GROUPS OF ORDER p^5 AND THEIR UNRAMIFIED BRAUER GROUPS

PRIMOŽ MORAVEC

ABSTRACT. We prove that if p is a prime, $p > 3$, and G is a group of order p^5 not belonging to the 10-th isoclinism family, then the unramified Brauer group of G is trivial.

1. Introduction

Let G be a finite group and V a faithful representation of G over $\mathbb{C}$. Then there is a natural action of G upon the field of rational functions $\mathbb{C}(V)$. A problem posed by Emmy Noether [19] asks as to whether the field of G-invariant functions $\mathbb{C}(V)^G$ is purely transcendental over $\mathbb{C}$, i.e., whether the quotient space V/G is *rational*. A question related to the above mentioned is whether V/G is *stably rational*, that is, whether there exist independent variables $x_1, \dots, x_r$ such that $\mathbb{C}(V)^G(x_1, \dots, x_r)$ becomes a pure transcendental extension of $\mathbb{C}$. This problem has close connection with Lüroth's problem [21] and the inverse Galois problem [24, 22]. It is known that the stable rationality of V/G does not depend upon the choice of V, but only on the group G. Saltman [22] found examples of groups G of order p^9 such that V/G is not stably rational over $\mathbb{C}$. His main method was application of the unramified cohomology group $\mathrm{H}^2_{\mathrm{nr}}(\mathbb{C}(V)^G, \mathbb{Q}/\mathbb{Z})$ as an obstruction. A version of this invariant had been used before by Artin and Mumford [1] who constructed unirational varieties over $\mathbb{C}$ that were not rational. Bogomolov [3] further explored this cohomology group. He proved that $\mathrm{H}^2_{\mathrm{nr}}(\mathbb{C}(V)^G, \mathbb{Q}/\mathbb{Z})$ is canonically isomorphic to

$$\mathrm{B}_0(G) = \bigcap_{\substack{A \leq G,\\ A \text{ abelian}}} \ker \mathrm{res}^G_A, \tag{1.0.1}$$

where $\mathrm{res}^G_A : \mathrm{H}^2(G, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^2(A, \mathbb{Q}/\mathbb{Z})$ is the usual cohomological restriction map. The group $\mathrm{B}_0(G)$ is a subgroup of the *Schur multiplier* $\mathrm{H}^2(G, \mathbb{Q}/\mathbb{Z})$ of G. Kunyavskiĭ [14] coined the term the *Bogomolov multiplier* of G for the group $\mathrm{B}_0(G)$. Bogomolov used the above description to find new examples of groups G of order p^6 with $\mathrm{H}^2_{\mathrm{nr}}(\mathbb{C}(V)^G, \mathbb{Q}/\mathbb{Z}) \neq 0$. Subsequently, Bogomolov, Maciel and Petrov [5] showed that $\mathrm{B}_0(G) = 0$ when G is a finite simple group of Lie type A_ℓ, whereas Kunyavskiĭ [14] recently proved that $\mathrm{B}_0(G) = 0$ for every quasisimple or almost simple group G.

The Bogomolov multiplier of a given finite group is hard to compute. To illustrate this, we note that Chu, Hu, Kang, and Kunyavskiĭ [7] only recently computed Bogomolov multipliers of all groups of order 64. In our paper [17], we have obtained a new description of $\mathrm{B}_0(G)$ for a finite group G. It relies on the notion of the *nonabelian exterior square* $G \wedge G$ of the group G (see Section 2 for definition and further details). To state the main result of [17], denote by $\mathrm{M}(G)$ be the kernel

Date: October 3, 2012.
2010 *Mathematics Subject Classification.* 13A50, 14E08, 14M20, 12F12.
Key words and phrases. Unramified Brauer group, Bogomolov multiplier.

of the commutator homomorphism $\kappa : G \wedge G \to [G,G]$, and $\mathrm{M}_0(G) = \langle x \wedge y \mid x, y \in G, [x,y] = 1\rangle$. Then $\mathrm{B}_0(G)$ is isomorphic to $\mathrm{M}(G)/\mathrm{M}_0(G)$. One of the advantages of this approach is that it provides a purely combinatorial description of $\mathrm{B}_0(G)$ in terms of presentation of G. Among other things, this leads to a Hopf-type formula for $\mathrm{B}_0(G)$, and an efficient algorithm for computing $\mathrm{B}_0(G)$ when G is a finite solvable group. This method enables fast computer calculations of $\mathrm{B}_0(G)$ for a reasonably large finite solvable group G, cf. [17] for further details.

In his seminal paper [3], Bogomolov claimed that if G is a group of order p^5, where p is a prime, then $\mathrm{B}_0(G)$ is trivial. Moreover, it had been conjectured that if G is a group of order p^5, then $\mathbb{C}(V)^G$ is always purely transcendental over $\mathbb{C}$. This is indeed true for $p = 2$ and was confirmed by Chu, Hu, Kang, and Prokhorov [8]. On the other hand, we showed [17] that there are precisely three groups of order 3^5 with nontrivial Bogomolov multiplier. This shows that both Bogomolov's claim and the above conjecture are false. More recently, Hoshi and Kang [11] proved that for every odd prime p there exist groups G of order p^5 with $\mathrm{B}_0(G) \neq 0$. They posed a problem of classification of all such groups G.

The purpose of this paper is to find all groups of order p^5 with trivial Bogomolov multiplier. A complete list of all groups of order p^5 is known by the work of James [13]. In his paper, the nonabelian groups of order p^5 are divided into isoclinism families Φ_k, $2 \le k \le 10$. Hoshi and Kang [11] proved the following result.

Theorem 1.1 (Hoshi, Kang, [11])**.** *Let p be an odd prime and let G be a group of order p^5 that belongs to the family Φ_{10}. Then $\mathrm{B}_0(G) \neq 0$.*

Our main result shows that these groups are essentially the only ones having non-trivial Bogomolov multiplier:

Theorem 1.2. *Let G be a group of order p^5, $p > 3$. If G does not belong to the family Φ_{10}, then $\mathrm{B}_0(G) = 0$.*

The case $p = 3$ is already dealt with in [11] by theoretical means, and in [17] using computational methods. It turns out that a group order 3^5 has non-trivial Bogomolov multiplier if and only if it belongs to Φ_{10}. For the sake of simplicity of the proofs we therefore assume that $p > 3$. The main step in the proof of the above theorem is Proposition 3.2 which ensures a sufficient condition for a p-group of nilpotency class ≤ 3 to have trivial Bogomolov multiplier. This condition can be read off from a polycyclic presentation of the group. The result we obtain covers the families Φ_2, Φ_3, Φ_4, and Φ_6, which form the major part of groups in question. The remaining families are then dealt with separately. The techniques we use are purely combinatorial, and do not require any cohomological machinery. The fact that every group in Φ_{10} has nontrivial unramified Brauer group follows from Hoshi and Kang [11].

In 1987, Bogomolov [3] announced a full classificiation of groups of order p^6 with nontrivial B_0, yet no such classification has appeared. We note here that all groups of order p^6 have been listed by James [13]. Our techniques could be applied and extended to find all groups among these that have nontrivial unramified Brauer group, but the calculations would become quite lengthy.

We have recently become aware of a paper by Hoshi, Kang, and Kunyavskiĭ [12] where a full classification of groups of order p^5 with non-trivial Bogomolov multiplier has appeared. In fact, their results are more general and also deal with isomorphisms of the corresponding fields. We note here that our techniques are essentially different, and our argument is more elementary.

Notations. Let G be a group and $x, y \in G$. We use the notation $x^y = y^{-1}xy$ for conjugation from the right. The commutator $[x,y]$ of elements x and y is defined by

$[x, y] = x^{-1}y^{-1}xy = x^{-1}x^y$. Commutators of higher weight are defined inductively by $[x_1, \dots, x_n] = [[x_1, \dots, x_{n-1}], x_n]$ for $x_1, \dots, x_n \in G$. We use shorthand notation $[x, {}_n y]$ for the commutator $[x, y, \dots, y]$ with n copies of y. When referring to the classification of groups of order p^5, we closely follow the notations of James [13].

2. The nonabelian exterior square of a group

We recall the definition and basic properties of the nonabelian exterior square of a group. The reader is referred to [6, 15] for more thorough accounts on the theory and its generalizations. Let G be a group. We form the group $G \wedge G$, generated by the symbols $m \wedge n$, where $m, n \in G$, subject to the following relations:

$$m_1 m \wedge n = (m_1^m \wedge n^m)(m \wedge n), \tag{2.0.1}$$

$$m \wedge n_1 n = (m \wedge n)(m^n \wedge n_1^n), \tag{2.0.2}$$

$$m \wedge m = 1, \tag{2.0.3}$$

for all $m, m_1, n, n_1 \in G$. The group $G \wedge G$ is said to be the *nonabelian exterior square* of G. By definition, the commutator map $\kappa : G \wedge G \to [G, G]$, given by $g \wedge h \mapsto [g, h]$, is a well defined homomorphism of groups. Clearly $\mathrm{M}(G) = \ker \kappa$ is central in $G \wedge G$, and G acts trivially via diagonal action on $\mathrm{M}(G)$. Miller [15] proved that there is a natural isomorphism between $\mathrm{M}(G)$ and $\mathrm{H}_2(G, \mathbb{Z})$.

An alternative way of obtaining $G \wedge G$ is the following. Let G be a group and let G^φ be an isomorphic copy of G via the mapping $\varphi : g \mapsto g^\varphi$ for all $g \in G$. We define the group $\tau(G)$ to be

$$\tau(G) = \langle G, G^\varphi \mid [g, h^\varphi]^x = [g^x, (h^x)^\varphi] = [g, h^\varphi]^{x^\varphi}, [g, g^\varphi] = 1 \, \forall x, g, h \in G \rangle.$$

The groups G and G^φ embed into $\tau(G)$. In the rest of the paper we consider the group $[G, G^\varphi]$ as a subgroup of $\tau(G)$. Then the map $\phi : G \wedge G \to [G, G^\varphi]$ defined by $(g \wedge h)^\phi = [g, h^\varphi]$ for all g and h in G is an isomorphism. This was proved independently by Ellis and Leonard [9] and Rocco [20], and later further explored by Blyth and Morse [2].

The advantage of the above description of $G \wedge G$ is the ability of using the full power of the commutator calculus instead of computing with elements of $G \wedge G$. The following lemma collects various properties of $\tau(G)$ and $[G, G^\varphi]$ that will be used in the proof of the main result.

Lemma 2.1 (Blyth and Morse [2])**.** *Let G be a group.*

(a) *If G is nilpotent of class c, then $\tau(G)$ is nilpotent of class at most $c + 1$.*
(b) *If $[G, G]$ is nilpotent of class c, then $[G, G^\varphi]$ is nilpotent of class c or $c + 1$.*
(c) *If G is nilpotent of class ≤ 2, then $[G, G^\varphi]$ is abelian.*
(d) *$[g, h^\varphi] = [g^\varphi, h]$ for all $g, h \in G$.*
(e) *$[g, h, k^\varphi] = [g, h^\varphi, k] = [g^\varphi, h, k] = [g^\varphi, h^\varphi, k] = [g^\varphi, h, k^\varphi] = [g, h^\varphi, k^\varphi]$ for all $g, h, k \in G$.*
(f) *$[[g_1, h_1^\varphi], [g_2, h_2^\varphi]] = [[g_1, h_1], [g_2, h_2]^\varphi]$ for all $g_1, h_1, g_2, h_2 \in G$.*
(g) *If g and h are commuting elements of G of orders m and n, respectively, then the order of $[g, h^\varphi]$ divides* $\gcd(m, n)$.

Let G be a finite solvable group. Then G is *polycyclic*, i.e., it has a subnormal series $G = G_1 \triangleright G_2 \triangleright \cdots \triangleright G_{n+1} = 1$ such that every factor G_i/G_{i+1} is cyclic of order r_i. A *polycyclic generating sequence* of G is a sequence $g_1, \dots, g_n$ of elements of G such that $G_i = \langle G_{i+1}, g_i \rangle$ for all $1 \leq i \leq n$. The value r_i is called the *relative order* of g_i. With respect to a given polycyclic generating sequence $g_1, \dots, g_n$, each element g of G can be represented in a unique way (*normal form*) as a product $g = g_1^{e_1} g_2^{e_2} \cdots g_n^{e_n}$ with exponents $e_i \in \{0, \dots, r_i - 1\}$. Given a polycyclic generating sequence, the group G can be presented by a *polycyclic presentation*, cf. [23] for further details.

The following lemma will be basic in our considerations.

Lemma 2.2 (Proposition 20 of [2]). *Let G be a finite solvable group with a polycyclic generating sequence $g_1,\dots,g_n$. Then the group $[G,G^\varphi]$, considered as a subgroup of $\tau(G)$, is generated by the set $\{[g_i,g_j^\varphi]\mid i,j=1,\dots,n, i>j\}$.*

3. Groups of order p^5

Groups of order p^5 were classified by James [13]. He compiled a list of polycyclic presentations of these groups, and divided the non-abelian ones into families denoted by $\Phi_2,\dots,\Phi_{10}$, according to isoclinism. In the case $p=3$, James uses the notation Δ instead of Φ.

As already mentioned, there are no groups G of order 2^5 with $\mathrm{B}_0(G)\neq 0$, see [8]. Also, there are precisely three groups of order 3^5 with nontrivial Bogomolov multiplier. These groups are described in [11] and [17]. According to James's notation, they are the groups $\Delta_{10}(2111)a_i$, $i=1,2,3$. From here on we assume that $p\geq 5$. In this section we prove Theorem 1.2 using the theory developed in [17]. The main idea is the following. Suppose that G is a finite group. Let ϕ be the natural isomorphism between $G\wedge G$ and $[G,G^\varphi]$. Let $\kappa^*:[G,G^\varphi]\to[G,G]$ be the composite of ϕ^{-1} and κ, and denote $\mathrm{M}^*(G)=\mathrm{M}(G)^\phi=\ker\kappa^*$ and $\mathrm{M}_0^*(G)=\mathrm{M}_0(G)^\phi$. Then $\mathrm{B}_0(G)$ is clearly isomorphic to $\mathrm{M}^*(G)/\mathrm{M}_0^*(G)$ by [17]. In order to prove that $\mathrm{B}_0(G)=0$ for a given group G it suffices to show that $\mathrm{M}^*(G)=\mathrm{M}_0^*(G)$. This can be achieved by finding a generating set of $\mathrm{M}^*(G)$ consisting solely of elements of $\mathrm{M}_0^*(G)$.

Lemma 3.1. *Let G be a nilpotent group of class ≤ 3. Then*

$$[x,y^n]=[x,y]^n[x,y,y]^{\binom{n}{2}}[x,y,y,y]^{\binom{n}{3}}$$

for all $x,y\in\tau(G)$ and every positive integer n.

Proof. Since the class of G is at most 3, it follows that $\tau(G)$ is nilpotent of class ≤ 4 by Lemma 2.1 (a). In particular if $x,y\in\tau(G)$, then $\langle x,y\rangle$ is metabelian. The assertion now follows easily by induction on n, see, for example, Lemma 3 of Hogan and Kappe [10]. □

In the following, we say that an element g of a polycyclic generating sequence of G is *absolute* if its relative order is equal to the order of g.

Proposition 3.2. *Let G be a p-group, $p>3$, and suppose that G is nilpotent of class ≤ 3. Let $g_1,\dots,g_n$ be a polycyclic generating sequence of G. Suppose that all nontrivial commutators $[g_i,g_j]$, where $i>j$, are different absolute elements of the polycyclic generating sequence. Then $\mathrm{B}_0(G)=0$.*

Proof. The group $[G,G^\varphi]$ is generated by the set $\mathcal{G}=\{[g_i,g_j^\varphi]\mid i>j\}$. As above, let κ^* be the commutator map $[G,G^\varphi]\to[G,G]$, and suppose that $[g_{i_1},g_{j_1}],\dots,[g_{i_\ell},g_{j_\ell}]$ are the nontrivial elements of $\mathcal{G}^{\kappa^*}$. Since $[G,G^\varphi]$ is nilpotent of class ≤ 2 by Lemma 2.1 (b), every word $w\in[G,G^\varphi]$ can be written as

$$w=\prod_{k=1}^{\ell}[g_{i_k},g_{j_k}^\varphi]^{n_k}\cdot\tilde{w},$$

where $\tilde{w}\in\mathrm{M}_0^*(G)$. Mapping w with κ^*, we get

$$w^{\kappa^*}=\prod_{k=1}^{\ell}[g_{i_k},g_{j_k}]^{n_k}.$$

Let p^{r_k} be the order of $[g_{i_k},g_{j_k}]$. Since these commutators are different absolute terms of the polycyclic generating sequence of G, it follows that $w\in\mathrm{M}^*(G)=\ker\kappa^*$

if and only if p^{r_k} divides n_k for every i. We have that $[g_{i_k}, g_{j_k}^{\varphi}]^{p^{r_k}}$ belongs to $\mathrm{M}^*(G)$. On the other hand,

$$\begin{aligned}
[g_{i_k}, g_{j_k}^{\varphi}]^{p^{r_k}} &= [g_{i_k}^{p^{r_k}}, g_{j_k}^{\varphi}][g_{j_k}^{\varphi}, g_{i_k}, g_{i_k}]^{\binom{p^{r_k}}{2}}[g_{j_k}^{\varphi}, g_{i_k}, g_{i_k}, g_{i_k}]^{\binom{p^{r_k}}{3}} \\
&= [g_{i_k}^{p^{r_k}}, g_{j_k}^{\varphi}][[g_{j_k}^{\varphi}, g_{i_k}]^{\binom{p^{r_k}}{2}}, g_{i_k}][[g_{j_k}^{\varphi}, g_{i_k}]^{\binom{p^{r_k}}{3}}, g_{i_k}, g_{i_k}].
\end{aligned}$$

Since $p > 3$, it follows that $[[g_{j_k}^{\varphi}, g_{i_k}]^{\binom{p^{r_k}}{2}}, g_{i_k}]$ and $[[g_{j_k}^{\varphi}, g_{i_k}]^{\binom{p^{r_k}}{3}}, g_{i_k}, g_{i_k}]$ belong to $\mathrm{M}_0^*(G)$. In addition to that, since G is nilpotent of class ≤ 3, we get $[g_{j_k}, g_{i_k}^{p^{r_k}}] = [g_{j_k}, g_{i_k}]^{p^{r_k}}[g_{j_k}, g_{i_k}, g_{i_k}]^{\binom{p^{r_k}}{2}} = [[g_{j_k}, g_{i_k}]^{\binom{p^{r_k}}{2}}, g_{i_k}] = 1$, hence $[g_{i_k}^{p^{r_k}}, g_{j_k}^{\varphi}] \in \mathrm{M}_0^*(G)$. Thus we conclude that $[g_{i_k}, g_{j_k}^{\varphi}]^{p^{r_k}}$ belongs to $\mathrm{M}_0^*(G)$. By the above we clearly have that $\mathrm{M}^*(G) = \langle [g_{i_k}, g_{j_k}^{\varphi}]^{p^{r_k}} \mid k = 1, \ldots, \ell \rangle \, \mathrm{M}_0^*(G) = \mathrm{M}_0^*(G)$ and therefore $\mathrm{B}_0(G) = 0$. □

Proposition 3.2 can be applied to groups of order p^5. For example, the group $\Phi_6(221)a$ has a polycyclic presentation $\langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_i] = \beta_i = \alpha_i^p, \beta^p = \beta_i^p = 1, (i = 1, 2) \rangle$ that satisfies the assumptions of Proposition 3.2, hence $\mathrm{B}_0(\Phi_6(221)a) = 0$. By closely inspecting other polycyclic presentations of groups of order p^5 in [13], we obtain the following.

Corollary 3.3. *Let G be a group of order p^5, $p > 3$. If G belongs to one of the families Φ_2, Φ_3, Φ_4 or Φ_6, then $\mathrm{B}_0(G) = 0$.*

We deal with the remaining families separately. At first we exhibit those with trivial Bogomolov multiplier.

Proposition 3.4. *Let G be a group of order p^5, $p > 3$. If G belongs to the family Φ_5, then $\mathrm{B}_0(G) = 0$.*

Proof. There are two groups in the family Φ_5, by James's notations [13] we denote them by $\Phi_5(2111)$ and $\Phi_5(1^5)$. They are both nilpotent of class 2. We only prove the result for $G = \Phi_5(2111)$, the proof for $\Phi_5(1^5)$ is almost identical. The group G has a polycyclic presentation

$$G = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta \mid [\alpha_1, \alpha_2] = [\alpha_3, \alpha_4] = \alpha_1^p = \beta, \alpha_2^p = \alpha_3^p = \alpha_4^p = \beta^p = 1 \rangle,$$

where all the relations of the form $[x, y] = 1$ between the generators have been omitted from the list. A generating set of the group $[G, G^{\varphi}]$ can be found by Lemma 2.2. It consists of $[\alpha_1, \alpha_2^{\varphi}]$, $[\alpha_3, \alpha_4^{\varphi}]$, $[\alpha_1, \alpha_3^{\varphi}]$, $[\alpha_1, \alpha_4^{\varphi}]$, $[\alpha_2, \alpha_3^{\varphi}]$, $[\alpha_2, \alpha_4^{\varphi}]$, and $[\alpha_i, \beta^{\varphi}]$, $i = 1, 2, 3, 4$. Apart from the first two, all of these generators belong to $\mathrm{M}_0^*(G)$. Since $[G, G^{\varphi}]$ is abelian, every element $w \in [G, G^{\varphi}]$ can be written as $w = [\alpha_1, \alpha_2^{\varphi}]^m [\alpha_3, \alpha_4^{\varphi}]^n \tilde{w}$, where $\tilde{w} \in \mathrm{M}_0^*(G)$. Then $w^{\kappa^*} = \beta^{m+n}$, hence $w \in \mathrm{M}^*(G)$ if and only if p divides $m + n$. Now, since $\tau(G)$ is nilpotent of class ≤ 3, we get

$$1 = [\alpha_1^{\varphi}, \alpha_2^p] = [\alpha_1^{\varphi}, \alpha_2]^p [\alpha_1^{\varphi}, \alpha_2, \alpha_2]^{\binom{p}{2}} = [\alpha_1^{\varphi}, \alpha_2]^p,$$

hence $[\alpha_1, \alpha_2^{\varphi}]$ is of order p. Similarly, $[\alpha_3, \alpha_4^{\varphi}]$ is of order p. It follows that

$$\mathrm{M}^*(G) = \langle [\alpha_1, \alpha_2^{\varphi}][\alpha_3, \alpha_4^{\varphi}]^{-1} \rangle \, \mathrm{M}_0^*(G).$$

Note that $1 = [\alpha_1, \alpha_2][\alpha_4, \alpha_3] = [\alpha_1\alpha_4, \alpha_2\alpha_3]$, hence $[\alpha_1\alpha_4, (\alpha_2\alpha_3)^{\varphi}] \in \mathrm{M}_0^*(G)$. Expanding the latter using the class restriction and Lemma 2.1, we get

$$\begin{aligned}
[\alpha_1\alpha_4, (\alpha_2\alpha_3)^{\varphi}] &= [\alpha_1, \alpha_2^{\varphi}\alpha_3^{\varphi}][\alpha_1, \alpha_2^{\varphi}\alpha_3^{\varphi}, \alpha_4][\alpha_4, \alpha_2^{\varphi}\alpha_3^{\varphi}] \\
&= [\alpha_1, \alpha_2^{\varphi}\alpha_3^{\varphi}][\alpha_1, \alpha_2\alpha_3, \alpha_4^{\varphi}][\alpha_4, \alpha_3^{\varphi}][\alpha_4, \alpha_2^{\varphi}][\alpha_4, \alpha_2^{\varphi}, \alpha_3^{\varphi}] \\
&= [\alpha_1, \alpha_3^{\varphi}][\alpha_1, \alpha_2^{\varphi}][\alpha_1, \alpha_2, \alpha_3^{\varphi}][\alpha_1, \alpha_2, \alpha_4^{\varphi}][\alpha_1, \alpha_3, \alpha_4^{\varphi}] \\
&\quad \cdot [\alpha_4, \alpha_3^{\varphi}][\alpha_4, \alpha_2^{\varphi}][\alpha_4, \alpha_2, \alpha_3^{\varphi}] \\
&= [\alpha_1, \alpha_3^{\varphi}][\alpha_1, \alpha_2^{\varphi}][\beta, \alpha_3^{\varphi}][\beta, \alpha_4^{\varphi}][\alpha_4, \alpha_3^{\varphi}][\alpha_4, \alpha_2^{\varphi}].
\end{aligned}$$

Note that $[\alpha_1,\alpha_3^\varphi]$, $[\beta,\alpha_3^\varphi]$, $[\beta,\alpha_4^\varphi]$, and $[\alpha_4,\alpha_2^\varphi]$ belong to $\mathrm{M}_0^*(G)$. From here it follows that $[\alpha_1,\alpha_2^\varphi][\alpha_3,\alpha_4^\varphi]^{-1}\in \mathrm{M}_0^*(G)$, hence $\mathrm{B}_0(G)=0$. □

Proposition 3.5. *Let G be a group of order p^5, $p>3$. If G belongs to the family Φ_7, then $\mathrm{B}_0(G)=0$.*

Proof. The family Φ_7 consists of five groups. James [13] denotes them by $\Phi_7(2111)a$, $\Phi_7(2111)b_r$, where $r=1$ or r is the smallest non-quadratic residue mod p, $\Phi_7(2111)c$, and $\Phi_7(1^5)$. These groups are all nilpotent of class 3.

At first we deal with the case $G=\Phi_7(1^5)$. The group G has a polycyclic presentation

$$G=\langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta\mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_1,\beta]=\alpha_3,\\ \alpha^p=\alpha_1^{(p)}=\alpha_{i+1}^p=\beta^p=1,(i=1,2)\rangle,$$

where $\alpha_1^{(p)}$ denotes $\alpha_1^p\alpha_2^{\binom{p}{2}}\alpha_3^{\binom{p}{3}}$. Since $p>3$, the relation $\alpha_1^{(p)}=1$, together with other power relations, implies $\alpha_1^p=1$.

The group $[G,G^\varphi]$ is generated modulo $\mathrm{M}_0^*(G)$ by $[\alpha_1,\alpha^\varphi]$, $[\alpha_2,\alpha^\varphi]$, and $[\alpha_1,\beta^\varphi]$. Since the nilpotency class of $[G,G^\varphi]$ is at most 2, every element $w\in[G,G^\varphi]$ can be written as $w=[\alpha_1,\alpha^\varphi]^{m_1}[\alpha_2,\alpha^\varphi]^{m_2}[\alpha_1,\beta^\varphi]^{m_3}\tilde{w}$, where $\tilde{w}\in\mathrm{M}_0^*(G)$. This gives $w^{\kappa^*}=\alpha_2^{m_1}\alpha_3^{m_2+m_3}$, therefore $w\in\mathrm{M}^*(G)$ if and only if p divides m_1 and m_2+m_3. By Lemma 3.1 we have for $i=1,2$ that

$$\begin{aligned}1&=[\alpha^\varphi,\alpha_i^p]\\ &=[\alpha^\varphi,\alpha_i]^p[\alpha^\varphi,\alpha_i,\alpha_i]^{\binom{p}{2}}[\alpha^\varphi,\alpha_i,\alpha_i,\alpha_i]^{\binom{p}{3}}\\ &=[\alpha^\varphi,\alpha_i]^p[\alpha_{i+1},\alpha_i^\varphi]^{\binom{p}{2}}\\ &=[\alpha^\varphi,\alpha_i]^p,\end{aligned}$$

and similarly $[\alpha_i,\beta^\varphi]^p=1$. From here it follows that $\mathrm{M}^*(G)$ is generated modulo $\mathrm{M}_0^*(G)$ by $[\alpha_2,\alpha^\varphi][\alpha_1,\beta^\varphi]^{-1}$, thus we need to show that $[\alpha_2,\alpha^\varphi][\alpha_1,\beta^\varphi]^{-1}\in\mathrm{M}_0^*(G)$. At first we observe that $[\alpha_2\beta,\alpha\alpha_1]=[\alpha_2,\alpha_1]^\beta[\alpha_2,\alpha]^{\alpha_1\beta}[\beta,\alpha_1][\beta,\alpha]^{\alpha_1}=1$, hence $[\alpha_2\beta,(\alpha\alpha_1)^\varphi]\in\mathrm{M}_0^*(G)$. Expansion gives

$$\begin{aligned}[\alpha_2\beta,(\alpha\alpha_1)^\varphi]&=[\alpha_2,\alpha_1^\varphi]^\beta[\alpha_2,\alpha^\varphi]^{\alpha_1^\varphi\beta}[\beta,\alpha_1^\varphi][\beta,\alpha^\varphi]^{\alpha_1^\varphi}\\ &=[\alpha_2,\alpha_1^\varphi]^\beta[\alpha_2,\alpha^\varphi][\alpha_2,\alpha^\varphi,\alpha_1^\varphi\beta][\beta,\alpha_1^\varphi][\beta,\alpha^\varphi]^{\alpha_1^\varphi}.\end{aligned}$$

Observe that $[\alpha_2,\alpha_1^\varphi]$, $[\alpha_2,\alpha^\varphi,\alpha_1^\varphi\beta]$, and $[\beta,\alpha^\varphi]$ all belong to $\mathrm{M}_0^*(G)$. Thus we conclude that $[\alpha_2,\alpha^\varphi][\alpha_1,\beta^\varphi]^{-1}=[\alpha_2,\alpha^\varphi][\beta,\alpha_1^\varphi]\in\mathrm{M}_0^*(G)$, as required.

The proofs for $\Phi_7(2111)a$ and $\Phi_7(2111)c$ are almost the same and are thus omitted. In the case of $G=\Phi_7(2111)b_r$, a polycyclic presentation of G is

$$G=\langle\alpha,\alpha_1,\alpha_2,\alpha_3,\beta\mid[\alpha_1,\alpha]=\alpha_2,[\alpha_2,\alpha]=\alpha_3,[\alpha_1,\beta]^r=\alpha_3^r=\alpha_1^{(p)},\\ \alpha^p=\alpha_2^p=\alpha_3^p=\beta^p=1\rangle,$$

where r is as above. Write $\bar\beta=\beta^r$. Then the group G is generated by $\alpha,\alpha_1,\alpha_2,\alpha_3,\bar\beta$, the relation $\beta^p=1$ is equivalent to $\bar\beta^p=1$, and the relation $[\alpha_1,\beta]^r=\alpha_3^r$ is equivalent to $[\alpha_1,\bar\beta]=\alpha_3^r$. It is now easy to adapt the above argument to show that $\mathrm{B}_0(G)=0$, hence we skip the details. □

Proposition 3.6. *Let G be a group of order p^5, $p>3$. If G belongs to the family Φ_8, then $\mathrm{B}_0(G)=0$.*

Proof. The only group in the family Φ_8 is

$$G=\Phi_8(32)=\langle\alpha_1,\alpha_2,\beta\mid[\alpha_1,\alpha_2]=\beta=\alpha_1^p,\beta^{p^2}=\alpha_2^{p^2}=1\rangle.$$

G is nilpotent of class 3, and $[G, G^\varphi]$ is nilpotent of class ≤ 2, generated modulo $\mathrm{M}_0^*(G)$ by $[\alpha_1, \alpha_2^\varphi]$. We have

$$\begin{aligned} 1 &= [\alpha_1^\varphi, \alpha_2^{p^2}] \\ &= [\alpha_1^\varphi, \alpha_2]^{p^2} [\alpha_1^\varphi, \alpha_2, \alpha_2]^{\binom{p^2}{2}} [\alpha_1^\varphi, \alpha_2, \alpha_2, \alpha_2]^{\binom{p^2}{3}} \\ &= [\alpha_1^\varphi, \alpha_2]^{p^2} [\beta, \alpha_2^\varphi]^{\binom{p^2}{2}} [\beta, \alpha_2, \alpha_2^\varphi]^{\binom{p^2}{3}} \\ &= [\alpha_1^\varphi, \alpha_2]^{p^2}. \end{aligned}$$

Every element $w \in [G, G^\varphi]$ can be written as $w = [\alpha_1, \alpha_2^\varphi]^m \tilde{w}$ with $\tilde{w} \in \mathrm{M}_0^*(G)$. Thus w belongs to $\mathrm{M}^*(G)$ if and only if $\beta^m = 1$, which in turn is equivalent to $p^2 | m$. It follows that $\mathrm{M}^*(G) = \mathrm{M}_0^*(G)$, hence $\mathrm{B}_0(G) = 0$. □

The families Φ_9 and Φ_{10} consist of groups G of order p^5 that are nilpotent of class 4. Since every group of order p^5 is metabelian, it follows that $[G, G^\varphi]$ is nilpotent of class ≤ 2. Besides, $\tau(G)$ is nilpotent of class ≤ 5. When expanding commutators in $\tau(G)$, we will need the following lemma.

Lemma 3.7 (Lemma 9 of [16])**.** *Let H be a nilpotent group of class ≤ 5, n a positive integer and $x, y \in H$. Then*

$$[x^n, y] = [x, y]^n [x, y, x]^{\binom{n}{2}} [x, y, x, x]^{\binom{n}{3}} [x, y, x, x, x]^{\binom{n}{4}} [x, y, x, [x, y]]^{\sigma(n)},$$

where $\sigma(n) = n(n-1)(2n-1)/6$.

Proposition 3.8. *Let G be a group of order p^5, $p > 3$. If G belongs to the family Φ_9, then $\mathrm{B}_0(G) = 0$.*

Proof. The groups belonging to the family Φ_9 are denoted by $\Phi_9(2111)a$, $\Phi_9(2111)b_r$, where $r + 1 = 1, 2, \ldots, \gcd(p-1, 3)$, and $\Phi_9(1^5)$.

All the groups in Φ_9 have a polycyclic generating sequence $\alpha, \alpha_1, \ldots, \alpha_4$ satisfing the relations $[\alpha_i, \alpha] = \alpha_{i+1}$, where $i = 1, 2, 3$. These are the only nontrivial commutator relations between these generators. Thus the group $[G, G^\varphi]$ is generated modulo $\mathrm{M}_0^*(G)$ by $[\alpha_i, \alpha^\varphi]$, where $i = 1, 2, 3$. Since the nilpotency class of $[G, G^\varphi]$ is at most 2, every element w of $[G, G^\varphi]$ can be written as $w = \prod_{i=1}^3 [\alpha_i, \alpha^\varphi]^{m_i} \tilde{w}$, where $\tilde{w} \in \mathrm{M}_0^*(G)$. We have that $w^{\kappa^*} = \prod_{i=1}^3 \alpha_{i+1}^{m_i}$. In all cases, the elements α_{i+1}, where $i = 1, 2, 3$, have order p, hence $w \in \mathrm{M}^*(G)$ if and only if $p | m_i$, $i = 1, 2, 3$.

In the cases when $G = \Phi_9(1^5)$ or $G = \Phi_9(2111)a$, the power relations imply that $\alpha_1^p = 1$. For $i = 1, 2, 3$ we get by Lemma 3.7 that

$$\begin{aligned} 1 &= [\alpha_i^p, \alpha^\varphi] \\ &= [\alpha_i, \alpha^\varphi]^p [\alpha_i, \alpha^\varphi, \alpha_i]^{\binom{p}{2}} [\alpha_i, \alpha^\varphi, {}_2\alpha_i]^{\binom{p}{3}} [\alpha_i, \alpha^\varphi, {}_3\alpha_i]^{\binom{p}{4}} [\alpha_i, \alpha^\varphi, \alpha_i, [\alpha_i, \alpha^\varphi]]^{\sigma(p)} \\ &= [\alpha_i, \alpha^\varphi]^p [\alpha_{i+1}, \alpha_i^\varphi]^{\binom{p}{2}} [\alpha_{i+1}, \alpha_i, \alpha_i^\varphi]^{\binom{p}{3}} [\alpha_{i+1}, \alpha_i, \alpha_i, \alpha_i^\varphi]^{\binom{p}{4}} [\alpha_{i+1}, \alpha_i^\varphi, [\alpha_i, \alpha^\varphi]]^{\sigma(p)} \\ &= [\alpha_i, \alpha^\varphi]^p. \end{aligned}$$

From here it follows that $\mathrm{M}^*(G) = \mathrm{M}_0^*(G)$, hence $\mathrm{B}_0(G) = 0$.

When $G = \Phi_9(2111)b_r$, the defining relations imply that $\alpha^p = \alpha_1^{p^2} = 1$. By the above calculations we see that $[\alpha_{i+1}, \alpha^\varphi]$, $i = 1, 2, 3$, are elements of order p. On the other hand, expansion of $1 = [\alpha^p, \alpha_1^\varphi]$ yields that $[\alpha_1, \alpha^\varphi]$ has order p as well. Therefore $\mathrm{M}^*(G) = \mathrm{M}_0^*(G)$, and thus $\mathrm{B}_0(G) = 0$. □

This finishes the proof of Theorem 1.2. We remark here that a recent result of [18] shows that groups within an isoclinism family have isomorphic unramified Brauer groups. This has been generalized by Bogomolov and Böhning [4] who showed that isoclinic groups have stably equivalent generically free linear quotients.

References

[1] M. Artin, and D. Mumford, *Some elementary examples of unirational varieties which are not rational*, Proc. London. Math. Soc. (3) **25** (1972), 75–95.

[2] R. D. Blyth, and R. F. Morse, *Computing the nonabelian tensor square of polycyclic groups*, J. Algebra **321** (2009), no. 8, 2139–2148.

[3] F. A. Bogomolov, *The Brauer group of quotient spaces by linear group actions*, Izv. Akad. Nauk SSSR Ser. Mat **51** (1987), no. 3, 485–516.

[4] F. Bogomolov, and C. Böhning, *Isoclinism and stable cohomology of wreath products*, arXiv:1204.4747v1, 2012.

[5] F. A. Bogomolov, J. Maciel, T. Petrov, *Unramified Brauer groups of finite simple groups of type A_ℓ*, Amer. J. Math. **126** (2004), 935–949.

[6] R. Brown, and J.-L. Loday, *Van Kampen theorems for diagrams of spaces*, Topology **26** (1987), no. 3, 311–335.

[7] H. Chu, S. Hu., M. Kang, and B. E. Kunyavskiĭ, *Noether's problem and the unramified Brauer group for groups of order 64*, Int. Math. Res. Not. IMRN **12** (2010), 2329–2366.

[8] H. Chu, S. Hu., M. Kang, and Y. G. Prokhorov, *Noether's problem for groups of order 32*, J. Algebra **320** (2008), 3022–3035.

[9] G. Ellis, and F. Leonard, *Computing Schur multipliers and tensor products of finite groups*, Proc. Roy. Irish Acad. Sect. A **95** (1995), no. 2, 137–147.

[10] G. T. Hogan, and W. P. Kappe, *On the H_p-problem for finite p-groups*, Proc. Amer. Math. Soc. **20** (1969), no. 2, 450–454.

[11] A. Hoshi, and M. Kang, *Unramified Brauer groups for groups of order p^5*, ArXiv:1109.2966v1, 2011.

[12] A. Hoshi, M. Kang, and B. E. Kunyavskiĭ, *Noether problem and unramified Brauer groups*, ArXiv:1202.5812v1, 2012.

[13] R. James, *The groups of order p^6 (p an odd prime)*, Math. Comp. **34** (1980), 613–637.

[14] B. E. Kunyavskiĭ, *The Bogomolov multiplier of finite simple groups*, Cohomological and geometric approaches to rationality problems, 209–217, Progr. Math., 282, Birkhäuser Boston, Inc., Boston, MA, 2010.

[15] C. Miller, *The second homology of a group*, Proc. Amer. Math. Soc. **3** (1952), 588–595.

[16] P. Moravec, *On the exponent semigroups of finite p-groups*, J. Group Theory **11** (2008), no. 4, 511-524.

[17] P. Moravec, *Unramified groups of finite and infinite groups*, Amer. J. Math., to appear.

[18] P. Moravec, *Unramified Brauer groups and isoclinism*, arXiv:1203.2422v1, 2012.

[19] E. Noether, *Gleichungen mit vorgeschriebener Gruppe*, Math. Ann. **78** (1916), 221–229.

[20] N. R. Rocco, *A construction related to the nonabelian tensor square of a group*, Bol. Soc. Brasil Mat. (N.S.) **22** (1991), no. 1, 63–79.

[21] I. R. Šafarevič, *The Lüroth problem*, Proc. Steklov Inst. Math. **183** (1991), 241–246.

[22] D. J. Saltman, *Noether's problem over an algebraically closed field*, Invent. Math. **77** (1984), 71–84.

[23] C. C. Sims, *Computation with finitely presented groups*, Encyclopedia of Mathematics and its Applications, vol. 48, Cambridge University Press, 1994.

[24] R. G. Swan, *Noether's problem in Galois theory*, in 'Emmy Noether in Bryn Mawr', Springer-Verlag, Berlin, 1983.

Department of Mathematics, University of Ljubljana, Jadranska 21, 1000 Ljubljana, Slovenia

E-mail address: `primoz.moravec@fmf.uni-lj.si`