(F/[R, F]) \cap R/[R, F]\rangle = \langle \mathrm{K}(F) \cap R\rangle[R, F]/[R, F] = \langle \mathrm{K}(F) \cap R\rangle/[R, F]$. Thus we have proved the following Hopf-type formula for $\tilde{\mathrm{B}}_0(G)$:

Proposition 3.8. *Let G be a group given by a free presentation $G = F/R$. Then*

$$\tilde{\mathrm{B}}_0(G) \cong \frac{\gamma_2(F) \cap R}{\langle \mathrm{K}(F) \cap R\rangle}.$$

This formula enables, in principle, explicit calculations of $\tilde{\mathrm{B}}_0(G)$, given a free presentation of G. For example, a word w in a free group F is said to be a *commutator word* if $w = [u, v]$ for some $u, v \in F$. We have the following result:

Corollary 3.9. *Let $\mathfrak{V}$ be a variety of groups defined by a commutator word w. If G is a $\mathfrak{V}$-relatively free group, then $\tilde{\mathrm{B}}_0(G) = 0$.*

Proof. Let w be an n-variable commutator word. G can be presented as a quotient $F/\mathfrak{V}(F)$ of a free group F by the verbal subgroup $\mathfrak{V}(F) = \langle w(f_1, \ldots, f_n) \mid f_1, \ldots, f_n \in F\rangle$ of F. Note that $\mathfrak{V}(F) \le \gamma_2(F)$ and $\langle \mathrm{K}(F) \cap \mathfrak{V}(F)\rangle = \mathfrak{V}(F)$. By Proposition 3.8 we get the result. □

On the other hand, there exist relatively free groups G with $\tilde{\mathrm{B}}_0(G) \neq 0$, cf. Example 3.5 and Section 8.

Another interpretation of $\tilde{\mathrm{B}}_0(G)$ for finite groups G can be obtained via covering groups. Covering groups of a given group $G = F/R$ may not be unique, yet their derived subgroups are all naturally isomorphic to $\gamma_2(F)/[R, F]$. Under this identification we have the following result.

Proposition 3.10. *Let G be a finite group and H its covering group. Let Z be a central subgroup of H such that $Z \le \gamma_2(H)$, $Z \cong \mathrm{M}(G)$ and $H/Z \cong G$. Then*

$$\tilde{\mathrm{B}}_0(G) \cong \frac{Z}{\langle \mathrm{K}(H) \cap Z\rangle}.$$

In particular, $\tilde{\mathrm{B}}_0(G) = 0$ if and only if every element of Z can be represented as a product of commutators that all belong to Z.

We note here that a special case of Proposition 3.10 formed one of the crucial steps in proving the main results of [4] and [18].

One of the main features of the homological description of $\mathrm{B}_0(G)$ is a five term exact sequence associated to the short exact sequence $1 \to N \to G \to G/N \to 1$ of groups. This sequence is an unramified Brauer group analogue of the well known five term homology sequence, cf [5, p. 46].

Theorem 3.11. *Let G be a group and N a normal subgroup of G. Then we have the following exact sequence:*

$$\tilde{\mathrm{B}}_0(G) \longrightarrow \tilde{\mathrm{B}}_0(G/N) \longrightarrow \frac{N}{\langle \mathrm{K}(G) \cap N]\rangle} \longrightarrow G^{\mathrm{ab}} \longrightarrow (G/N)^{\mathrm{ab}} \longrightarrow 0.$$

Proof. Let G have a free presentation $G = F/R$, and let SR/R be the corresponding free presentation of N. Then Proposition 3.8 implies that $\tilde{\mathrm{B}}_0(G) \cong (\gamma_2(F) \cap R)/\langle \mathrm{K}(F) \cap R\rangle$ and $\tilde{\mathrm{B}}_0(G/N) \cong (\gamma_2(F) \cap RS)/\langle \mathrm{K}(F) \cap RS\rangle$. The canonical epimorhism $\rho : G \to G/N$ induces a homomorphism $\rho^\sharp : \tilde{\mathrm{B}}_0(G) \to \tilde{\mathrm{B}}_0(G/N)$. From the above Hopf formulae it follows that

$$\ker \rho^\sharp = \frac{R \cap \langle \mathrm{K}(F) \cap RS \rangle}{\langle \mathrm{K}(F) \cap R \rangle}$$

and

$$\mathrm{im}\, \rho^\sharp = \frac{\gamma_2(F) \cap \langle \mathrm{K}(F) \cap RS \rangle R}{\langle \mathrm{K}(F) \cap RS \rangle}.$$

It is straightforward to verify that $\langle \mathrm{K}(G) \cap N \rangle = \langle \mathrm{K}(F) \cap RS \rangle R/R$, therefore $N/\langle \mathrm{K}(G) \cap N \rangle \cong RS/\langle \mathrm{K}(F) \cap RS \rangle R$. Thus there is a natural map $\sigma : \tilde{\mathrm{B}}_0(G/N) \to N/\langle \mathrm{K}(G) \cap N \rangle$. We have that $\ker \sigma = \mathrm{im}\, \rho^\sharp$ and

$$\mathrm{im}\, \sigma = \frac{(\gamma_2(F) \cap RS)R}{\langle \mathrm{K}(F) \cap RS \rangle R} = \frac{\gamma_2(F)R \cap RS}{\langle \mathrm{K}(F) \cap RS \rangle R} = \frac{\gamma_2(G) \cap N}{\langle \mathrm{K}(G) \cap N \rangle}.$$

Furthermore, there is a natural map $\pi : N/\langle \mathrm{K}(G) \cap N \rangle \to G^{\mathrm{ab}}$ whose kernel is equal to $\mathrm{im}\, \sigma$, and $\mathrm{im}\, \pi = N\gamma_2(G)/\gamma_2(G)$. Finally, there is a surjective homomorphism $G^{\mathrm{ab}} \to (G/N)^{\mathrm{ab}}$ whose kernel is equal to $\mathrm{im}\, \pi$. From here our assertion readily follows. □

The proof of Theorem 3.11 also yields another exact sequence that is an analogue of the corresponding sequence for Schur multipliers obtained by Blackburn and Evens [2]. More precisely, we have:

Proposition 3.12. *Let G be a group given by a free presentation $G = F/R$ and let $N = SR/R$ be a normal subgroup of G. Then the sequence*

$$0 \to \frac{R \cap \langle \mathrm{K}(F) \cap RS \rangle}{\langle \mathrm{K}(F) \cap R \rangle} \to \tilde{\mathrm{B}}_0(G) \to \tilde{\mathrm{B}}_0(G/N) \to \frac{N \cap \gamma_2(G)}{\langle \mathrm{K}(G) \cap N] \rangle} \to 0$$

is exact.

The above result has the following group theoretical consequence:

Corollary 3.13. *Let G be a finite group and S the solvable radical of G, i.e., the largest solvable normal subgroup of G. Then $S \cap \gamma_2(G) = \langle S \cap \mathrm{K}(G) \rangle$.*

Proof. The factor group G/S does not contain proper nontrivial abelian normal subgroups, i.e., it is semisimple. By a result of Kunyavskiĭ [18] we conclude that $\tilde{\mathrm{B}}_0(G/S) = 0$. From Proposition 3.12 we get the desired result. □

4. The ‘commutativity-preserving’ nonabelian exterior product of groups

The nonabelian exterior square of a group encodes crucial information on the Schur multiplier of the group. In this section we introduce a related construction that plays a similar role when considering the functor $\tilde{\mathrm{B}}_0$.

Let G be a group and M and N normal subgroups of G. We form the group $M \curlywedge N$, generated by the symbols $m \curlywedge n$, where $m \in M$ and $n \in N$, subject to the following relations:

$$\begin{aligned} mm' \curlywedge n &= ({}^m m' \curlywedge {}^m n)(m \curlywedge n), \\ m \curlywedge nn' &= (m \curlywedge n)({}^n m \curlywedge {}^n n'), \\ x \curlywedge y &= 1, \end{aligned} \tag{4.0.1}$$

for all $m, m' \in M$ $n, n' \in N$, and all $x \in M$ and $y \in N$ with $[x, y] = 1$. If we denote $\mathrm{M}_0(M, N) = \langle m \wedge n \mid m \in M, n \in N, [m, n] = 1 \rangle$, then we have that $M \curlywedge N = (M \wedge N)/\mathrm{M}_0(M, N)$.

Let L be a group. A function $\phi : M \times N \to L$ is called a $\tilde{\mathrm{B}}_0$-*pairing* if for all $m, m' \in M$, $n, n' \in N$, and for all $x \in M$, $y \in N$ with $[x, y] = 1$,

$$\begin{aligned}\phi(mm', n) &= \phi({}^m m', {}^m n)\phi(m, n),\\ \phi(m, nn') &= \phi(m, n)\phi({}^n m, {}^n n'),\\ \phi(x, y) &= 1.\end{aligned}$$

Clearly a $\tilde{\mathrm{B}}_0$-pairing ϕ determines a unique homomorphism of groups $\phi^* : M \curlywedge N \to L$ such that $\phi^*(m \curlywedge n) = \phi(m, n)$ for all $m \in M$, $n \in N$. An example of a $\tilde{\mathrm{B}}_0$-pairing is the commutator map $M \times N \to [M, N]$. It induces a homomorphism $\tilde{\kappa} : M \curlywedge N \to [M, N]$ such that $\tilde{\kappa}(m \curlywedge n) = [m, n]$ for all $m \in M$ and $n \in N$. We denote the kernel of this homomorphism by $\tilde{\mathrm{B}}_0(M, N)$.

In the case when $M = N = G$, we have that $\mathrm{M}_0(G, G) = \mathrm{M}_0(G)$ and $\tilde{\mathrm{B}}_0(G, G) = \tilde{\mathrm{B}}_0(G)$. We therefore have a central extension

$$0 \longrightarrow \tilde{\mathrm{B}}_0(G) \longrightarrow G \curlywedge G \xrightarrow{\tilde{\kappa}} \gamma_2(G) \longrightarrow 1\ ,$$

where $\tilde{\kappa}$ is the commutator map. Thus one can interpret $\tilde{\mathrm{B}}_0(G)$ as a measure of the extent to which relations among commutators in G fail to be consequences of 'universal' commutator relations given by the images of relations (4.0.1) under the commutator map.

Proposition 4.1. *Let M and N be normal subgroups of a group G. Let $K \leq M \cap N$ be a normal subgroup of G. Then $M/K \curlywedge N/K \cong (M \curlywedge N)/J$, where $J = \langle m \curlywedge n \mid m \in M, n \in N, [m, n] \in K\rangle$.*

Proof. The map $M/K \times N/K \to (M \curlywedge N)/J$ given by $(mK, nK) \mapsto (m \curlywedge n)J$ is well defined and a $\tilde{\mathrm{B}}_0$-pairing, hence it induces a homomorphism $\varphi : M/K \curlywedge N/K \to (M \curlywedge N)/J$. On the other hand, we have a canonical $\tilde{\mathrm{B}}_0$-pairing $M \times N \to M/K \curlywedge N/K$ that induces a homomorphism $M \curlywedge N \to M/K \curlywedge N/K$. Under this homomorphism J gets mapped to 1, hence we have a homomorphism $\psi : (M \curlywedge N)/J \to M/K \curlywedge N/K$ whose inverse is φ. □

Schur [29], cf. also [15, Kapitel V], developed the theory of stem extensions. Here we indicate the role $\tilde{\mathrm{B}}_0(G)$ and $G \curlywedge G$ within the theory. Let G be a finite group and denote by (E, π, A) the central extension

$$1 \longrightarrow A \longrightarrow E \xrightarrow{\pi} G \longrightarrow 1 \tag{4.1.1}$$

of G. If the transgression homomorphism $\mathrm{tra} : \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^2(G, \mathbb{Q}/\mathbb{Z})$ is injective, then we say that (E, π, A) is a *stem extension* of G, and that the group $B = \mathrm{im}\,\mathrm{tra}$ is *produced* by (E, π, A). One can show that a central extension (E, π, A) of G is a stem extension if and only if $A \leq \gamma_2(E)$. In this case we have that $B \cong A$.

By a well known result of Schur [29], every subgroup of $\mathrm{H}^2(G, \mathbb{Q}/\mathbb{Z})$ is produced by some stem extension of G. This, in particular, applies to $\mathrm{B}_0(G)$. In terms of its homological counterpart $\tilde{\mathrm{B}}_0(G)$, we obtain the following result.

Theorem 4.2. *Let G be a finite group. Let (E, π, A) be a stem extension that produces $\tilde{\mathrm{B}}_0(G)$. Then $|\gamma_2(E)| = |G \curlywedge G|$. Furthermore, there exists a stem extension (E, π, A) of G producing $\tilde{\mathrm{B}}_0(G)$ such that $\gamma_2(E) \cong G \curlywedge G$.*

Proof. Let $G = F/R$ be a free presentation of G. By Proposition 4.1 we have that $G \curlywedge G \cong (F \curlywedge F)/J$, where $J = \langle x \curlywedge y \mid x, y \in F, [x, y] \in R\rangle$. Since the centralizer of every nontrivial element of F is cyclic, we have $F \curlywedge F = F \wedge F$. As $\mathrm{H}_2(F, \mathbb{Z}) = 0$, the commutator map $\kappa : F \wedge F \to \gamma_2(F)$ is an isomorphism. From here we conclude that $G \curlywedge G \cong \gamma_2(F)/\langle \mathrm{K}(F) \cap R\rangle$.

Let (E,π,A) be a stem extension of $G = F/R$ producing $\tilde{\mathrm{B}}_0(G)$. We have that $A \cong \tilde{\mathrm{B}}_0(G)$. Let $\{x_1,\ldots,x_n\}$ be the set of free generators of F. For every $1 \le i \le n$ choose $e_i \in E$ such that $\pi(e_i) = x_iR$. As $A \le Z(E) \cap \gamma_2(E)$, we conclude that A is contained in the Frattini subgroup $\mathrm{Frat}(E)$ of E, cf. [15, Satz III.3.12]. Thus $e_1,\ldots,e_n$ generate E. From here it follows that there is an epimorphism $\sigma : F \to E$ such that $\sigma(x_i) = e_i$ for all $i = 1,\ldots,n$. Denote $C = \ker\sigma$. It is straightforward to see that $C \le R$. Since $\pi(\sigma(x)) = xR$ for every $x \in F$, we have that $\sigma(R) = A$ and $\sigma^{-1}(A) = R$. From here we obtain $[R,F] \le C$. We claim that $\sigma(R \cap \gamma_2(F)) = A$. For, if $a \in A = A \cap \gamma_2(E) = \sigma(R) \cap \sigma(\gamma_2(F))$, then we can write $a = \sigma(r) = \sigma(\omega)$ for some $r \in R$ and $\omega \in \gamma_2(F)$. It follows that $\omega r^{-1} \in C \le R$, hence $\omega \in R \cap \gamma_2(F)$, as required. If $\bar\sigma$ is the restriction of σ to $R \cap \gamma_2(F)$, then $\ker\bar\sigma = C \cap \gamma_2(F)$. Therefore we have $(R \cap \gamma_2(F))/(C \cap \gamma_2(F)) \cong A \cong (R \cap \gamma_2(F))/\langle R \cap \mathrm{K}(F)\rangle$. This in particular shows that $|C \cap \gamma_2(F) : [R,F]| = |\langle R \cap \mathrm{K}(F)\rangle : [R,F]|$. From here we obtain $|\gamma_2(E)| = |\gamma_2(F) : C \cap \gamma_2(F)| = |\gamma_2(F) : [R,F]|/|C \cap \gamma_2(F) : [R,F]| = |\gamma_2(F) : [R,F]|/|\langle R \cap \mathrm{K}(F)\rangle : [R,F]| = |\gamma_2(F) : \langle R \cap \mathrm{K}(F)\rangle| = |G \curlywedge G|$.

It remains to construct a stem extension (E,π,A) of $G = F/R$ producing $\tilde{\mathrm{B}}_0(G)$ such that $\gamma_2(E) \cong G \curlywedge G$. Denote $B = (R \cap \gamma_2(F))/\langle R \cap \mathrm{K}(F)\rangle$ and $T = R/\langle R \cap \mathrm{K}(F)\rangle$. Then $T/B \cong R/(R \cap \gamma_2(F))$ is free abelian, hence B is complemented in T. Denote its complement by $\bar C = C/\langle R \cap \mathrm{K}(F)\rangle$, and put $E = F/C$, $A = R/C$. Let $\pi : E \to G$ be the canonical epimorphism. Then $\ker\pi = A$. As $[R,F] \le \langle R \cap \mathrm{K}(F)\rangle \le C$, it follows that (E,π,A) is a central extension of G. We have $A \cong T/\bar C = B\bar C/\bar C \cong C(R \cap \gamma_2(F))/C$, therefore $A \le \gamma_2(E)$. This shows that (E,π,A) is a stem extension of G. As $\gamma_2(E) \cong \gamma_2(F)/(C \cap \gamma_2(F)) = \gamma_2(F)/(C \cap (R \cap \gamma_2(F))) = \gamma_2(F)/\langle R \cap \mathrm{K}(F)\rangle \cong G \curlywedge G$, the assertion is proved. □

If a group G is perfect, then $G \wedge G$ is the universal central extension of G. This can be deduced readily, cf. [22, Theorem 5.7]. A similar description can be obtained for $G \curlywedge G$. We say that a central extension (E,π,A) of a group G is *commutativity-preserving* (*CP*) if commuting elements of G lift to commuting elements in E. A CP extension (U,ϕ,A) of a group G is said to be *CP-universal* if for every CP extension (E,ψ,B) of G there exists a homomorphism $\chi : U \to E$ that factors through G, i.e., $\psi\chi = \phi$. It is straightforward to see that a group G admits, up to isomorphism, at most one CP-universal central extension.

The following results have their direct counterparts in the theory of universal central extensions. The proofs follow along the lines of those of [22, Chapter 5].

Proposition 4.3. *A CP extension (U,ϕ,A) of a group G is CP-universal if and only if U is perfect, and every CP extension of U splits.*

Proof. Assume first that U is perfect, and that every CP extension of U splits. Let (E,ψ,B) be an arbitrary CP extension of G. Form $U \times_G E = \{(u,e) \in U \times E \mid \phi(u) = \psi(e)\}$, and let $\pi : U \times_G E \to U$ be the projection to the first factor. Then $(U \times_G E, \pi, \ker\pi)$ is a central extension of U, obviously a CP one. Thus it splits and therefore the section $\sigma : U \to U \times_G E$ induces a homomorphism $\chi : U \to E$. Since U is perfect, χ is uniquely determined [22, Lemma 5.4].

Conversely, suppose that (U,ϕ,A) is a CP-universal central extension of G. Then U is perfect by [22, Lemma 5.5]. Let (X,ψ,B) be a CP extension of U. Then $(X,\phi\psi,\ker\phi\psi)$ is a central extension of G. Take $x,y \in G$ with $[x,y] = 1$. Since the extension (U,ϕ,A) is CP, x and y have commuting lifts $x',y' \in U$ with respect to ϕ. The central extension (X,ψ,B) of U is also CP, hence x' and y' have commuting lifts $x'',y'' \in X$ with respect to ψ. This shows that $(X,\phi\psi,\ker\phi\psi)$ is a CP extension of G. By the assumption, there exists a homomorphism $\chi : U \to X$ that factors through G. We have that $\psi\chi$ is the identity map, hence the extension (X,ψ,B) of G splits. □

Proposition 4.4. *A group G admits a CP-universal central extension if and only if it is perfect. In the latter case, $(G \curlywedge G, \tilde{\kappa}, \tilde{\mathrm{B}}_0(G))$ is the CP-universal central extension of G.*

Proof. Let G be a perfect group. Suppose G is given by the free presentation $G = F/R$, and denote $K = \langle \mathrm{K}(F) \cap R \rangle$. We have a canonical surjection $\rho : F/K \to F/R$, and $\ker \rho = R/K$ is central F/K. By [22, Lemma 5.6], the group $\gamma_2(F)/K$, together with the appropriate restriction of ρ, is a perfect central extension of $\gamma_2(G) = G$. Let x and y be commuting elements of G. Then there exist $f_1, f_2 \in \gamma_2(F)$ such that $x = f_1 R$, $y = f_2 R$, and $[f_1, f_2] \in \mathrm{K}(F) \cap R \subseteq K$. This shows that the above central extension of G is CP. We claim that it is also CP-universal. Let (E, ψ, A) be another CP extension of G. As F is free, there exists a homomorphism $\tau : F \to X$ such that $\psi\tau = \rho$. Take an arbitrary $[f_1, f_2] \in \mathrm{K}(F) \cap R$, where $f_1, f_2 \in F$. Since $\rho(f_1)$ and $\rho(f_2)$ commute, there exist commuting lifts $e_1, e_2 \in E$ of these with respect to ψ. We can write $\tau(f_i) = e_i z_i$ for some $z_i \in A \leq Z(E)$, $i = 1, 2$. Then $\tau([f_1, f_2]) = [e_1 z_1, e_2 z_2] = 1$, hence τ induces a homomorphism $\chi : F/K \to E$. The restriction of χ to $\gamma_2(F)/K$ gives the required map. The second statement follows from the proof of Theorem 4.2.

The converse is obvious. □

5. Nilpotent groups of class 2

The first examples of finite p-groups G with $\mathrm{B}_0(G) \neq 0$ were found within the groups that are nilpotent of class 2, cf. [28, 3]. In this section we find a new description of $\mathrm{B}_0(G)$ for an arbitrary group G of class 2. This is achieved via the group $G \curlywedge G$.

Let the group G be nilpotent of class 2 and consider $G \curlywedge G$. As $\gamma_2(G) \leq Z(G)$, it follows that $[x, y] \curlywedge z = 1$ for all $x, y, z \in G$. In particular, $G \curlywedge G$ is an abelian group; in fact, it is easy to see that even the group $G \wedge G$ is abelian. It also follows that

$$^{z}(x \curlywedge y) = {}^{z}x \curlywedge {}^{z}y = [z, x]x \curlywedge {}^{z}y = x \curlywedge [z, y]y = x \curlywedge y,$$

therefore G acts trivially on $G \curlywedge G$. Thus the defining relations (4.0.1) of $G \curlywedge G$ show that the mapping $G \times G \to G \curlywedge G$ defined by $(x, y) \mapsto x \curlywedge y$ is bilinear. By the above argument, this map induces a well defined bilinear mapping $G^{\mathrm{ab}} \times G^{\mathrm{ab}} \to G \curlywedge G$ given by $(\bar{x}, \bar{y}) \mapsto x \curlywedge y$, where $\bar{x} = x\gamma_2(G)$ and $\bar{y} = y\gamma_2(G)$. This in turn induces a surjective group homomorphism $\Psi : G^{\mathrm{ab}} \wedge G^{\mathrm{ab}} \to G \curlywedge G$ given by $\bar{x} \wedge \bar{y} \mapsto x \curlywedge y$. Similarly, there is a well defined commutator map $G^{\mathrm{ab}} \times G^{\mathrm{ab}} \to \gamma_2(G)$ defined by $(\bar{x}, \bar{y}) \mapsto [x, y]$. Since G is of class 2, the latter mapping is also bilinear, hence it induces a surjective homomorphism $\Phi : G^{\mathrm{ab}} \wedge G^{\mathrm{ab}} \to \gamma_2(G)$. We have that $\Phi = \tilde{\kappa}\Psi$.

Proposition 5.1. *Let G be a group of class 2. Then $\tilde{\mathrm{B}}_0(G)$ is isomorphic to $\ker \Phi / \ker \Psi$.*

Proof. Clearly $\ker \Psi \leq \ker \Phi$. Let τ be the restriction of Ψ to $\ker \Phi$. Take any $k \in \ker \tilde{\kappa}$. There exists $t \in G^{\mathrm{ab}} \wedge G^{\mathrm{ab}}$ such that $\Psi(t) = k$. Then $\Phi(t) = 0$, hence τ maps $\ker \Phi$ onto $\ker \tilde{\kappa}$. Besides, $\ker \tau = \ker \Psi$, thus $\ker \Phi / \ker \Psi \cong \ker \tilde{\kappa} \cong \tilde{\mathrm{B}}_0(G)$. □

Remark 5.2. We also have group homomorphisms $\Psi_1 : G^{\mathrm{ab}} \otimes G^{\mathrm{ab}} \to G \curlywedge G$ and $\Phi_1 : G^{\mathrm{ab}} \otimes G^{\mathrm{ab}} \to \gamma_2(G)$, defined similarly as above. It can be shown that $\tilde{\mathrm{B}}_0(G) \cong \ker \Phi_1 / \ker \Psi_1$.

Bogomolov [3] found a rather detailed description of $\mathrm{B}_0(G)$ when G is a p-group of class 2 such that G^{ab} is an elementary abelian p-group. Here we propose an alternative approach via the Blackburn-Evens theory [2]. First note that both $\gamma_2(G)$ and $G \curlywedge G$ are elementary abelian p-groups. Denote $V = G^{\mathrm{ab}}$ and $W = \gamma_2(G)$. We can consider V and W as vector spaces over $\mathbb{F}_p$. For $v_1, v_2 \in V$ denote $(v_1, v_2) =$

$[x_1, x_2]$ where $v_i = x_i\gamma_2(G)$. This gives us a bilinear map $V \times V \to W$. Let X_1 be the subspace of $V \otimes W$ spanned by all $v_1 \otimes (v_2, v_3) + v_2 \otimes (v_3, v_1) + v_3 \otimes (v_1, v_2)$, where $v_i \in V$. Furthermore, define the map $f : V \to W$ by $f(g\gamma_2(G)) = g^p$, and let X_2 be the subspace of $V \otimes W$ spanned by all $v \otimes f(v)$, where $v \in V$. Put $X = X_1 + X_2$. Straightforward verification, cf. [2], shows that the map $\sigma : V \wedge V \to (V \otimes W)/X$ given by $\sigma(v_1 \wedge v_2) = v_1 \otimes f(v_2) + \binom{p}{2} v_2 \otimes (v_1, v_2) + X$ is well defined and $\mathbb{F}_p$-linear. As both $V \wedge V$ and $(V \otimes W)/X$ are elementary abelian p-groups, there exists an elementary abelian p-group M^* with $N \leq M^*$ such that

$$N \cong (V \otimes W)/X \quad \text{and} \quad M^*/N \cong V \wedge V. \tag{5.2.1}$$

Theorem 5.3. *Let G be a finite group of class 2 such that G^{ab} is an elementary abelian p-group. Under the isomorphisms given by* (5.2.1)*, let M/N correspond to* $\ker \Phi$ *in M^*/N, and M_0/N correspond to* $\ker \Psi$ *in M^*/N. Then $\tilde{\mathrm{B}}_0(G) \cong M/M_0$ and $\mathrm{M}_0(G) \cong M_0$.*

Proof. By a result of Blackburn and Evens [2], $M \cong \mathrm{M}(G)$. Proposition 5.1 implies that $\tilde{\mathrm{B}}_0(G) \cong \ker \Phi / \ker \Psi \cong M/M_0$. Since both M and M_0 are elementary abelian, it follows from Theorem 3.2 that $\mathrm{M}_0(G) \cong M_0$. This concludes the proof. □

6. Split extensions and Frobenius groups

In this section all the groups are finite. Let $G = N \rtimes Q$ be a split extension of the group N by Q. Then the Schur multiplier of G can be described by a result of Tahara [33], see also [17, p. 28]. We have that $\mathrm{H}^2(G, \mathbb{Q}/\mathbb{Z})$ is naturally isomorphic to $\mathrm{H}^2(Q, \mathbb{Q}/\mathbb{Z}) \oplus \bar{\mathrm{H}}^2(G, \mathbb{Q}/\mathbb{Z})$, where $\bar{\mathrm{H}}^2(G, \mathbb{Q}/\mathbb{Z}) = \ker \mathrm{res}^G_Q$. Moreover, $\bar{\mathrm{H}}^2(G, \mathbb{Q}/\mathbb{Z})$ fits into the following exact sequence:

$$\begin{aligned} 0 \to \mathrm{H}^1(Q, \mathrm{H}^1(N, \mathbb{Q}/\mathbb{Z})) \to \bar{\mathrm{H}}^2(G, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^2(N, \mathbb{Q}/\mathbb{Z})^Q \\ \to \mathrm{H}^2(Q, \mathrm{H}^1(N, \mathbb{Q}/\mathbb{Z})). \end{aligned} \tag{6.0.1}$$

A description in terms of the nonabelian exterior products is obtained as follows. The commutator map $G \wedge N \to [N, G]$ is a homomorphism of groups. Denote its kernel by $\mathrm{M}(G, N)$. The group $\mathrm{M}(G, N)$ is said to be the *Schur multiplier of the pair* (G, N). Ellis [12] proved that $\mathrm{M}(G) \cong \mathrm{M}(G, N) \oplus \mathrm{M}(Q)$, and $\mathrm{M}(G, N) \cong \ker(\mathrm{M}(G) \to \mathrm{M}(Q))$. Here $\mathrm{M}(G, N)$ is embedded into $\mathrm{M}(G)$ via the the restriction ι_1 of the natural homomorphism $G \wedge N \to G \wedge G$, and the embedding $\iota_2 : \mathrm{M}(Q) \hookrightarrow \mathrm{M}(G)$ is induced by the split surjection $G \rightleftarrows Q$.

Our aim is to describe $\tilde{\mathrm{B}}_0(G)$ in the case when G is a split extension of N by Q. At first we define a subgroup $\bar{\mathrm{M}}_0(G, N)$ of $G \wedge N$ by

$$\bar{\mathrm{M}}_0(G, N) = \langle (a \wedge m)^{-1} (b \wedge n)(n \wedge m) \mid a, b \in G, m, n \in N, [a, b] = 1, {}^b nm = {}^a mn \rangle.$$

It is straightforward to verify that $\mathrm{M}_0(G, N) \leq \bar{\mathrm{M}}_0(G, N) \leq \mathrm{M}(G, N)$.

Theorem 6.1. *Let $G = N \rtimes Q$. Then $\tilde{\mathrm{B}}_0(G) \cong \mathrm{M}(G, N)/\bar{\mathrm{M}}_0(G, N) \oplus \tilde{\mathrm{B}}_0(Q)$.*

Proof. Let $x, y \in G$ commute. We can write $x = n_1^{-1} q_1$ and $y = n_2^{-1} q_2$ for some $n_1, n_2 \in N$ and $q_1, q_2 \in Q$. From $[x, y] = 1$ we obtain ${}^{n_1^{-1}}[q_1, n_2] \cdot {}^{n_1^{-1} n_2^{-1}}[q_1, q_2] \cdot [n_1^{-1}, y] = 1$. As $N \cap Q = 1$, we conclude that $[q_1, q_2] = 1$. Therefore $q_1 \wedge q_2$ as an element of $G \wedge G$ belongs to $\iota_2 \mathrm{M}_0(Q)$, hence it is central in $G \wedge G$ and G acts trivially upon it. Now we have

$$\begin{aligned} x \wedge y &= {}^{n_1^{-1}}(q_1 \wedge n_2^{-1}) \cdot {}^{n_1^{-1} n_2^{-1}}(q_1 \wedge q_2) \cdot (n_1^{-1} \wedge n_2^{-1}) \cdot {}^{n_2^{-1}}(n_1^{-1} \wedge q_2) \\ &= {}^{n_1^{-1} n_2^{-1}} \left({}^{n_2}(q_1 \wedge n_2^{-1}) \cdot {}^{n_1 n_2}(n_1^{-1} \wedge n_2^{-1}) \cdot {}^{[n_2, n_1] n_1}(n_1^{-1} \wedge q_2) \right) (q_1 \wedge q_2) \\ &= {}^{n_1^{-1} n_2^{-1}} \left((q_1 \wedge n_2)^{-1} (q_2 \wedge n_1)(n_1 \wedge n_2) \right) (q_1 \wedge q_2). \end{aligned}$$

From $[x,y]=[q_1,q_2]=1$ we obtain that $[n_2,q_1][q_2,n_1][n_1,n_2]=1$, which is equivalent to ${}^{q_2}n_1n_2={}^{q_1}n_2n_1$. It follows from here that $(q_1\wedge n_2)^{-1}(q_2\wedge n_1)(n_1\wedge n_2)\in \iota_1\bar{\mathrm{M}}_0(G,N)$. This shows that $\mathrm{M}_0(G)\leq \iota_1\bar{\mathrm{M}}_0(G,N)\oplus\iota_2\,\mathrm{M}_0(Q)$. Conversely, it is clear that $\iota_2\,\mathrm{M}_0(Q)\leq \mathrm{M}_0(G)$. Take any generator $\omega=(a\wedge m)^{-1}(b\wedge n)(n\wedge m)$ of $\bar{\mathrm{M}}_0(G,N)$. Then we have that $[a,b]=1$ and ${}^bnm={}^amn$. The above calculation shows that ${}^{n^{-1}m^{-1}}\omega=(n^{-1}a\wedge m^{-1}b)(b\wedge a)$. By our assumptions we have $[n^{-1}a,m^{-1}b]=1$, therefore $\omega\in\mathrm{M}_0(G)$. From here we can finally conclude that $\mathrm{M}_0(G)=\iota_1\bar{\mathrm{M}}_0(G,N)\oplus\iota_2\,\mathrm{M}_0(Q)$, and this proves the assertion. □

The structure of $\tilde{\mathrm{B}}_0(G)$ can further be refined when G is a Frobenius group. A *Frobenius group* [15, p. 496] is a transitive permutation group such that no non-trivial element fixes more than one point and some non-trivial element fixes a point. The subgroup Q of a Frobenius group G fixing a point is called the *Frobenius complement*. By a theorem of Frobenius [15, p. 496], the set

$$N=G\setminus\bigcup_{g\in G}{}^g(Q\setminus\{1\})$$

is a normal subgroup in G called the *Frobenius kernel* N. We have that $G=N\rtimes Q$, and Q acts fixed-point-freely upon N. We have that $Q\cap{}^gQ=1$ for every $g\in G\setminus Q$, and so if $\{g_1,\dots,g_r\}$ is a left transversal of Q in G then we have a *Frobenius partition*

$$G=N\,\dot\cup\,{}^{g_1}Q\,\dot\cup\,\cdots\,\dot\cup\,{}^{g_r}Q, \tag{6.1.1}$$

where the word 'partition' means that the intersection of two different components is 1.

At first we describe the Schur multiplier of a Frobenius group by refining the above mentioned result of Tahara.

Proposition 6.2. *Let G be a Frobenius group with Frobenius kernel N and complement Q. Then* $\mathrm{H}^2(G,\mathbb{Q}/\mathbb{Z})\cong\mathrm{H}^2(N,\mathbb{Q}/\mathbb{Z})^Q\cong\mathrm{M}(G,N)$.

Proof. By Tahara's result we have $\mathrm{H}^2(G,\mathbb{Q}/\mathbb{Z})\cong\mathrm{H}^2(Q,\mathbb{Q}/\mathbb{Z})\oplus\bar{\mathrm{H}}^2(G,\mathbb{Q}/\mathbb{Z})$. The Sylow p-subgroups of Q are cyclic if p is odd, and either cyclic or generalized quaternion groups if $p=2$ [15, Hauptsatz V.8.7]. Thus $\mathrm{H}^2(P,\mathbb{Q}/\mathbb{Z})=0$ for every Sylow p-subgroup P of Q and every prime p dividing the order of Q. It follows from here that $\mathrm{H}^2(Q,\mathbb{Q}/\mathbb{Z})=0$. It remains to show that $\bar{\mathrm{H}}^2(G,\mathbb{Q}/\mathbb{Z})\cong\mathrm{H}^2(N,\mathbb{Q}/\mathbb{Z})^Q$. By [15, Satz V.8.3] we have $\gcd(|N|,|Q|)=1$, which clearly implies $\mathrm{H}^i((Q,\mathrm{H}^1(N,\mathbb{Q}/\mathbb{Z}))=1$ for all $i\geq 1$, hence the exact sequence (6.0.1) gives the isomorphism $\mathrm{H}^2(G,\mathbb{Q}/\mathbb{Z})\cong\mathrm{H}^2(N,\mathbb{Q}/\mathbb{Z})^Q$. The fact that the latter is isomorphic to $\mathrm{M}(G,N)$ follows from the above mentioned result of Ellis. □

Moving on to $\tilde{\mathrm{B}}_0(G)$, where G is a Frobenius group, we first need to describe the structure of commuting pairs in G. In the Frobenius case, these are particularly well behaved, as the following result shows.

Lemma 6.3. *Let G be a Frobenius group with the Frobenius kernel N and complement Q. Let $x,y\in G$ commute. Then either $x,y\in N$ or there exists $g\in G$ such that both x and y belong to ${}^gQ$.*

Proof. Let G have a Frobenius partition as given by (6.1.1). Suppose that $[x,y]=1$ for $x,y\in G\setminus\{1\}$. We may further suppose that at least one of these elements does not belong to N. Assume first that $x\in N$ and $y\notin N$. Without loss of generality we can write $y={}^{g_1}q$ for some $q\in Q$. We have ${}^{xg_1}q={}^{g_1}q$, therefore ${}^{g_1^{-1}x}q=q$. This can be rewritten as ${}^q\left({}^{g_1^{-1}}x\right)={}^{g_1^{-1}}x$. Since Q acts fixed-point-freely on N, we conclude that $q=1$ or $x=1$, a contradiction.

Assume now that x and y belong to different conjugates of Q. Without loss of generality we may assume that $x \in Q$ and $y \in {}^{g_1}Q$ where $g_1 \notin Q$. We can write $y = {}^{g_1}q$, where $q \in Q$ and $g_1 = f_1 q_1$ with $f_1 \in F \setminus \{1\}$ and $q_1 \in Q$. Denote $\tilde{q} = {}^{q_1}q$. From ${}^{x}y = y$ we conclude that ${}^{f_1^{-1}xf_1}\tilde{q} = \tilde{q} \in Q \cap {}^{f_1^{-1}xf_1}Q$. As $\tilde{q} \neq 0$, we obtain that $f_1^{-1}xf_1 \in Q$, hence $x \in {}^{f_1}Q$. But $Q \cap {}^{f_1}Q = 1$, and this is contrary to the assumption that $x \neq 1$. This concludes the proof. □

Corollary 6.4. *Let G be a Frobenius group with the Frobenius kernel N. Then*
$$\tilde{\mathrm{B}}_0(G) \cong \frac{\mathrm{M}(G,N)}{\mathrm{im}(\mathrm{M}_0(N) \to \mathrm{M}(G,N))}.$$

Proof. Denote $N_0 = \mathrm{im}(\mathrm{M}_0(N) \to \mathrm{M}(G,N))$. Let $x, y \in G$ and suppose that $[x,y] = 1$. By Lemma 6.3 we either have that $x, y \in N$ or there exists $g \in G$ such that $x = {}^{g}q_1$ and $y = {}^{g}q_2$ for some $q_1, q_2 \in Q$. We clearly have that $[q_1, q_2] = 1$, hence $x \wedge y = {}^{g}q_1 \wedge {}^{g}q_2 = {}^{g}(q_1 \wedge q_2) = q_1 \wedge q_2$. This shows that $\mathrm{M}_0(G) = \langle x \wedge y \mid [x,y] = 1, \text{either } (x,y) \in N \times N \text{ or } (x,y) \in Q \times Q\rangle$. In view of the above notations we can thus write $\mathrm{M}_0(G) = \iota_1 N_0 \oplus \iota_2 \mathrm{M}_0(Q)$. As $\mathrm{H}^2(Q, \mathbb{Q}/\mathbb{Z}) = 0$, we have that $\mathrm{M}_0(Q) = 0$, and hence the result. □

Corollary 6.5. *Let G be a Frobenius group with the Frobenius kernel N. Then*
$$\mathrm{B}_0(G) = \bigcap_{A \in \mathcal{C}} \ker \mathrm{res}^G_A,$$
where $\mathcal{C}$ is the family of all bicyclic subgroups of N.

Proof. Denote $B_0 = \bigcap_{A \in \mathcal{C}} \ker \mathrm{res}^G_A$. By a result of Bogomolov [3] we have that $\mathrm{B}_0(G) = \bigcap_{A \in \mathcal{B}} \ker \mathrm{res}^G_A$, where $\mathcal{B}$ is the collection of all bicyclic subgroups of G, hence $\mathrm{B}_0(G) \leq B_0$. Now let $\gamma \in B_0$. Fix an arbitrary $B = \langle x, y \rangle \in \mathcal{B}$. If $x, y \in N$, then $B \in \mathcal{C}$, and thus $\mathrm{res}^G_B \gamma = 0$. Otherwise, Lemma 6.3 implies that there exists $g \in G$ such that $x = {}^{g}q_1$ and $y = {}^{g}q_2$ for some $q_1, q_2 \in Q$. Clearly we have $[q_1, q_2] = 1$. As $\mathrm{H}^2({}^{g}Q, \mathbb{Q}/\mathbb{Z}) = 0$, we have $\mathrm{H}^2(G, \mathbb{Q}/\mathbb{Z}) = \ker \mathrm{res}^G_{{}^{g}Q} \leq \ker \mathrm{res}^G_B$, hence we again have $\mathrm{res}^G_B \gamma = 0$. We conclude that $\gamma \in \mathrm{B}_0(G)$. □

Corollary 6.6. *Let G be a Frobenius group with abelian Frobenius kernel. Then $\mathrm{B}_0(G) = 0$.*

Proof. Let N be the Frobenius kernel of G and Q a complement of N in G. As N is abelian, application of Corollary 6.5 gives $\mathrm{B}_0(G) = \ker \mathrm{res}^G_N$. Thus it suffices to show that the map res^G_N is injective. Let $\mathrm{cor}^G_N : \mathrm{H}^2(N, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^2(G, \mathbb{Q}/\mathbb{Z})$ be the cohomological corestriction map. Let p be a prime dividing $|N|$, and denote the restriction of the map res^G_N to the p-part $\mathrm{H}^2(G, \mathbb{Q}/\mathbb{Z})_p$ of $\mathrm{H}^2(G, \mathbb{Q}/\mathbb{Z})$ by $\mathrm{res}^G_N(p)$. Similarly, let $\mathrm{cor}^G_N(p)$ be the restriction of cor^G_N to $\mathrm{H}^2(N, \mathbb{Q}/\mathbb{Z})_p$. Then $\mathrm{cor}^G_N(p)\,\mathrm{res}^G_N(p) : \mathrm{H}^2(G, \mathbb{Q}/\mathbb{Z})_p \to \mathrm{H}^2(G, \mathbb{Q}/\mathbb{Z})_p$ is multiplication by $n = |G : N| = |Q|$. As p is coprime to n, it follows that $\mathrm{res}^G_N(p)$ is injective for every p dividing $|N|$. Therefore res^G_N is injective, as required. □

7. The functor $\tilde{\mathrm{B}}_0$ in K-theory

In this section, the role of the functor $\tilde{\mathrm{B}}_0$ within K-theory is outlined. We first briefly recall some of the basic notions of K-theory. For unexplained notations and further account we refer to Milnor's book [22]. Throughout this section let Λ be a ring with 1. The group $\mathrm{GL}(\Lambda)$ is the direct limit of the chain $\mathrm{GL}(1,\Lambda) \subset \mathrm{GL}(2,\Lambda) \subset \cdots$, where $\mathrm{GL}(n,\Lambda)$ is embedded in $\mathrm{GL}(n+1,\Lambda)$ via $A \mapsto \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}$. Denote by $\mathrm{E}(\Lambda)$ the subgroup of $\mathrm{GL}(\Lambda)$ generated by all elementary matrices, and let $\mathrm{St}(\Lambda)$ be the Steinberg group. Then the K_1 and K_2 functors are given by $\mathrm{K}_1\,\Lambda = \mathrm{GL}(\Lambda)/\,\mathrm{E}(\Lambda)$

and $\mathrm{K}_2\,\Lambda = \ker(\Phi : \mathrm{St}(\Lambda) \twoheadrightarrow \mathrm{E}(\Lambda))$, respectively. It is known that $\mathrm{K}_2\,\Lambda$ is precisely the center of $\mathrm{St}(\Lambda)$, $\mathrm{K}_2\,\Lambda \cong \mathrm{H}_2(\mathrm{E}(\Lambda),\mathbb{Z})$, and the sequence

$$1 \longrightarrow \mathrm{K}_2\,\Lambda \longrightarrow \mathrm{St}(\Lambda) \longrightarrow \mathrm{GL}(\Lambda) \longrightarrow \mathrm{K}_1\,\Lambda \longrightarrow 1$$

is exact.

The fact that $\mathrm{K}_2\,\Lambda$ can be identified with $\mathrm{H}_2(\mathrm{E}(\Lambda),\mathbb{Z})$ suggests the following definition. For a ring Λ set $\tilde{\mathrm{B}}_0\,\Lambda = \tilde{\mathrm{B}}_0(\mathrm{E}(\Lambda))$. This clearly defines a covariant functor from **Ring** to **Ab**. The group $\tilde{\mathrm{B}}_0\,\Lambda$ fits into the exact sequence

$$1 \longrightarrow \tilde{\mathrm{B}}_0\,\Lambda \longrightarrow \mathrm{E}(\Lambda) \curlywedge \mathrm{E}(\Lambda) \longrightarrow \mathrm{E}(\Lambda) \longrightarrow 1.$$

Thus $\tilde{\mathrm{B}}_0\,\Lambda$ can be considered a measure of the extent to which relations among commutators in $\mathrm{GL}(\Lambda)$ fail to be consequences of 'universal' relations of $\mathrm{E}(\Lambda)\curlywedge\mathrm{E}(\Lambda)$. Another description of $\tilde{\mathrm{B}}_0\,\Lambda$ can be obtained via the Steinberg group. Denote $\mathrm{M}_0\,\Lambda = \langle \mathrm{K}(\mathrm{St}(\Lambda)) \cap \mathrm{K}_2\,\Lambda\rangle$. Then we have the following result.

Theorem 7.1. *Let Λ be a ring. Then $\mathrm{E}(\Lambda) \curlywedge \mathrm{E}(\Lambda)$ is naturally isomorphic to $\mathrm{St}(\Lambda)/\,\mathrm{M}_0\,\Lambda$, and $\tilde{\mathrm{B}}_0\,\Lambda \cong \mathrm{K}_2\,\Lambda/\,\mathrm{M}_0\,\Lambda$.*

Proof. The group $\mathrm{St}(\Lambda)$ is the universal central extension of $\mathrm{E}(\Lambda)$ [22, Theorem 5.10]. Since $\mathrm{St}(\Lambda)$ is perfect, it follows from [21] that $\mathrm{St}(\Lambda) \cong \mathrm{E}(\Lambda) \wedge \mathrm{E}(\Lambda)$. The isomorphism $\psi : \mathrm{E}(\Lambda) \wedge \mathrm{E}(\Lambda) \to \mathrm{St}(\Lambda)$ can be chosen so that we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{M}(\mathrm{E}(\Lambda)) \arrow[r] \arrow[d, "\psi|_{\mathrm{M}(\mathrm{E}(\Lambda))}"', "\cong"] & \mathrm{E}(\Lambda) \wedge \mathrm{E}(\Lambda) \arrow[r, "\kappa"] \arrow[d, "\psi"', "\cong"] & \mathrm{E}(\Lambda) \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathrm{K}_2\,\Lambda \arrow[r] & \mathrm{St}(\Lambda) \arrow[r, "\Phi"] & \mathrm{E}(\Lambda) \arrow[r] & 1
\end{tikzcd}$$

From here we get that

$$\mathrm{E}(\Lambda) \curlywedge \mathrm{E}(\Lambda) = (\mathrm{E}(\Lambda) \wedge \mathrm{E}(\Lambda))/\,\mathrm{M}_0(\mathrm{E}(\Lambda)) \cong \mathrm{St}(\Lambda)/\psi(\mathrm{M}_0(\mathrm{E}(\Lambda))).$$

As $\psi(\mathrm{M}_0(\mathrm{E}(\Lambda))) = \langle [x,y] \mid x,y \in \mathrm{St}(\Lambda), [\Phi(x),\Phi(y)] = 1\rangle = \langle [x,y] \mid x,y \in \mathrm{St}(\Lambda), [x,y] \in \mathrm{K}_2\,\Lambda\rangle = \mathrm{M}_0\,\Lambda$, we get the result. □

Theorem 7.1 thus shows that $\tilde{\mathrm{B}}_0\,\Lambda$ is the obstruction to $\mathrm{K}_2\,\Lambda$ being generated by commutators. Alternatively, let $A, B \in \mathrm{E}(\Lambda)$ commute, and choose $a, b \in \mathrm{St}(\Lambda)$ such that $A = \Phi(a)$ and $B = \Phi(b)$. Define $A \star B = [a,b] \in \mathrm{K}_2\,\Lambda$ to be the *Milnor element* induced by A and B, cf. [22, p. 63]. The following is then straightforward.

Proposition 7.2. *Let Λ be a ring. Then $\mathrm{M}_0\,\Lambda = \langle A\star B \mid A, B \in \mathrm{E}(\Lambda), [A,B] = 1\rangle$. Thus $\tilde{\mathrm{B}}_0\,\Lambda = 0$ if and only if $\mathrm{K}_2\,\Lambda$ is generated by Milnor's elements.*

The question as to whether $\mathrm{K}_2\,\Lambda$ is generated by Milnor's elements for every ring Λ was posed by Bass, cf. Problem 3 of [10]. As the group $\mathrm{E}(\Lambda)$ is perfect, the problem is equivalent to the question whether or not every CP extension of $\mathrm{E}(\Lambda)$ is trivial, cf. Proposition 4.4.

Now let $\{x_{ij}^{\lambda} \mid i,j \in \mathbb{N}, \lambda \in \Lambda\}$ be the standard generating set of $\mathrm{St}(\Lambda)$. For $u \in \Lambda^{\times}$ define $w_{ij}(u) = x_{ij}^{u} x_{ji}^{-u^{-1}} x_{ij}^{u}$ and $h_{ij}(u) = w_{ij}(u)w_{ij}(-1)$. For $u, v \in \Lambda^{\times}$ with $uv = vu$ let $\{u,v\} = [h_{ij}(u), h_{ij}(v)]$ be the *Steinberg symbol.* It is known that $\mathrm{K}_2\,\mathbb{Z}$ is generated by the Steinberg symbol $\{-1,-1\}$, cf. [22, Corollary 10.2]. This, together with Theorem 7.1, implies that $\tilde{\mathrm{B}}_0\,\mathbb{Z} = 0$. Similarly, we have the following.

Corollary 7.3. *Let Λ be a commutative semilocal ring. Then $\tilde{\mathrm{B}}_0\,\Lambda = 0$.*

Proof. By a result of Stein and Dennis [31, Theorem 2.7], $\mathrm{K}_2\,\Lambda$ is generated by the Steinberg symbols $\{u,v\}$, where $u, v \in \Lambda^{\times}$, hence the result follows from Theorem 7.1. □

Our next goal is to compute $\tilde{\mathrm{B}}_0(\mathrm{GL}(\Lambda))$ for an arbitrary ring Λ. Dennis [9, Corollary 8] showed that $\mathrm{H}_2(\mathrm{GL}(\Lambda),\mathbb{Z}) \cong \mathrm{K}_2\,\Lambda \oplus \mathrm{H}_2(\mathrm{GL}(\Lambda)^{\mathrm{ab}},\mathbb{Z})$. The drawback is that the splitting is non-canonical. Instead of that, we use a variant of the functor H_2 defined by Dennis. Given a group G, let $G\tilde{\wedge}G$ be the group generated by symbols $x\tilde{\wedge}y$, where $x,y\in G$ are subject to the relations analogous to (2.0.1) and (2.0.2) in the definition of the nonabelian exterior square $G\wedge G$ of the group G, and the relation (2.0.3) is replaced by

$$(x\tilde{\wedge}y)(y\tilde{\wedge}x) = 1 \tag{7.3.1}$$

for all $x,y\in G$. We clearly have the commutator homomorphism $\hat{\kappa} : G\tilde{\wedge}G \to \gamma_2(G)$ given by $x\tilde{\wedge}y \mapsto [x,y]$. Denote $\tilde{\mathrm{H}}_2(G) = \ker\hat{\kappa}$. The latter group has a topological interpretation. Namely, it follows from [6] that $\tilde{\mathrm{H}}_2(G) \cong \pi_4(\Sigma^2\,\mathrm{K}(G,1))$, where $\mathrm{K}(G,1)$ is the classifying space of G. Let $\tilde{\mathrm{M}}_0(G) = \langle x\tilde{\wedge}y \mid x,y\in G, [x,y]=1\rangle$. Then the defining relations of $G\tilde{\wedge}G$ imply that $(G\tilde{\wedge}G)/\tilde{\mathrm{M}}_0(G) \cong G \curlywedge G$ and $\tilde{\mathrm{H}}_2(G)/\tilde{\mathrm{M}}_0(G) \cong \tilde{\mathrm{B}}_0(G)$. If G is perfect, then $(G\tilde{\wedge}G,\hat{\kappa},\tilde{\mathrm{H}}_2(G))$ is the universal central extension of G.

In our context it is crucial to note that there is a canonical split exact sequence

$$1 \longrightarrow \tilde{\mathrm{H}}_2(\mathrm{E}(\Lambda)) \longrightarrow \tilde{\mathrm{H}}_2(\mathrm{GL}(\Lambda)) \longrightarrow \tilde{\mathrm{H}}_2(\mathrm{GL}(\Lambda)^{\mathrm{ab}}) \longrightarrow 1, \tag{7.3.2}$$

cf. [9, Theorem 7]. This facilitates the proof of the following result:

Theorem 7.4. *Let Λ be a ring. Then $\tilde{\mathrm{B}}_0\,\Lambda$ is naturally isomorphic to $\tilde{\mathrm{B}}_0(\mathrm{GL}(\Lambda))$.*

Proof. Let $G = \mathrm{GL}(\Lambda)$ and $E = \mathrm{E}(\Lambda)$. By [9, Theorem 7] we have that $G\tilde{\wedge}G$ is naturally isomorphic to $(E\tilde{\wedge}E)\times(G^{\mathrm{ab}}\tilde{\wedge}G^{\mathrm{ab}})$. Explicitly, there is a pairing $\star : G\times G \to E\tilde{\wedge}E \cong \mathrm{St}(\Lambda)$ that extends the Milnor pairing defined above. This was found by Grayson, see [14] for the details. It turns out [14, p. 27] that the map $\star$ preserves the relations (2.0.1), (2.0.2) and (7.3.1), hence it induces a well defined homomorphism $\star : G\tilde{\wedge}G \to E\tilde{\wedge}E$. We have that $G^{\mathrm{ab}}\tilde{\wedge}G^{\mathrm{ab}} = \tilde{\mathrm{H}}_2(G^{\mathrm{ab}})$, and the pairing $\circ : G\times G \to \tilde{\mathrm{H}}_2(G^{\mathrm{ab}})$ given by $a\circ b = (a\oplus 1)\gamma_2(G)\curlywedge(1\oplus b)\gamma_2(G)$ induces a homomorphism $\circ : G\tilde{\wedge}G \to \tilde{\mathrm{H}}^2(G^{\mathrm{ab}})$. It can be proved [9] that $x\tilde{\wedge}y = (x\star y)(x\circ y)$ for every $x,y\in G$. From the definition of $\star$ it follows that if x and y commute, then $x\star y\in \mathrm{K}_2\,\Lambda$, and the elements $x\circ y$ generate $\tilde{\mathrm{H}}^2(G^{\mathrm{ab}})$. Therefore $\tilde{\mathrm{M}}_0(G) = \tilde{M}_0(E)\times\tilde{\mathrm{H}}^2(G^{\mathrm{ab}})$. This gives the result. □

One of the fundamental results in K-theory is that if Λ is a ring and I an ideal of Λ, then the sequence

$$\mathrm{K}_2(\Lambda,I) \longrightarrow \mathrm{K}_2\,\Lambda \xrightarrow{\tau} \mathrm{K}_2(\Lambda/I) \xrightarrow{\partial} \mathrm{K}_1(\Lambda,I) \longrightarrow \mathrm{K}_1\,\Lambda \longrightarrow \cdots \tag{7.4.1}$$

is exact [22, Theorem 6.2]. In the rest of the section we derive a similar sequence for $\tilde{\mathrm{B}}_0$. To this end, denote $\mathrm{J}(\Lambda,I) = \partial(\mathrm{M}_0(\Lambda/I))$ and $\mathrm{T}(\Lambda,I) = \tau^{-1}(\mathrm{M}_0(\Lambda/I))$. Then we have the following.

Proposition 7.5. *Let Λ be a ring and I an ideal of Λ. Then the sequence*

$$1 \longrightarrow \frac{\mathrm{T}(\Lambda,I)}{\mathrm{M}_0\,\Lambda} \longrightarrow \tilde{\mathrm{B}}_0\,\Lambda \longrightarrow \tilde{\mathrm{B}}_0(\Lambda/I) \longrightarrow \frac{\mathrm{K}_1(\Lambda,I)}{\mathrm{J}(\Lambda,I)} \longrightarrow \mathrm{K}_1\,\Lambda \longrightarrow \cdots$$

is exact.

Proof. The canonical homomorphism $\tau : \mathrm{K}_2\,\Lambda \to \mathrm{K}_2(\Lambda/I)$ induces a homomorphism $\tau^\sharp : \tilde{\mathrm{B}}_0\,\Lambda \to \tilde{\mathrm{B}}_0(\Lambda/I)$, whose kernel is precisely $\mathrm{T}(\Lambda,I)/\mathrm{M}_0\,\Lambda$. By definition, the connecting homomorphism $\partial : \mathrm{K}_2(\Lambda/I) \to \mathrm{K}_1(\Lambda,I)$ induces a natural map $\partial^\sharp : \tilde{\mathrm{B}}_0(\Lambda/I) \to \mathrm{K}_1(\Lambda,I)/\mathrm{J}(\Lambda,I)$, and we have that $\ker\partial^\sharp = \mathrm{M}_0(\Lambda/I)\ker\partial/\mathrm{M}_0(\Lambda/I) = \operatorname{im}\tau^\sharp$. Using the fact that the sequence (7.4.1) is exact, we see that the canonical homomorphism $\sigma : \mathrm{K}_1(\Lambda,I) \to \mathrm{K}_1\,\Lambda$ induces a well defined homomorphism $\sigma^\sharp$:

$\mathrm{K}_1(\Lambda, I)/\mathrm{J}(\Lambda, I) \to \mathrm{K}_1\,\Lambda$. We have that $\operatorname{im}\sigma^\sharp = \operatorname{im}\sigma$, and $\ker\sigma^\sharp = \ker\sigma/\mathrm{J}(\Lambda, I) = \operatorname{im}\partial/\mathrm{J}(\Lambda, I) = \operatorname{im}\partial^\sharp$. This concludes the proof. □

8. Computing $\tilde{\mathrm{B}}_0(G)$

A group G is said to be *polycyclic* if it has a subnormal series $1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$ such that every factor G_{i+1}/G_i is cyclic. A finite group is polycyclic if and only if it is solvable. Computations with polycyclic groups are very efficient, since several algorithmic problems are decidable within this class [30].

Recently Eick and Nickel [11] developed efficient algorithms for computing nonabelian exterior squares and Schur multipliers of (possibly infinite) polycyclic groups. Given a polycyclic group G, one can compute its nonabelian exterior square $G \wedge G$, the crossed pairing $\lambda : G \times G \to G \wedge G$ given by $\lambda(x, y) = x \wedge y$, and the commutator map $\kappa : G \wedge G \to \gamma_2(G)$. The Schur multiplier $\mathrm{H}_2(G, \mathbb{Z})$ is then computed as $\mathrm{M}(G) = \ker\kappa$.

Let G be a finite solvable group. In order to compute $\tilde{\mathrm{B}}_0(G)$ it suffices to efficiently compute $\mathrm{M}_0(G) = \langle x \wedge y \mid x, y \in G, [x, y] = 1\rangle$ as a subgroup of $\mathrm{M}(G)$. One would have to compute the set $\mathcal{C}_G = \{(x, y) \in G \times G \mid [x, y] = 1\}$ of all commuting pairs in G and then to compute $\mathrm{M}_0(G)$ as the group generated by $\{\lambda(x, y) \mid (x, y) \in \mathcal{C}_G\}$. It turns out that this is computationally inefficient. The first improvement is to observe that if $(x, y) \in \mathcal{C}_G$, then also $({}^z x, {}^z y) \in \mathcal{C}_G$ for every $z \in G$. On the other hand, since G acts trivially on $\mathrm{M}(G)$, we have that ${}^z x \wedge {}^z y = {}^z(x \wedge y) = x \wedge y$, therefore it suffices to determine the conjugacy classes $C_1, \ldots, C_k$ and choose representatives $c_i \in C_i$, $i = 1, \ldots, k$. Then $\mathrm{M}_0(G) = \langle c_i \wedge x \mid c_i \in C_i, x \in C_G(c_i), i = 1, \ldots, k\rangle$. This can further be improved. For $x \in G$ consider the map $\varphi : C_G(x) \to \ker\kappa$ given by $y \mapsto x \wedge y$. Let $y, z \in C_G(x)$. Then $x \wedge yz = (x \wedge y)({}^y x \wedge {}^y z) = (x \wedge y)(x \wedge z)$, as G acts trivially on $\ker\kappa$. Thus φ is a homomorphism. It follows from here that if $\mathcal{X}_i$ is a generating set of $C_G(c_i)$, $i = 1, \ldots, k$, then

$$\mathrm{M}_0(G) = \langle c_i \wedge x \mid c_i \in C_i, x \in \mathcal{X}_i, i = 1, \ldots, k\rangle.$$

This formula enables efficient computation of $\mathrm{M}_0(G)$, as it provides a reasonably small set of generators of this group. The algorithm has been implemented in GAP [13]. It allows us to compute $\tilde{\mathrm{B}}_0(G)$ and $G \curlywedge G$ for finite solvable groups G. A file of the GAP functions and commands for computing $\tilde{\mathrm{B}}_0(G)$ can be found at the author's website [23].

Computer experiments reveal that there are no groups G of order 32 with $\mathrm{B}_0(G) \neq 0$. This coincides with the hand calculations done by Chu, Hu, Kang and Prokhorov [8]. Next, there are nine groups G of order 64 with $\mathrm{B}_0(G) \neq 0$. If we denote the i-th group in the GAP library of all groups of order n by $G_n(i)$, then our computations using the above algorithm show that we have $B(G_{64}(i)) \neq 0$ for $i \in \{149, 150, 151, 170, 171, 172, 177, 178, 182\}$. In fact, in all these cases $\mathrm{B}_0(G_{64}(i))$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. This confirms the calculations of Chu, Hu, Kang, and Kunyavskiĭ [7, Theorem 10.8].

Bogomolov [3, Lemma 4.11] stated that if G is a group with $G^{\mathrm{ab}} \cong \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ and $\mathrm{B}_0(G) \neq 0$, then $p > 3$ and $|G|$ has to be at least p^7. His methods also imply that if G is a p-group with $\mathrm{B}_0(G) \neq 0$, then $|G| \geq p^6$, cf. [4, Corollary 2.11]. On the other hand, our computations show that if $i \in \{28, 29, 30\}$, then $G_{243}(i)^{\mathrm{ab}} \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ and $\mathrm{B}_0(G_{243}(i)) \cong \mathbb{Z}/3\mathbb{Z}$. These can be double-checked by hand calculations using the methods of [7], and thus contradict both of the above Bogomolov's claims. We only sketch here the relevant computations with the group $G_{243}(28)$.

Example 8.1. Denote $G = G_{243}(28)$. This group has the following polycyclic presentation:

$$G = \langle g_1, g_2, g_3, g_4, g_5 \mid g_1^3 = 1,\ g_2^3 = g_4^2,\ g_3^3 = g_5^2,\ g_4^3 = 1,\ g_5^3 = 1,\ [g_2, g_1] = g_3, \\ [g_3, g_1] = g_4,\ [g_3, g_2] = g_5,\ [g_4, g_1] = g_5,\ [g_i, g_j] = 1 \text{ for other } i > j\rangle.$$

Computations with GAP show that $G \wedge G$ is isomorphic to $G_{243}(34)$, and is generated by the set $\{g_2 \wedge g_1,\ g_3 \wedge g_1,\ g_3 \wedge g_2,\ g_4 \wedge g_1\}$. Denote $w = (g_2 \wedge g_3)(g_4 \wedge g_1)$. We have that $|w| = 9$, and since $[g_2, g_3][g_4, g_1] = 1$, it follows that $w \in \mathrm{M}(G)$. Further inspection of $G \wedge G$ reveals that $\mathrm{M}(G) = \langle w \rangle$ and $\mathrm{M}_0(G) \cong \langle w^3 \rangle$, therefore $\mathrm{B}_0(G) \cong \mathbb{Z}/3\mathbb{Z}$.

We have managed to find all solvable groups G of order ≤ 729, apart from the orders 512, 576 and 640, with $\mathrm{B}_0(G) \neq 0$. The numbers of such groups are given in Table 1. As for the timings, it takes, for example, about seven seconds to compute $\mathrm{B}_0(G)$ for a given group G of order 729. We note here that the algorithm works well even for reasonably larger solvable groups. For example, the free 2-generator Burnside group $B(2, 4)$ of exponent 4 has order 2^{12}, and our algorithm returns $\mathrm{B}_0(B(2, 4)) \cong \mathbb{Z}/2\mathbb{Z}$.

n	# of groups of order n	# of G with $\mathrm{B}_0(G) \neq 0$
64	267	9
128	2328	230
192	1543	54
243	67	3
256	56092	5953
320	1640	54
384	20169	1820
448	1396	54
486	261	3
704	1387	54
729	504	85

TABLE 1. Numbers of groups G with $\mathrm{B}_0(G) \neq 0$.

REFERENCES

[1] M. Artin, and D. Mumford, *Some elementary examples of unirational varieties which are not rational*, Proc. London. Math. Soc. (3) **25** (1972), 75–95.

[2] N. Blackburn, and L. Evens, *Schur multipliers of p-groups*, J. Reine Angew. Math. **309** (1979), 100–113.

[3] F. A. Bogomolov, *The Brauer group of quotient spaces by linear group actions*, Izv. Akad. Nauk SSSR Ser. Mat **51** (1987), no. 3, 485–516.

[4] F. A. Bogomolov, J. Maciel, T. Petrov, *Unramified Brauer groups of finite simple groups of type A_ℓ*, Amer. J. Math. **126** (2004), 935–949.

[5] K. S. Brown, *Cohomology of groups*, Springer-Verlag, New York, Heidelberg, Berlin, 1982.

[6] R. Brown, and J.-L. Loday, *Van Kampen theorems for diagrams of spaces*, Topology **26** (1987), no. 3, 311–335.

[7] H. Chu, S. Hu., M. Kang, and B. E. Kunyavskiĭ, *Noether's problem and the unramified Brauer group for groups of order 64*, Int. Math. Res. Not. IMRN **12** (2010), 2329–2366.

[8] H. Chu, S. Hu., M. Kang, and Y. G. Prokhorov, *Noether's problem for groups of order 32*, J. Algebra **320** (2008), 3022–3035.

[9] R. K. Dennis, *In search of new "homology" functors having a close relationship to K-theory*, Preprint, Cornell University, Ithaca, NY, 1976.

[10] R. K. Dennis, and M. R. Stein, *The functor* K_2*: a survey of computations and problems*, Algebraic K-theory, II: "Classical" algebraic K-theory and connections with arithmetic (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), pp. 243–280. Lecture Notes in Math. Vol. 342, Springer, Berlin, 1973.

[11] B. Eick, and W. Nickel, Computing the Schur multiplicator and the nonabelian tensor square of a polycyclic group, J. Algebra **320** (2008), 927–944.

[12] G. Ellis, *The Schur multiplier of a pair of groups*, Appl. Categ. Struct. **6** (1998), 355–371.

[13] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.4.12*; 2008, (`http://www.gap-system.org`).

[14] D. R. Grayson, K_2 *and the K-theory of automorphisms*, J. Algebra **58** (1979), 12–30.

[15] B. Huppert, *Endliche Gruppen*, Springer-Verlag, Berlin, 1967.

[16] S. V. Ivanov, *The free Burnside groups of sufficiently large exponents*, Internat. J. Algebra Comput. **4** (1994), 1–308.

[17] G. Karpilovsky, *The Schur multiplier*, London Mathematical Society Monographs. New Series, 2. The Clarendon Press, Oxford University Press, New York, 1987.

[18] B. E. Kunyavskiĭ, *The Bogomolov multiplier of finite simple groups*, Cohomological and geometric approaches to rationality problems, 209–217, Progr. Math., 282, Birkhäuser Boston, Inc., Boston, MA, 2010.

[19] R. C. Lyndon, *Cohomology theory of groups with a single defining relation*, Ann. of Math. (2) **52** (1950), 650–665.

[20] W. Magnus, A. Karrass, and D. Solitar, *Combinatorial group theory*, Dover Publications, New York, 1976.

[21] C. Miller, *The second homology of a group*, Proc. Amer. Math. Soc. **3** (1952), 588–595.

[22] J. Milnor, *Introduction to algebraic K-theory*, Princeton University Press and University of Tokio Press, Princeton, New Jersey, 1971.

[23] P. Moravec, *GAP code for computing the unramified Brauer groups of finite solvable groups*; 2010, (`www.fmf.uni-lj.si/~moravec/b0g.g`).

[24] B. B. Newman, *Some results on one-relator groups*, Bull. Amer. Math. Soc. **74** (1968), 568–571.

[25] E. Noether, *Gleichungen mit vorgeschriebener Gruppe*, Math. Ann. **78** (1916), 221–229.

[26] A. Yu. Ol'shanskii, *Geometry of defining relations in groups*, Kluwer Academic Publishers, Dordrecht, Boston, London, 1991.

[27] I. R. Šafarevič, *The Lüroth problem*, Proc. Steklov Inst. Math. **183** (1991), 241–246.

[28] D. J. Saltman, *Noether's problem over an algebraically closed field*, Invent. Math. **77** (1984), 71–84.

[29] I. Schur, *Untersuchungen über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen*, J. Reine Angew. Math. **132** (1907), 85–137.

[30] C. C. Sims, *Computation with finitely presented groups*, Encyclopedia of Mathematics and its Applications, vol. 48, Cambridge University Press, 1994.

[31] M. R. Stein, and R. K. Dennis, K_2 *of radical ideals and semi-local rings revisited*, Algebraic K-theory, II: "Classical" algebraic K-theory and connections with arithmetic (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), pp. 281–303. Lecture Notes in Math. Vol. 342, Springer, Berlin, 1973.

[32] R. G. Swan, *Noether's problem in Galois theory*, in 'Emmy Noether in Bryn Mawr', Springer-Verlag, Berlin, 1983.

[33] K. Tahara, *On the second cohomology groups of semidirect products*, Math. Z. **129** (1972), 365–379.

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