On power endomorphisms of n-central groups

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Abstract. A group G is said to be n-central if the factor group $G/Z(G)$ is of exponent n. We improve a result of Gupta and Rhemtulla by showing that every 4-central group is 16-abelian and every 6-central group is 36-abelian. There are examples of finite groups which show that these bounds are best possible. Consequently, we completely describe the structure of exponent semigroups of free non-cyclic n-central groups for $n = 2, 3, 4, 6$. We obtain a characterization of metabelian p-central groups and a classification of finitely generated 2-central groups. We compute the nilpotency class of the free metabelian 4-central group of arbitrary finite rank.
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1. Introduction

The groups of finite exponent play a prominent role in group theory. Two of the most important problems that occur in connection with these groups are the so called Burnside problem and the restricted Burnside problem. For a survey about these two problems the reader is refered to the M. Vaughan–Lee's book [18].

The purpose of this paper is the study of center-by-finite-exponent groups. More precisely, a group G is said to be n-central if $G/Z(G)$ is a group of exponent n. This type of groups was initially considered by N. D. Gupta and A. H. Rhemtulla [7]. In particular, they noted that if an integer n is such that the free two-generator Burnside group $B(2, n)$ of exponent n is finite, then there exists an integer $1 < f(n) \leq |B(2, n)|$ such that the map $x \mapsto x^{f(n)}$ is a group endomorphism of every n-central group. Saying that a group G is k-abelian if $(xy)^k = x^k y^k$ for any $x, y \in G$, we can state the following question of [7]: Does there exist for a given positive integer n, an integer $f(n) > 1$ such that every n-central group is $f(n)$-abelian? The answer to this question is negative in general, as it was shown by S. I. Adjan [1]. On the other hand, the answer is certainly positive for $n = 2, 3, 4, 6$ by the solution of Burnside problem for these exponents. Apart from the crude bound $f(n) = |B(2, n)|$, the following result can be found in [7].

Theorem A ([7], Theorem 1)**.**

(a) *Every* 2*-central group is* 4*-abelian.*
(b) *Every* 3*-central group is* 9*-abelian.*
(c) *Every* 4*-central group is* 32*-abelian.*

There is a question whether the bounds for $f(2)$, $f(3)$ and $f(4)$ in Theorem A are the best possible. It turns out that this is true for 2-central and 3-central groups. However, the bound for 4-central groups can be improved.

Theorem 1.1. *Every* 4*-central group is* 16*-abelian.*

The result is best possible in a sense that there exists a 4-central group (even finite and metabelian) which is not k-abelian for any $1 < k < 16$.

Beside that, there appears to be no reasonably good estimate for $f(6)$ in [7] or elsewhere. Using computer calculations, we are able to construct the free two-generator 6-central group. This gives us the following result.

Theorem 1.2. *Every* 6*-central group is* 36*-abelian.*

Here it is worth mentioning that there exists a finite nilpotent 6-central group which is not k-abelian for $1 < k < 36$. By investigating the structure of nilpotent 6-central groups in more detail, we also give a characterization of nilpotent 6-central groups in terms of the powers of certain Engel words; see Corollary 4.3.

In [19, 20] E. I. Zel'manov solved the celebrated restricted Burnside problem by proving that there is a largest finite quotient $R(r, n)$ of the free r-generator group $B(r, n)$ of exponent n. Now, let G be a locally soluble or locally finite n-central group. Let $a, b \in G$. Then $\langle a, b\rangle/Z(\langle a, b\rangle)$ is a homomorphic image of $R(2, n)$, hence $\langle a, b\rangle'$ is a finite group of exponent e dividing $|R(2, n)|$ by Schur's theorem [16, Part 1, Theorem 4.12]. Now it is easy to see that $(ab)^k = a^k b^k$ for $k = ne$, therefore every locally soluble or locally finite n-central group is $g(n)$-abelian for some $g(n) > 1$. Focusing on n-central metabelian groups, we prove that every n-central metabelian group is $2n^2$-abelian; if n is odd, then such a group is even n^2-abelian (Proposition 2.8). This result is best possible at least in the case when n is an odd prime. If $h(n) > 1$ is the least integer such that every metabelian n-central group is $h(n)$-abelian, then one might conjecture that $h(n) \geq n^2$ for all n's. However, this fails to be true. Namely, it turns out that $h(8) = 32$; this is proved in Section 5.

Dealing with soluble n-central groups, one can reduce the problem to a consideration of finite soluble n-central groups; see Theorem 2.5. As a consequence, we obtain the following characterization of metabelian p-central groups.

Theorem 1.3. *Let p be a prime and let G be a metabelian group. Then G is p-central if and only if the exponent of G' divides p and G is nilpotent of class $\leq p$.*

This is an extension of Theorem 13 of [13] which gives a characterization of metabelian p-central p-groups. It also generalizes the well–known result of Meier-Wunderli [16, Part 2, pp. 50] that a metabelian group of exponent p is nilpotent of class at most p. A similar result in this direction, Corollary 2.7, is a classification of finitely generated 2-central groups, which depends on a classification of finite 2-central groups obtained in [5].

Another question arising in the study of metabelian n-central groups is the following. Suppose that F_r is the free metabelian 4-central group of rank r. What is the nilpotency class of F_r? From a result of [8] it follows directly that F_r is of class $\leq r + 3$ when $r = 2, 3$, and is of class $\leq r + 2$ when $r > 3$. Lifting the identities, which hold in the free r-generator group of exponent four, to F_r and using some commutator calculations, we compute the exact class of F_r.

Theorem 1.4. *Let $r > 1$ and let F_r be the r-generator free metabelian* 4*-central group. Then F_r is nilpotent and the class of F_r is* 5 *when $r = 2, 3$, and is $r + 1$ when $r > 3$.*

The paper is organized as follows. In Section 2 we derive some general properties of soluble n-central groups. Section 3 is devoted to 4-central groups. There we prove Theorem 1.1 and Theorem 1.4. In sections 4 and 5 we deal with 6-central groups and metabelian 8-central groups. Using the results of L.-C. Kappe [12], we completely describe the arithmetic structure of exponent semigroups of free non-cyclic n-central groups in Section 6. We also deal with the following generalization of n-central groups. Define a group G to be (k,n)-central if the factor group $G/Z_k(G)$ is of exponent n. The results obtained for n-central groups can be extended to the class of (k,n)-central groups in a natural way. This is briefly mentioned in Section 7.

The notation is mainly taken from [16] and [18]. The standard commutator identities [16, Part 1, Section 2.1] will be used without further reference.

2. Soluble n-central groups

In the beginning we mention some well–known identities which hold in metabelian groups; for a proof see [13].

Lemma 2.1. *Let G be a metabelian group, $x,y,z \in G$ and $c,d \in G'$. Then we have:*

(a) $[c,x,y] = [c,y,x]$.
(b) $[x,y,z] = [y,x,z]^{-1}$.
(c) $[cd,x] = [c,x][d,x]$.
(d) $[x,y,z][y,z,x][z,x,y] = 1$ *(Jacobi identity).*
(e) $[x,y^n] = \prod_{1\le i\le n} [x,{}_iy]^{\binom{n}{i}}$.
(f) $(xy^{-1})^n = x^n \cdot \prod_{0<i+j<n} [x,{}_iy,{}_jx]^{\binom{n}{i+j+1}} \cdot y^{-n}$.

Here is a basic characterization of n-central groups:

Lemma 2.2. *Let G be a group. The following statements are equivalent.*

(a) *G is an n-central group.*
(b) $G^n \le Z(G)$.
(c) $[x^n,y] = 1$ *for any $x,y \in G$.*
(d) $(xy)^n = (yx)^n$ *for any $x,y \in G$.*

The proof of this lemma is elementary and will be omitted.

Our first result provides a starting point for the calculation of exponents of terms of the lower central series in a metabelian n-central group.

Lemma 2.3. *Let G be a metabelian n-central group. Then the exponent of G' divides n^2 and $\gamma_3(G)$ is of exponent dividing n.*

Proof. Let $x,y \in G$. Since $[x,y]^n$ belongs to the center of G, we have $[x,y,z]^n = [[x,y]^n,z] = 1$ for any $z \in G$. Now we have

$$1 = [x,y^n]^n = \prod_{i=1}^{n} [x,{}_iy]^{n\binom{n}{i}} = [x,y]^{n^2},$$

hence the assertion is proved. $\square$

Corollary 2.4. *Let G be a soluble n-central group. Then the derived subgroup G' has a finite exponent dividing $n^{2(d-1)}$, where d is the derived length of G.*

Proof. We prove this by induction on the derived length d of group G. By Lemma 2.3 we may assume that $d > 2$. Since $G^{(d-2)}$ is metabelian, it follows from Lemma 2.3 that $\exp G^{(d-1)}$ divides n^2. By the assumption, the group $(G/G^{(d-1)})' = G'/G^{(d-1)}$ also has a finite exponent dividing $n^{2(d-2)}$, hence the exponent of G' divides $n^{2(d-1)}$. □

The next result shows that the structure of finite soluble n-central groups tells us the whole story about finitely generated soluble n-central groups. More precisely, we have:

Theorem 2.5. *Let G be a finitely generated soluble group of derived length d and let n be a positive integer. The group G is n-central if and only if G is isomorphic to a subgroup of the direct product of a finite soluble n-central group of derived length $\leq d$ and a free abelian group of finite rank.*

Proof. Let G be a finitely generated soluble n-central group. Then $G/Z(G)$ is finite, hence G' is finite by Schur's theorem [16, Part 1, Theorem 4.12]. Therefore the group G is an FC-group [17, Theorem 1.1]. From a proof of a result of Černikov – see, for instance, [17, Theorem 1.7] – it follows that there is a natural embedding of G into $G/A \times G/T$, where T is the torsion subgroup of group G and A is a maximal torsion–free subgroup of G contained in the center of G. The group G/A is finite, soluble of derived length $\leq d$ and n-central, whereas G/T is a free abelian group of finite rank; see [17] for the details. The converse statement is obvious. □

Since every finite p^k-central group is clearly nilpotent, we immediately have the following.

Corollary 2.6. *Let G be a finitely generated soluble group, let p be a prime and let k be a positive integer. The group G is p^k-central if and only if G is isomorphic to a subgroup of the direct product of a finite soluble p^k-central p-group and an abelian group of finite rank.*

By [13, Theorem 13], a metabelian p-group G is p-central if and only if $G'^p = \gamma_{p+1}(G) = 1$. We give a proof of Theorem 1.3 which shows that we may leave out the assumption of being a p-group:

Proof of Theorem 1.3. For $p = 2$, this is proved in [13, Theorem 7], therefore we may assume $p > 2$. Suppose that G is a metabelian p-central group. Let $x, y \in G$. The 2-generator group $\langle x, y\rangle$ has a derived subgroup of exponent dividing p by Corollary 2.6 and [13, Theorem 13], hence $[x, y]^p = 1$ and therefore $G'^p = 1$. Let H be any p-generator subgroup of the group G. Using Corollary 2.6 and [13, Theorem 13] once again, we conclude that H is nilpotent of class $\leq p$. By a well–known result of Heineken [11], G is nilpotent of class $\leq p$. The converse statement follows directly from the identity (e) of Lemma 2.1. □

In [5], there is a classification of finite 2-central groups. Since every 2-central group is metabelian [13, Theorem 7], there is a way to classify finitely generated 2-central groups as follows.

Corollary 2.7. *Let G be a finitely generated group. Then G is 2-central if and only if G is isomorphic to a subgroup of $A \times B$, where A is abelian group of finite rank and B is a subgroup of the direct product of groups each of which is the central product of some of the following groups: cyclic 2-groups, $M_n = \langle x, y \mid x^{2^{n-1}} = y^2 = 1,\ [x,y] = x^{2^{n-2}} \rangle$ for various values of n or the quaternion group of order 8.*

Recall that every locally soluble n-central group is $|R(2,n)|$-abelian. In case of metabelian groups, we can substantially improve this result.

Proposition 2.8. *Let G be a metabelian n-central group. Then G is $2n^2$-abelian. If n is odd, then G is also an n^2-abelian group.*

Proof. Let $x, y \in G$. Then we have

$$(xy^{-1})^n = x^n y^{-n} [x,y]^{\binom{n}{2}} \cdot \prod_{1<i+j<n} [x, {}_i y, {}_j x]^{\binom{n}{i+j+1}}.$$

Since $G^n \leq Z(G)$ and $2n\binom{n}{2}$ is divisible by n^2, we obtain $(xy^{-1})^{2n^2} = x^{2n^2} y^{-2n^2}$ by Lemma 2.3. If we further assume that n is odd, then n divides $\binom{n}{2}$, so we get, using Lemma 2.3, $(xy^{-1})^{n^2} = x^{n^2} y^{-n^2}$, hence G is n^2-abelian. □

When p is a prime, there exist finite metabelian p-central groups which are not k-abelian for any $1 < k < p^2$. To see this, set $F = C_p * C_p$ and let $G = F/F'^p F'' \gamma_{p+1}(F)$. This is a finite metabelian p-group of exponent p^2. By Theorem 1.3, G is a p-central group. If G were p-abelian, it would also be $(p-1)$-Engel by [13, Theorem 11], but this is not the case.

3. 4-central groups

Every 4-central group is 32-abelian by Theorem A. The proof of this result in [7] does not rely on the solution of the Burnside problem for exponent four. Using the information about the structure of groups of exponent 4, we prove that every 4-central group is 16-abelian.

Proof of Theorem 1.1. Let G be a 4-central group and let $a, b \in G$. The group $\langle a,b \rangle / Z(\langle a,b \rangle)$ is a homomorphic image of $B(2,4)$, hence $\langle a,b \rangle$ is nilpotent of class 6 at most [18]. Pick $w, x, y, z \in \{a, b\}$. Using the class restriction, we get $1 = [[w,x,y]^4, z] = [w,x,y,z]^4$, hence

$$\gamma_4(\langle a,b \rangle)^4 = 1. \tag{1}$$

From [18, Section 6.3] it also follows that $[a,b]^2 \in \gamma_4(G)Z(G)$, hence $[[a,b]^2, c] \in \gamma_5(G)$ for any $c \in G$. Since $[[a,b]^2, c] = [a,b,c]^2 [a,b,c,[a,b]]$, we get

$$[a,b,c]^2 \in \gamma_5(G) \tag{2}$$

for any $a, b, c \in G$. Now, let $x_i \in \{a, b\}$, $i = 1, \ldots, 5$. Because of the class restriction for $\langle a,b \rangle$ and (2) we obtain

$$[x_1, x_2, x_3, x_4, x_5]^2 = [[x_1, x_2, x_3]^2, x_4, x_5] = 1,$$

hence

$$\gamma_5(\langle a,b \rangle)^2 = 1. \tag{3}$$

The group $\langle a, b\rangle/\gamma_5(\langle a, b\rangle)$ is metabelian. Using (1) and (2), we get

$$1 = [a, b^4] \equiv [a, b]^4 \mod \gamma_5(\langle a, b\rangle).$$

Observing (3), we obtain $[a, b]^8 = 1$. Since $\langle a, b\rangle$ is nilpotent of class ≤ 6, the commutator collection process [9, pp. 65–66] gives

$$(ab)^{16} \equiv a^{16}b^{16}[b, a]^{\binom{16}{2}}[b, a, a]^{\binom{16}{3}}[b, a, b]^{\binom{16}{2}+2\binom{16}{3}} \mod \gamma_4(\langle a, b\rangle)^4.$$

Using (1), (2) and (3), we get $(ab)^{16} = a^{16}b^{16}$, hence G is 16-abelian. □

Turning our attention to 4-central metabelian groups, we first determine the exponents of terms of the lower central series for such a group.

Proposition 3.1. *Let G be a metabelian 4-central group. Then $G'^8 = \gamma_3(G)^4 = \gamma_4(G)^2 = 1$.*

Proof. Since every 4-central group satisfies the law $[x, y]^8 = 1$ (see the proof of Theorem 1.1), we have $G'^8 = 1$. By Lemma 2.3 we also get $\gamma_3(G)^4 = 1$. Since $G/Z(G)$ is a metabelian group of exponent 4, we obtain $\gamma_3(G/Z(G))^2 = 1$ by [8]. This implies $\gamma_4(G)^2 = 1$. □

Corollary 3.2. *Let G be a metabelian group. The following assertions are equivalent:*

(a) *G is a 4-central group.*
(b) *$G'^8 = \gamma_3(G)^4 = \gamma_4(G)^2 = 1$ and $[x, y]^4[x, y, y]^2[x, y, y, y, y] = 1$ for all $x, y \in G$.*
(c) *$[x, y, y]^4 = [x, y, y, y]^2 = [x, y]^4[x, y, y]^2[x, y, y, y, y] = 1$ for all $x, y \in G$.*

Proof. This follows from Proposition 3.1 and from the identity $[x, y^4] = [x, y]^4[x, y, y]^6[x, y, y, y]^4[x, y, y, y, y]$. □

Next, we give an example of a finite metabelian 4-central group with two generators, which is nilpotent of class 5 and is not k-abelian for any $1 < k < 16$.

Example 3.3. Let D be the largest abelian quotient of the group

$$F = \langle x, y_1, y_2, z_1, z_2, z_3, w_1, w_2, w_3, w_4 \mid x^4 = w_2w_3, y_1^2 = w_1w_2w_3, y_2^2 = w_2w_3w_4, z_i^2 = w_j^2 = 1\rangle.$$

The group D may be viewed as a group generated by the commuting generators $x, y_1, y_2, z_1, z_2, z_3, w_1, w_2, w_3, w_4$ and the remaining relations in D are just those inherited from F. Let $A = [D]\langle a\rangle$ be the semidirect product of the group D by the cyclic group of order 16, where the action of a on the generators of D is given as follows.

$$[x, a] = y_1,\ [y_i, a] = z_i,\ [z_j, a] = w_j,\ [w_k, a] = 1. \tag{1}$$

The relations of A are those of D, (1) and $a^{16} = 1$. Next we form the split extension $G = [A]\langle b\rangle$ generated by the element b of order 16 which induces the following action on A.

$$[x, b] = y_2,\ [y_i, b] = z_{i+1},\ [z_j, b] = w_{j+1},\ [w_k, b] = 1,\ [a, b] = x^{-1}. \tag{2}$$

The relations of G are those of A, (2) and $b^{16} = 1$. We observe that $G = \langle a, b\rangle$, $|G| = 2^{19}$, G is metabelian and nilpotent of class 5 and $\exp(G) = 16$.

It is straightforward to prove that G is 4-central, yet it is not k-abelian for any $1 < k < 16$.

Now we begin with preparations for computing the nilpotency class of the free r-generator metabelian 4-central group F_r. The previous example, together with [8], shows that F_2 is nilpotent of class 5. Next we deal with the three-generator case.

Proposition 3.4. *The free three-generator metabelian 4-central group is nilpotent of class* 5.

Proof. Let $G = \langle a, b, c\rangle$ be a 3-generator metabelian 4-central group. By [8], G is nilpotent of class ≤ 6. In order to prove that the class of G is 5 at most, we have to prove that all the commutators of the form $[a, {}_i b, {}_j a, {}_k c]$, where $i > 0$, $j, k \geq 0$, $i + j + k = 5$, are trivial. Using the fact that the class of $\langle a, b\rangle$ is ≤ 5, we only have to deal with the following cases.

$$\begin{array}{ll} c_1 = [a, b, c, c, c, c] & c_6 = [a, b, b, a, c, c] \\ c_2 = [a, b, a, c, c, c] & c_7 = [a, b, b, a, a, c] \\ c_3 = [a, b, a, a, c, c] & c_8 = [a, b, b, b, c, c] \\ c_4 = [a, b, a, a, a, c] & c_9 = [a, b, b, b, a, c] \\ c_5 = [a, b, b, c, c, c] & c_{10} = [a, b, b, b, b, c] \end{array}$$

Using Proposition 3.1, we have

$$1 = [a, b, c^4] = [a, b, c]^4[a, b, c, c]^6[a, b, c, c, c]^4[a, b, c, c, c, c] = [a, b, c, c, c, c],$$

hence $c_1 = 1$. Similarly, from $[a, b^4, c] = 1$ we get $c_4 = c_{10} = 1$. Using the fact that $G/Z(G)$ is metabelian of exponent four, the following identities follow directly from [8, Lemma 3]:

$$[y, z, x, x, x, w] = 1. \tag{1}$$

$$[x, y, y, y, z, w] = [x, y^2, z^2, w]. \tag{2}$$

$$[x, y, z]^2 = [x, y, x, y, z][x, y, y, y, z][y, x, x, x, z]. \tag{3}$$

Using Proposition 3.1, we get $[x, y^2, z^2] = [x, y, z^2]^2[x, y, y, z^2] = [x, y, y, z, z]$, hence we can rewrite (2) as

$$[x, y, y, y, z, w] = [x, y, y, z, z, w]. \tag{4}$$

The identity (1) implies $c_2 = c_5 = 1$, whereas (4) implies $c_8 = c_5 = 1$, hence also $c_3 = 1$. Commuting (3) with w and using Proposition 3.1, we get $[x, y, x, y, z, w][x, y, y, y, z, w][y, x, x, x, z, w] = 1$. In particular, we get $c_6 = 1$. Replacing a with ca in $[a, b, b, a, c, c] = 1$, we obtain $[c, b, b, a, c, c] = 1$, which implies $c_7 = 1$, hence also $c_9 = 1$. This completes the proof that the nilpotency class of G does not exceed 5. On the other hand, the free 3-generator metabelian group of exponent four is a homomorphic image of the free 3-generator metabelian 4-central group F, hence the nilpotency class of F is at least 5 by [8]. This proves the result. □

It is noted in [4] that if G is a soluble group of exponent four, then the subgroup G^2 is nilpotent of class depending only on the derived length of G. A direct consequence of this result is that if G is a soluble 4-central group of derived length d, then G^2 is nilpotent of class depending only on d. In case of 4-central metabelian groups, we have the following.

Proposition 3.5. *Let G be a 4-central metabelian group. Then the subgroup G^2 is nilpotent of class ≤ 2.*

Proof. Let $a, b, c \in G$. Then we have $[a^2, b^2, c^2] = [a^2, b, c^2]^2[a^2, b, b, c^2] = [a, b, c^2]^4[a, b, a, c^2]^2[a, b, b, c^2]^2[a, b, a, b, c^2]$. Using Proposition 3.1, we obtain $[a^2, b^2, c^2] = [a, b, a, b, c^2] = [a, b, a, b, c]^2[a, b, a, b, c, c] = [a, b, a, b, c, c]$, which is trivial by Proposition 3.4. Since G is metabelian, we have $[x, y, z] = 1$ for any $x, y, z \in G^2$, hence the result. □

Proof of Theorem 1.4. First of all, the results of [8] imply that the class of F_r is $\leq r + 3$ when $r = 2, 3$, and is of class $\leq r + 2$ when $r > 3$. From the Example 3.3 and Proposition 3.4 it follows that the assertion is already proved for $r = 2, 3$. Furthermore, every 4-central metabelian group, being a center-by-exponent-four group, satisfies the following identities [8]:

$$[y, z, x, x, x, w] = 1. \tag{1}$$

$$[x, y, y, y, z, w] = [x, y^2, z^2, w] = [x, y, y, z, z, w] = [z, y, y, x, x, w]. \tag{2}$$

$$[x, y, x, y, z, w][x, y, y, z, z, w][y, x, x, z, z, w] = 1. \tag{3}$$

$$[x, y, z, z, u, v, w] = 1. \tag{4}$$

Now, let H be a four-generator 4-central metabelian group. By [8], we have to prove that all the commutators of weight 6 are trivial. Because of the class restriction of F_3 and (1)–(4), it is enough to consider the following commutators.

$$\begin{aligned} c_1 &= [x, y, z, z, w, w], \\ c_2 &= [x, y, x, y, z, w], \\ c_3 &= [x, y, y, z, z, w]. \end{aligned}$$

At first we notice that $c_1 = [x, y, z^2, w^2] \in [H', H^2, H^2] \leq \gamma_3(H^2) = 1$ by Proposition 3.5. For the remaining cases we make a use of the identity $[x, y, y, z, z][x, y, x, y, z] = [x, z, y, y, y][z, x, x, y, y]$ which holds in every metabelian 4-central group and can be proved by a routine expansion in F_3 (see also the remark at the end of the proof). Commuting this identity with w and using (2) and (3), we obtain $[y, x, x, z, z, w] = [x, y, x, y, z, w]$. Observing (3) once again yields $c_3 = 1$, hence also $c_2 = 1$. This, together with the fact that there are 4-generator metabelian groups of exponent 4 and class 5 [8], implies that the class of F_4 is 5.

Now we may assume that $r \geq 5$. By [8] we only need to prove that the class of F_r is $\leq r + 1$. Let $x_1, x_2, \ldots, x_r$ be the generators of F_r and consider the commutator $c = [a, b, y_1, \ldots, y_r]$, where $a, b, y_i \in \{x_1, x_2, \ldots, x_r\}$. If $y_i = y_j$ for some $i \neq j$, then $c = 1$ by (4), thus the only form to be considered is $c = [a, b, a, b, y_3, \ldots, y_r]$, where $y_3, \ldots, y_r$ are pairwise distinct. By (3) and (4) we have $[a, b, a, b, y_3, y_4, \ldots y_r] = [a, {}_3b, y_3, y_4, \ldots, y_r][b, {}_3a, y_3, y_4, \ldots, y_r] = 1$, which completes the proof. □

Remark. The nilpotency class of F_r, where r is small enough, can be calculated using the Nilpotent Quotient Algorithm [15] which is implemented as a package for GAP [6]. In this way one can also check the identities used in the proof of Theorem 1.4.

4. 6-central groups

If G is a 6-central group, then it is k-abelian, where $k = |B(2,6)| = 2^{28}3^{25}$. A better bound for k can be achieved using Corollary 2.4. Namely, if G is a two-generator 6-central group, then $G/Z(G)$ is a homomorphic image of $B(2,6)$. It can be read off from a power–commutator presentation of $B(2,6)$ (see [10]) that this group is soluble with the derived length 3, which yields that G is soluble of derived length ≤ 4, hence $\exp G'$ divides 6^6. It follows from here that every 6-central group is 6^7-abelian. However, even this bound is far from the best one. It turns out that every 6-central group is 36-abelian. We sketch here computer calculations which lead to the proof of this fact.

Proof of Theorem 1.2. Let F be the free group of rank 2. Then $F/[F^6, F]$ is isomorphic to the free 6-central group of rank two. Since $F/F^6 \cong B(2,6)$ is polycyclic, $F/[F^6, F]$ is also polycyclic. The polycyclic presentation of $B(2,6)$ is given in [10]. Suppose that this presentation is given as $F/F^6 = \langle x_1, \ldots, x_k | r_1 = s_1, \ldots, r_l = s_l \rangle$. Then $F/[F^6, F]$ has a presentation of the form $F/[F^6, F] = \langle x_1, \ldots, x_k, y_1, \ldots, y_l | r_1 = s_1 y_1, \ldots, r_l = s_l y_l, [x_i, y_j] = 1 \rangle$. Now the consistency check yields a polycyclic presentation for $F/[F^6, F]$; the computational tools for doing this are implemented in GAP [6] by W. Nickel (personal communication). Let $G = F/[F^6, F]$ and let $a, b \in G$. Since G is 6-central, we have $(ab)^6 \equiv a^6 b^6 \mod (G' \cap Z(G))$. The group $G' \cap Z(G)$ is finite and abelian. Computations with GAP show that $G' \cap Z(G) \cong C_3^4 \times C_6^{14}$, which yields $(ab)^{36} = a^{36} b^{36}$, hence G is 36-abelian. □

Now we determine the structure of nilpotent 6-central groups. We need the following technical result concerning nilpotent groups of exponent 6.

Lemma 4.1. *Let G be a nilpotent group of exponent 6. Then G is metabelian and nilpotent of class ≤ 3 and every two-generator subgroup is nilpotent of class ≤ 2. The derived subgroup G' is of exponent dividing three.*

Proof. Let $a, b \in G$ and let $H = \langle a, b \rangle$. We may assume that H is nilpotent of class ≤ 3. In order to show that H is nilpotent of class ≤ 2, we have to prove that $[a, b, b] = [a, b, a] = 1$.

By the assumption, H is metabelian. Hence we have $1 = [a, b^6] = [a, b]^6 [a, b, b]^{15} = [a, b, b]^3$. By the symmetry we also have $[a, b, a]^3 = 1$. Now, $1 = (ab^{-1})^6 = [a, b]^{15} [a, b, a]^{20} [a, b, b]^{20} = [a, b]^3 [a, b, a]^2 [a, b, b]^2$. Similarly, we have $1 = (ab^{-1})^{12} = [a, b]^{66} [a, b, a]^{220} [a, b, b]^{220} = [a, b, a][a, b, b]$, which implies $[a, b]^3 = 1$. Replacing a by ba in the identity $[a, b, a][a, b, b] = 1$, we get $[a, b, a][a, b, b]^a [a, b, b] = 1$, hence $[a, b, b] = 1$ and therefore $[a, b, a] = 1$.

Since every two-generator subgroup of G is nilpotent of class ≤ 2, this implies that G is nilpotent of class ≤ 3; see [16, Part 2, Theorem 7.15]. Hence G is metabelian and the identity $[a, b]^3 = 1$ implies $G'^3 = 1$. □

Now we have:

Corollary 4.2. *Let G be a nilpotent 6-central group.*

(a) *G is nilpotent of class ≤ 4.*
(b) *Every two-generator subgroup is metabelian and nilpotent of class three at most.*
(c) $\gamma_3(G)^3 = 1$.

(d) *G satisfies the law* $[x,y]^6 = 1$.

Proof. Applying Lemma 4.1 to the group $G/Z(G)$, we obtain (a), (b) and (c). Let $x, y \in G$. Observing $1 = [x, y^6] = [x,y]^6[x,y,y]^{15} = [x,y]^6$, we get (d). □

Corollary 4.3. *Let G be a nilpotent group. Then G is 6-central if and only if $[x,y]^6 = [x,y,y]^3 = [x,y,y,y] = 1$ for any $x, y \in G$.*

Example 4.4. Let $D = \langle x\rangle \times \langle y\rangle \times \langle z\rangle$, where x is of order 6 and y and z are of order 3. Let $A = [D]\langle a\rangle$ be the semidirect product of the group D by the cyclic group of order 36, where the action of a on the generators of D is given as follows.

$$[x,a] = z,\ [y,a] = [z,a] = 1.$$

Let $G = [A]\langle b\rangle$, where b is an element of order 36 acting on A in the following way.

$$[x,b] = y,\ [y,b] = [z,b] = 1,\ [a,b] = x.$$

The group G is metabelian of class 3 and order $2^5 \cdot 3^7$, the exponent of G is 36. It is easy to check that G is 6-central and is not k-abelian for any $1 < k < 36$.

5. Metabelian 8-central groups

By Proposition 2.8, every 8-central metabelian group is 128-abelian. Yet this estimate can be improved. Using a detailed analysis of metabelian groups of exponent eight in [3], we shall prove that every 8-central metabelian group is 32-abelian. Starting as usual, we prove the following lemma.

Lemma 5.1. *Let G be a metabelian 8-central group and let $H \leq G$ be a two-generator subgroup. Then $\gamma_6(H)^4 = \gamma_{10}(H)^2 = \gamma_{14}(H) = 1$ and $G'^{16} = \gamma_3(G)^8 = 1$.*

Proof. The equality $\gamma_3(G)^8 = 1$ follows from Lemma 2.3. Since $H/Z(H)$ is a 2-generator metabelian group of exponent 8, we have $\gamma_5(H/Z(H))^4 = \gamma_9(H/Z(H))^2 = \gamma_{13}(H/Z(H)) = 1$ by [3], hence $\gamma_6(H)^4 = \gamma_{10}(H)^2 = \gamma_{14}(H) = 1$. So we are left with the proof that $G'^{16} = 1$. Let $a, b \in G$. First we note that $[a,b,b^4]^4 = [a,b,b]^{16}[a,{}_3b]^{24}[a,{}_4b]^{16}[a,{}_5b]^4 = 1$, therefore $[a,b]^4$ commutes with b^4. Since G is 13-Engel, we have

$$\begin{aligned} 1 &= [a,{}_{13}b] \\ &= [a,b]^{(-1+b)^{12}} \\ &= [a,b]^{1-12b+66b^2-220b^3+495b^4-792b^5+924b^6-792b^7+495b^8-220b^9+66b^{10}-12b^{11}+b^{12}}. \end{aligned}$$

By 8-centrality this yields $1 = [a,b]^{1576+4b^2+4b^6}$. Using Lemma 2.3 and $[a,b]^{4b^6} = [a,b]^{4b^2}$, we get $[a,b]^{48} = 1$. On the other hand, we know that $[a,b]^{64} = 1$ by Lemma 2.3, hence $[a,b]^{16} = 1$ and the lemma is proved. □

Corollary 5.2. *Every metabelian 8-central group is 32-abelian.*

Proof. This follows from Lemma 5.1 and the expansion

$$(ab)^{32} = ((ab)^8)^4 = a^{32}b^{32} \cdot \prod_{0<i+j<8} [a,{}_ib^{-1},{}_ja]^{4\binom{8}{i+j+1}} = a^{32}b^{32}.$$

□

Example 5.3. Let D be a group generated by the mutually commuting generators $x, y_1, \ldots, y_6, z_1, \ldots, z_6, w_1, w_2, w_3$ and x_{ij}, where $1 \le i \le 8$, $1 \le j \le i+1$. The generators x_{8i}, y_j, z_k and w_l are involutions and we have the following additional relations:

$$\begin{array}{lll}
x^8 = c & x_{43}^2 = x_{83}x_{85}x_{87}y_3y_4z_5 & x_{64}^2 = z_3 \\
x_{11}^4 = x_{31}^2 x_{71} w_1 c & x_{44}^2 = x_{84}x_{86}x_{88}y_3y_4z_5 & x_{65}^2 = z_4 \\
x_{12}^4 = x_{34}^2 x_{78} w_3 c & x_{45}^2 = x_{85}x_{87}x_{89}y_4y_5z_3z_5z_6 & x_{66}^2 = z_5 \\
x_{21}^4 = x_{83}x_{85}y_2y_3z_1 & x_{51}^2 = y_1w_1 & x_{67}^2 = z_6 \\
x_{22}^4 = x_{84}x_{86}y_3y_4z_5 & x_{52}^2 = y_2w_2 & x_{71}^2 = w_1 \\
x_{23}^4 = x_{85}x_{87}y_4y_5z_3z_5z_6 & x_{53}^2 = y_3z_2z_3w_2 & x_{72}^2 = w_2 \\
x_{31}^4 = w_1 & x_{54}^2 = y_4w_2 & x_{73}^2 = w_2 \\
x_{32}^4 = w_2 & x_{55}^2 = y_5w_2 & x_{74}^2 = w_2 \\
x_{33}^4 = w_2 & x_{56}^2 = y_6w_3 & x_{75}^2 = w_2 \\
x_{34}^4 = w_3 & x_{61}^2 = z_1 & x_{76}^2 = w_2 \\
x_{41}^2 = x_{81}x_{83}x_{85}y_2y_3z_1 & x_{62}^2 = z_2 & x_{77}^2 = w_2 \\
x_{42}^2 = x_{82}x_{84}x_{86}y_3y_4z_5 & x_{63}^2 = z_4 & x_{78}^2 = w_3
\end{array}$$

Here we use the abbreviation $c = x_{32}^2 x_{33}^2 x_{72}x_{73}x_{74}x_{75}x_{76}x_{77}$. Let $A = [D]\langle a\rangle$ be the semidirect product of D with a cyclic group $\langle a\rangle$ where a induces the following automorphism of order 32 on D: $[x,a] = x_{11}$, $[w_k, a] = 1$ for $1 \le k \le 3$, $[x_{ij}, a] = x_{i+1,j}$ for $1 \le i \le 7$, $1 \le j \le i+1$ and

$$\begin{array}{lll}
[x_{81}, a] = y_1 & [x_{88}, a] = y_4 & [y_6, a] = z_5 \\
[x_{82}, a] = y_2 & [x_{89}, a] = y_5 & [z_1, a] = w_1 \\
[x_{83}, a] = y_3z_2z_3 & [y_1, a] = z_1 & [z_2, a] = w_2 \\
[x_{84}, a] = y_4z_3z_5 & [y_2, a] = z_2 & [z_3, a] = w_2 \\
[x_{85}, a] = y_3z_3z_4 & [y_3, a] = z_4 & [z_4, a] = w_2 \\
[x_{86}, a] = y_4z_4z_5 & [y_4, a] = z_3 & [z_5, a] = w_2 \\
[x_{87}, a] = y_3 & [y_5, a] = z_4 & [z_6, a] = w_2
\end{array}$$

Let $G = [A]\langle b\rangle$, where b is an element of order 32 acting on A in the following way: $[a,b] = x^{-1}$, $[x,b] = x_{12}$, $[w_k, b] = 1$ for $1 \le k \le 3$, $[x_{ij}, b] = x_{i+1,j+1}$ for $1 \le i \le 7$, $1 \le j \le i+1$ and

$$\begin{array}{lll}
[x_{81}, b] = y_2 & [x_{88}, b] = y_5 & [y_6, b] = z_6 \\
[x_{82}, b] = y_3z_2z_3 & [x_{89}, b] = y_6 & [z_1, b] = w_2 \\
[x_{83}, b] = y_4z_3z_5 & [y_1, b] = z_2 & [z_2, b] = w_2 \\
[x_{84}, b] = y_3z_3z_4 & [y_2, b] = z_4 & [z_3, b] = w_2 \\
[x_{85}, b] = y_4z_4z_5 & [y_3, b] = z_3 & [z_4, b] = w_2 \\
[x_{86}, b] = y_3 & [y_4, b] = z_4 & [z_5, b] = w_2 \\
[x_{87}, b] = y_4 & [y_5, b] = z_5 & [z_6, b] = w_3
\end{array}$$

We notice that $G = \langle a, b\rangle$ is metabelian and nilpotent of class 13, $|G| = 2^{81}$ and $G^{32} = 1$. One can check that G is 8-central and that $(ab)^k \ne a^k b^k$ for any $1 < k < 32$.

6. Exponent semigroups of n-central groups for $n = 2, 3, 4, 6$

For a group G define

$$\mathcal{E}(G) = \{n \in \mathbb{Z} : (xy)^n = x^n y^n \text{ for all } x, y \in G\}.$$

These sets are semigroups under multiplication and they always contain zero. According to [12], the set $\mathcal{E}(G)$ is called the exponent semigroup of the group G. One of the results in [12] is an arithmetic characterization of the sets $\mathcal{E}(G)$ for an arbitrary group G. It is shown there that each of these sets always forms a so-called Levi system, which is, roughly speaking, a union of idempotent residue classes modulo a certain integer m, which depends on G. More precisely, let $q_1, q_2, \ldots, q_t$ be integers, $q_i > 1$ and $\gcd(q_i, q_j) = 1$ for $i \neq j$. Let $B(q_1, q_2, \ldots, q_t)$ be the set of integers which is the union of 2^t residue classes modulo q_i satisfying each a system of congruences $m \equiv \delta_i \mod q_i$, where $i = 1, \ldots, t$ and $\delta_i \in \{0, 1\}$. It is proved in [12] that each of the sets $\mathcal{E}(G)$ is equal either to $\mathbb{Z}$, $\{0, 1\}$ or to some $B(q_1, q_2, \ldots, q_t)$ with $q_i > 2$. This enables us to formulate the following result:

Theorem 6.1. *Let $n \in \{2, 3, 4, 6\}$ and let G be a free n-central group with two or more generators. Then $\mathcal{E}(G) = B(n^2)$.*

Proof. Let G be the free 2-central group with $r > 1$ generators. Then $m = 4$ is the smallest element of $\mathcal{E}(G)$. It follows from [12] that $k^2 \equiv k \mod 4$, hence either $k \equiv 0 \mod 4$ or $k \equiv 1 \mod 4$, which implies $\mathcal{E}(G) \subseteq B(4)$. Conversely, let t be any integer and $x, y \in G$. Then we have $(xy)^{4t} = x^{4t}y^{4t}$ and $(xy)^{4t+1} = x^{4t+1}y^{4t+1}$ which shows that $B(4) \subseteq \mathcal{E}(G)$. This proves our theorem for $n = 2$. For other n's the proof is very similar. There is only a slight difference in the case $n = 6$, since the congruence equation $k^2 \equiv k \mod 36$ has the following solutions: $k \equiv 0 \mod 36$, $k \equiv 1 \mod 36$, $k \equiv 9 \mod 36$ and $k \equiv 28 \mod 36$. But since there are 6-central groups which are neither 9-abelian nor 28-abelian (see Example 4.4), we can exclude the last two solutions. $\square$

7. (k, n)-central groups

A group G is said to be (k, n)-central if $G/Z_k(G)$ is of exponent n, that is, G satisfies the law $[x_1^n, x_2, \ldots, x_{k+1}] = 1$. It is clear that $(0, n)$-central groups are precisely the groups of exponent n, $(1, n)$-central groups are just n-central groups and the class of $(k, 1)$-central groups coincides with the class of all nilpotent groups of class $\leq k$. We have already noticed that n-central groups are very closely related to m-abelian groups. The situation is very similar for (k, n)-central groups, where the so called m-nilpotent groups [2] play an important role. For a given integer m and two group elements x, y define an m-commutator [14] of x and y by $[x, y]_m = (xy)^m y^{-m} x^{-m}$. There is a connection between commutators and m-commutators given by the identity

$$[x, y]_m = [y, x]^{y^{-1}x^{-2}} [y^2, x]^{y^{-2}x^{-3}} \cdots [y^{m-1}, x]^{y^{1-m}x^{-m}}.$$

For a group G define the m-center [2] by $Z(G; m) = \{c \in G : [g, c]_m = [c, g]_m = 1 \text{ for every } g \in G\}$. It is easily seen that $Z(G; m)$ is a characteristic subgroup of G. The upper m-center chain $Z_i(G; m)$ is defined inductively by the following rules: $Z_0(G; m) = 1$, $Z_{i+1}(G; m)/Z_i(G; m) = Z(G/Z_i(G; m); m)$. A group G is said to be m-nilpotent of class $\leq k$ if $Z_k(G; m) = G$. It is not difficult to see that a group G is m-nilpotent of class $\leq k$ if and only if $[\ldots[[x_1, x_2]_m, x_3]_m, \ldots, x_{k+1}]_m = 1$ for all $x_1, x_2, \ldots, x_{k+1} \in G$. Suppose now that G is a (k, n)-central group. For any $x_1, x_2 \in G$ we have $[x_1, x_2]_n \in G^n \leq Z_k(G)$, hence G is n-nilpotent of class $\leq k+1$. As in [7], we can ask the following question: If G is a (k,n)-central group, is there an integer $f(n)$ such that G is $f(n)$-nilpotent of class $\leq k$? Let $\mathfrak{C}$ be a class of groups closed with respect to the homomorphic images and suppose that every n-central group from $\mathfrak{C}$ is $f(n)$-abelian. Let G be a (k,n)-central group belonging to $\mathfrak{C}$. Then $G/Z_{k-1}(G)$ is n-central, hence it is $f(n)$-abelian. Let $m = f(n)$. We have $[x_1, x_2]_m \in Z_{k-1}(G)$, hence $[\ldots[[x_1, x_2]_m, x_3]_m, \ldots, x_{k+1}]_m = 1$. It follows that every (k,n)-central group from $\mathfrak{C}$ is $f(n)$-nilpotent of class $\leq k$. The results of the previous sections now yield:

Corollary 7.1.

(a) *If $n \in \{2,3,4,6\}$, then every (k,n)-central group is n^2-nilpotent of class $\leq k$.*
(b) *Every metabelian (k,n)-central group is $2n^2$-nilpotent of class $\leq k$; if n is odd, then we can replace $2n^2$ with n^2.*
(c) *Every metabelian $(k,8)$-central group is 32-nilpotent of class $\leq k$.*

In determining an explicit bound for $f(n)$ in some other cases, the following assertion is of some help:

Proposition 7.2. *Let G be a (k,n)-central group. If $\exp \gamma_{k+1}(G) = m < \infty$, then G is mn-nilpotent of class $\leq k$.*

Proof. Let $x_1, x_2, \ldots, x_{k+1} \in G$ and put $a = [\ldots[x_1, x_2]_{mn}, x_3]_{mn}, \ldots, x_k]_{mn}$. Clearly $a \in \gamma_k(G)$. In particular, we have $[a, x_{k+1}]_n^m = 1$, since $\gamma_{k+1}(G)$ is of exponent m. As G is (k,n)-central, it follows $[x_1, x_2]_{mn} \in Z_k(G)$, hence $a \in Z_2(G)$. This means that $[[a, x_{k+1}]_n, a^n]_m \in [G', Z_2(G)] = 1$. We also have $[[a, x_{k+1}]_n a^n, x_{k+1}^n]_m \in [\gamma_k(G), Z_k(G)] = 1$. But then

$$[a, x_{k+1}]_{mn} = [[a, x_{k+1}]_n a^n, x_{k+1}^n]_m \cdot [[a, x_{k+1}]_n, a^n]_m \cdot [a, x_{k+1}]_n^m = 1,$$

hence the proposition is proved. □

Finally, we give a version of Theorem 1.3 for (k,p)-central groups:

Theorem 7.3. *Let G be a metabelian group and p a prime. Then G is (k,p)-central if and only if $\gamma_{k+1}(G)^p = \gamma_{k+p}(G) = 1$.*

Proof. Suppose that G is (k,p)-central. Then $G/Z_{k-1}(G)$ is p-central, so we may apply Theorem 1.3 to obtain $\gamma_{k+p}(G) = 1$ and $G'^p \leq Z_{k-1}(G)$, which further yields $[x_1, x_2, \ldots, x_{k+1}]^p = [[x_1, x_2]^p, \ldots, x_{k+1}] = 1$, hence $\gamma_{k+1}(G)^p = 1$.

Conversely, suppose that $\gamma_{k+1}(G)^p = \gamma_{k+p}(G) = 1$ and let $x_0, x_1, \ldots, x_k \in G$. Using Lemma 2.1, we get $[x_0^p, x_1, \ldots, x_k] = \prod_{i=1}^{p} [x_1, {}_i x_0, x_2, \ldots, x_k]^{-\binom{p}{i}} = 1$ and the theorem is proved. □

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