COMPLETELY SIMPLE SEMIGROUPS WITH NILPOTENT STRUCTURE GROUPS

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Abstract. In this paper we find simple characterizations of completely simple semigroups with $\mathscr{H}$-classes nilpotent of class $\leq c$, and of completely simple semigroups whose core has $\mathscr{H}$-classes nilpotent of class $\leq c$. The notion of w-marginal completely regular semigroups is introduced, generalizing the concept of central semigroups. A law characterizing $[x_1, x_2, \ldots, x_{c+1}]$-marginal completely simple semigroups is obtained. Additionally, the least congruences corresponding to these classes are described. Our results extend the corresponding results obtained by Petrich and Reilly in the abelian case.

1. Introduction

The class $\mathscr{C}\mathscr{S}$ of completely simple semigroups is a variety of unary semigroups, that is, of semigroups with inversion. It is defined by the identities $x = xx^{-1}x$, $(x^{-1})^{-1} = x$, $xx^{-1} = x^{-1}x$ and $xx^{-1} = (xyx)(xyx)^{-1}$. According to the Rees theorem [6, Theorem III.2.6], every completely simple semigroup is isomorphic to a Rees matrix semigroup $\mathcal{M}(I, G, \Lambda; P)$, where I and Λ are nonempty sets, G is a group, and $P = (p_{\lambda i})$ is a $\Lambda \times I$-matrix with entries from G. This result suggests that the structure of a given completely simple semigroup S heavily depends on the properties of its structure group G, especially since all $\mathscr{H}$-classes of S are isomorphic to G. In [4], Petrich and Reilly characterized completely simple semigroups with abelian structure groups, whereas Rasin [7] determined all varieties of completely simple semigroups with abelian $\mathscr{H}$-classes. Petrich and Reilly [5] also determined all varieties of central completely simple semigroups, i.e., completely simple semigroups in which the product of any two idempotents lies in the center of the maximal subgroup containing it.

In this paper we consider the case when the $\mathscr{H}$-classes of a given completely simple semigroup are nilpotent of class $\leq c$. It is known that if $\mathscr{V}$ is a variety of groups, then the class $\mathscr{C}\mathscr{S}\mathrm{H}\mathscr{V}$ of completely simple semigroups with $\mathscr{H}$-classes in $\mathscr{V}$ is also a variety whose laws can be described in terms of $\mathscr{V}$. In general, these laws can be rather complicated. We find here a nice law characterizing the class $\mathscr{C}\mathscr{S}\mathrm{H}\mathscr{N}_c$, where $\mathscr{N}_c$ is the variety of all groups that are nilpotent of class $\leq c$. Our argument is based on the fact that the variety $\mathscr{N}_c$ can be described by a semigroup law found by Neumann and Taylor [3]. A similar procedure alllows us to find a law characterizing completely simple semigroups S with the core $C(S)$ in $\mathscr{C}\mathscr{S}\mathrm{H}\mathscr{N}_c$.

In [1], Phillip Hall introduced the notion of the w-marginal subgroup $w^*(G)$ of a group G for a given word w. In a special case when $w = x^{-1}y^{-1}xy$, $w^*(G)$ is precisely the center of G. This leads to a generalization of central semigroups. Given a group word w, we say that a completely regular semigroup S is w-marginal if the product of any two idempotents of S belongs to the w-marginal subgroup of the $\mathscr{H}$-class containing it. First we obtain some general characterizations of w-marginal semigroups. Then we treat a special case when w is the commutator word $\omega_c = [x_1, x_2, \ldots, x_{c+1}]$, and obtain a law characterizing completely simple ω_c-marginal semigroups.

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Simple characterizations of the class $\mathscr{C}\mathscr{S}\mathrm{H}\mathscr{N}_c$ and the class of completely simple ω_c-marginal semigroups suggest that there should be a relatively easy treatment of the lattices of varieties of $\mathscr{C}\mathscr{S}\mathrm{H}\mathscr{N}_c$ and completely simple ω_c-marginal semigroups modulo the subvarieties of $\mathscr{N}_c$. For $c = 1$ this has already been done in [5] and [7]. However, beyond this point relatively little is known. Note that the subvarieties of $\mathscr{N}_c$ are only known for $c \leq 3$ [2], and that the arguments of [5] and [7] heavily rely on the fact that the groups in question are abelian. It is to be expected that our results would be a first step towards determination of these lattices of varieties. This will be handled with separately.

2. Preliminaries and notations

In this section we briefly collect some facts on completely simple semigroups. A reference for this is for instance [6]. Recall that $S = \mathcal{M}(I, G, \Lambda; P)$ stands for a Rees matrix semigroup over the group G with $\Lambda \times I$ sandwich matrix $P = (p_{\lambda i})$. Elements of this semigroup are triples (i, g, λ), where $i \in I$, $g \in G$, $\lambda \in \Lambda$, and the multiplication is given by $(i, g, \lambda)(j, h, \mu) = (i, gp_{\lambda j}h, \mu)$. Moreover, P may be taken to be normalized, i.e., there exist $1 \in I$ and $1 \in \Lambda$ such that $p_{1i} = p_{\lambda 1} = e$ for all $i \in I$, $\lambda \in \Lambda$; here e is the identity element of G. For $x \in S$ let x^0 be the idempotent belonging to the $\mathscr{H}$-class H_x containing x, and x^{-1} denotes the inverse of x in the maximal subgroup H_x. If $x = (i, g, \lambda)$, then $H_x = \{(i, h, \lambda) : h \in G\}$, $x^0 = (i, p_{\lambda i}^{-1}, \lambda)$ and $x^{-1} = (i, p_{\lambda i}^{-1} g^{-1} p_{\lambda i}^{-1}, \lambda)$. With $\rho_{(r,N,\pi)}$ we denote the congruence on S corresponding to the admissible triple (r, N, π); see [6] for details. Semilattices $S = (Y; S_\alpha)$ of completely simple semigroups S_α are said to be completely regular semigroups.

Let G be a group. For $x_1, x_2, \ldots, x_n \in G$ we define the commutator $[x_1, x_2, \ldots, x_n]$ inductively by $[x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2$ and $[x_1, x_2, \ldots, x_n] = [[x_1, x_2, \ldots, x_{n-1}], x_n]$ for $n > 2$. Define the sequence $(\gamma_i(G))_i$ of subgroups of G by $\gamma_1(G) = G$ and $\gamma_{n+1}(G) = [\gamma_n(G), G] = \langle [a, b] : a \in \gamma_n(G), b \in G\rangle$ for $n \geq 1$. A group G is said to be nilpotent of class c if c is the smallest integer with the property that $\gamma_{c+1}(G) = \{e\}$.

For other unexplained notations we refer to [6] and [8].

3. General results

Lemma 3.1. *Let $w(x_1, \ldots x_n)$ be a word in the free group of rank n over $\{x_1, \ldots, x_n\}$. Let $S = \mathcal{M}(I, G, \Lambda; P)$ and choose $i \in I$ and $\lambda \in \Lambda$. Then*

$$w((i, g_1, \lambda), \ldots, (i, g_n, \lambda)) = (i, p_{\lambda i}^{-1} w(p_{\lambda i}g_1, \ldots, p_{\lambda i}g_n), \lambda)$$

for all $g_1, \ldots, g_n \in G$.

Proof. This follows from the equalities $(i, g, \lambda)(i, h, \lambda) = (i, p_{\lambda i}^{-1}(p_{\lambda i}g)(p_{\lambda i}h), \lambda)$ and $(i, g, \lambda)(i, h, \lambda)^{-1} = (i, g, \lambda)(i, p_{\lambda i}^{-1}h^{-1}p_{\lambda i}^{-1}, \lambda) = (i, p_{\lambda i}^{-1}(p_{\lambda i}g)(p_{\lambda i}h)^{-1}, \lambda)$, using induction on the length of w. □

Let $\mathscr{X}$ be a group theoretical class. A completely regular semigroup S is said to be an *over-$\mathscr{X}$ semigroup* if every $\mathscr{H}$-class of S belongs to $\mathscr{X}$. Denote by $\mathscr{C}\mathscr{R}\mathrm{H}\mathscr{X}$ the class of all completely regular over-$\mathscr{X}$ semigroups, and by $\mathscr{C}\mathscr{S}\mathrm{H}\mathscr{X}$ the class of all completely simple over-$\mathscr{X}$ semigroups. If $\mathscr{V}$ is a variety of groups determined by the laws $u_\sigma(x_1, \ldots, x_{i_\sigma}) = v_\sigma(x_1, \ldots, x_{i_\sigma})$, $\sigma \in \Sigma$, then $\mathscr{C}\mathscr{R}\mathrm{H}\mathscr{V}$ is a variety of completely regular semigroups determined by the laws $u_\sigma(e_\sigma x_1 e_\sigma, \ldots, e_\sigma x_{i_\sigma} e_\sigma) = v_\sigma(e_\sigma x_1 e_\sigma, \ldots, e_\sigma x_{i_\sigma} e_\sigma)$, where $\sigma \in \Sigma$ and $e_\sigma = (x_1 \ldots x_{i_\sigma})^0$ [6, Proposition II.7.2]. Similarly, [6, Theorem VIII.5.5] shows that $\mathscr{C}\mathscr{S}\mathrm{H}\mathscr{V}$ is a variety of completely simple semigroups determined by the laws of the form $u_\sigma(a^0x_1a^0, \ldots, a^0x_{i_\sigma}a^0) = v_\sigma(a^0x_1a^0, \ldots, a^0x_{i_\sigma}a^0)$, where $\sigma \in \Sigma$ and $a \notin \cup_{\sigma \in a} c(u_\sigma v_\sigma)$. Our first remark shows that we can choose another set of laws defining $\mathscr{C}\mathscr{S}\mathrm{H}\mathscr{V}$ without changing the content of the laws of $\mathscr{V}$.

Proposition 3.2. *Let $\mathscr{V}$ be a variety of groups characterized by the laws*

$$u_\sigma(x_1, x_2, \ldots, x_{n_\sigma}) = v_\sigma(x_1, x_2, \ldots, x_{n_\sigma}),$$

where $\sigma \in \Sigma$. Then a completely simple semigroup S is an over-$\mathscr{V}$ semigroup if and only if it satisfies the laws $u_\sigma(x_1, x_1^0x_2x_1^0, \dots, x_1^0x_{n_\sigma}x_1^0) = v_\sigma(x_1, x_1^0x_2x_1^0, \dots, x_1^0x_{n_\sigma}x_1^0)$, $\sigma \in \Sigma$.

Proof. Suppose first that the $\mathscr{H}$-classes of S belong to the variety $\mathscr{V}$. Let $x_i \in S$. Then $x_1 \mathscr{H} x_1^0x_ix_1^0$ for all i, hence S satisfies the laws $u_\sigma(x_1, x_1^0x_2x_1^0, \dots, x_1^0x_{n_\sigma}x_1^0) = v_\sigma(x_1, x_1^0x_2x_1^0, \dots, x_1^0x_{n_\sigma}x_1^0)$ for all $\sigma \in \Sigma$.

Conversely, let S satisfy $u_\sigma(x_1, x_1^0x_2x_1^0, \dots, x_1^0x_{n_\sigma}x_1^0) = v_\sigma(x_1, x_1^0x_2x_1^0, \dots, x_1^0x_{n_\sigma}x_1^0)$, where $\sigma \in \Sigma$. Suppose that $S = \mathcal{M}(I, G, \Lambda; P)$ with P normalized, and choose $x_i = (1, g_i, 1)$, where $g_i \in G$. Then we obtain $u_\sigma(g_1, g_2, \dots, g_{n_\sigma}) = v_\sigma(g_1, g_2, \dots, g_{n_\sigma})$ for all $\sigma \in \Sigma$ and $g_i \in G$, whence $G \in \mathscr{V}$. This concludes the proof. □

For a group G and a word $w(x_1, \dots, x_n)$ in a free group of rank n, denote by $w(G)$ the subgroup of G generated by all $w(g_1, \dots, g_n)$, where $g_1, \dots, g_n \in G$. According to [1], the group $w(G)$ is said to be the *verbal subgroup* of G corresponding to the word w. It is evident that $w(G)$ is a fully invariant subgroup of G.

Proposition 3.3. *Let $\mathscr{V} = [w = 1]$ be a variety of groups and let $S = \mathcal{M}(I, G, \Lambda; P)$ with P normalized. Then the admissible triple $(\epsilon, w(G), \epsilon)$ corresponds to the least $\mathscr{C}\mathscr{S}\mathrm{H}\mathscr{V}$-congruence on S.*

Proof. Let $\rho = \rho_{(\epsilon, w(G), \epsilon)}$. Then the structure group of S/ρ is isomorphic to $G/w(G)$, hence $S/\rho \in \mathscr{C}\mathscr{S}\mathrm{H}\mathscr{V}$. Let $\tau = \rho_{(r,N,\pi)}$ be another $\mathscr{C}\mathscr{S}\mathrm{H}\mathscr{V}$-congruence on S. Then $w((1, g_1, 1), \dots, (1, g_n, 1)) \;\tau\; (1, e, 1)$ for all $g_1, \dots, g_n \in G$, therefore we conclude that $(1, w(g_1, \dots, g_n), 1) \;\tau\; (1, e, 1)$ for all $g_1, \dots, g_n \in G$. By definition of an admissible triple it follows that $w(g_1, \dots, g_n) \in N$ for all $g_1, \dots, g_n \in G$, hence $w(G) \le N$. This shows that ρ is the least $\mathscr{C}\mathscr{S}\mathrm{H}\mathscr{V}$-congruence on S. □

For a variety of groups $\mathscr{V}$, let $\mathscr{C}\mathscr{S}\mathrm{CH}\mathscr{V}$ be the class of all completely simple semigroups with $C(S) \in \mathscr{C}\mathscr{S}\mathrm{H}\mathscr{V}$. It is not difficult to see that $S = \mathcal{M}(I, G, \Lambda; P)$ is in $\mathscr{C}\mathscr{S}\mathrm{CH}\mathscr{V}$ if and only if the group $\langle P \rangle$ belongs to $\mathscr{V}$. Our next result describes the least $\mathscr{C}\mathscr{S}\mathrm{CH}\mathscr{V}$-congruence on a given completely simple semigroup. For a given subgroup K of G denote by K^G the normal closure of K in G. The result can be stated as follows.

Corollary 3.4. *Let $\mathscr{V} = [w = 1]$ be a variety of groups and let $S = \mathcal{M}(I, G, \Lambda; P)$ with P normalized. Then the admissible triple $(\epsilon, w(\langle P \rangle)^G, \epsilon)$ corresponds to the least $\mathscr{C}\mathscr{S}\mathrm{CH}\mathscr{V}$-congruence on S.*

Proof. Let $\rho = \rho_{(\epsilon, w(\langle P \rangle)^G, \epsilon)}$. Then $S/\rho \cong \mathcal{M}(I, G/w(\langle P \rangle)^G, \Lambda; \bar{P})$, where $\bar{P}$ is the $\Lambda \times I$-matrix with (λ, i)-entry equal to $p_{\lambda i} w(\langle P \rangle)^G$ [6, Theorem III.4.6]. By [6, Lemma III.2.10] we have that $C(S/\rho)$ is isomorphic to $\mathcal{M}(I, \langle \bar{P} \rangle, \Lambda; \bar{P})$, hence S/ρ is a $\mathscr{C}\mathscr{S}\mathrm{CH}\mathscr{V}$-semigroup. To prove that ρ is the least $\mathscr{C}\mathscr{S}\mathrm{CH}\mathscr{V}$-congruence on S, let $\tau = \rho_{(r,N,\pi)}$ be an arbitrary $\mathscr{C}\mathscr{S}\mathrm{CH}\mathscr{V}$-congruence on S. For every $p_1, \dots, p_n$ in $\langle P \rangle$ we have that $(1, w(p_1, \dots, p_n), 1) \;\tau\; (1, e, 1)$, whence $w(\langle P \rangle) \le N$. As N is a normal subgroup of G, we also have that $w(\langle P \rangle)^G \le N$. This concludes the proof. □

Suppose w is a group word in n variables and let G be a group. Then the set of all $a \in G$ satisfying

$$w(g_1, \dots, g_{i-1}, g_i a, g_{i+1}, \dots, g_n) = w(g_1, \dots, g_{i-1}, g_i, g_{i+1}, \dots, g_n)$$

for all $g_j \in G, i = 1, \dots, n$ is a characteristic subgroup $w^*(G)$ of G. According to Hall [1], $w^*(G)$ is said to be the *w-marginal subgroup* of G. Clearly, G belongs to the variety of groups satisfying the law $w = 1$ if and only if $w^*(G) = G$. For example, when $w = \omega_c = [x_1, \dots, x_c, x_{c+1}]$, the marginal subgroup $w^*(G)$ is equal to $Z_c(G)$, the c-th term of the upper central series of G [1].

Let w be a group word. A completely regular semigroup S is said to be *w-marginal* if the product of any two idempotents of S belongs to the w-marginal subgroup of the $\mathscr{H}$-class containing it. In particular, when $w = [x, y]$, the class of w-marginal semigroups is precisely the class of central semigroups.

Proposition 3.5. *Let $S = (Y; S_\alpha)$ be a completely regular semigroup and w a word in a free group. The following assertions are equivalent.*

(a) *S is w-marginal.*
(b) *S_α is w-marginal for all $\alpha \in Y$.*
(c) *$C(S) \cap H_e \leq w^*(H_e)$ for all $e \in E(S)$.*

Proof. Clearly (a) implies (b). To see that (b) implies (a), let $e, f \in E(S)$ and put $x = (ef)^0$. By [6, Lemma II.6.3] there exist $e_1, \dots, e_n \in E(R_x)$ and $f_1, \dots, f_n \in E(L_x)$ such that $e_i f_i \in H_x$ for $i = 1, \dots, n$, and $ef = \prod_{1 \leq i \leq n} (e_i f_i)^{\epsilon_i}$, where $\epsilon_i \in \{-1, 1\}$. Clearly $e_i f_i \in D_x = J_x$, hence also $(e_i f_i)^{\epsilon_i} \in J_x$ for all $i = 1, \dots, n$. This shows that $(e_i f_i)^{\epsilon_i}$ all belong to the same semilattice component S_α. By hypothesis, $(e_i f_i)^{\epsilon_i} \in w^*(H_x)$, hence also $ef \in w^*(H_x)$.

Next we prove that (b) implies (c). Let $e \in E(S)$ and $a \in C(S) \cap H_e$. Applying [6, Lemma II.6.3] again, we can write $a = \prod_{1 \leq i \leq n} (e_i f_i)^{\epsilon_i}$, where $e_1, \dots, e_n \in E(R_e)$, $f_1, \dots, f_n \in E(L_e)$ and $\epsilon_i \in \{-1, 1\}$. As $(e_i f_i)^{\epsilon_i}$ all belong to the same semilattice component S_α, we have that $(e_i f_i)^{\epsilon_i} \in w^*(H_e)$, hence $a \in w^*(H_e)$.

Finally, if $e, f \in E(S)$, then $ef \in H_{(ef)^0} \cap C(S)$, hence (c) clearly implies (b). □

The above result shows that w-marginality is essentially a property of completely simple semigroups. In this case, the following additional characterization is obtained.

Theorem 3.6. *Let $S = \mathcal{M}(I, G, \Lambda; P)$ with P normalized and let w be a group word. Then S is w-marginal if and only if $\langle P \rangle \leq w^*(G)$.*

Proof. Let S be w-marginal and let $a = (i, e, 1)$ and $b = (1, e, \mu)$. Then $ab = (i, e, \mu)$ belongs to the w-marginal subgroup of the group $H_{i\mu} = \{(i, g, \mu) : g \in G\}$. By Lemma 3.1 we conclude that the equation $w(p_{\mu i} g_1, \dots, p_{\mu i} g_{t-1}, p_{\mu i} g_t p_{\mu i}, p_{\mu i} g_{t+1}, \dots, p_{\mu i} g_n) = w(p_{\mu i} g_1, \dots, p_{\mu i} g_{t-1}, p_{\mu i} g_t, p_{\mu i} g_{t+1}, \dots, p_{\mu i} g_n)$ holds for all $g_1, \dots, g_n \in G$ and all $t = 1, \dots, n$. Replacing g_j by $p_{\mu i}^{-1} g_j$ for all $j = 1, \dots, n$, we get that the above condition is equivalent to $w(g_1, \dots, g_{t-1}, g_t p_{\mu i}, g_{t+1}, \dots, g_n) = w(g_1, \dots, g_{t-1}, g_t, g_{t+1}, \dots, g_n)$. This implies that $p_{\mu i} \in w^*(G)$, thus also $\langle P \rangle \leq w^*(G)$.

Conversely, let $a = (i, p_{\lambda i}^{-1}, \lambda)$ and $b = (j, p_{\mu j}^{-1}, \mu)$ be arbitrary idempotents of S. Then $ab = (i, p_{\lambda i}^{-1} p_{\lambda j} p_{\mu j}^{-1}, \mu)$. By the assumption we have that $p_{\lambda i}^{-1} p_{\lambda j} p_{\mu j}^{-1} \in w^*(G)$. Let $h_k = (i, g_k, \mu)$ be arbitrary elements of $H_{i\mu} = \{(i, g, \mu) : g \in G\}$. Using Lemma 3.1, we get $w(h_1, \dots, h_{t-1}, h_t ab, h_{t+1}, \dots, h_n) = w(h_1, \dots, h_{t-1}, h_t, h_{t+1}, \dots, h_n)$ for all $t = 1, \dots, n$, hence $ab \in w^*(H_{i\mu})$. Thus S is w-marginal. □

Corollary 3.7. *Let $S = \mathcal{M}(I, G, \Lambda; P)$ with P normalized and let $w(x_1, \dots, x_n)$ be a group word. Then the least w-marginal congruence on S corresponds to the admissible triple (ϵ, N, ϵ) where N is the normal closure in G of the subgroup of G generated by all*

$$w(g_1, \dots, g_{t-1}, g_t p_{\lambda i}, g_{t+1}, \dots, g_n) \cdot w(g_1, \dots, g_{t-1}, g_t, g_{t+1}, \dots, g_n)^{-1},$$

where $i \in I$, $\lambda \in \Lambda$, $g_1, \dots, g_n \in G$, $t = 1, \dots, n$.

Proof. Let $\rho = \rho_{(\epsilon, N, \epsilon)}$, where N is as above. By definition we have that

$$w(g_1, \dots, g_{t-1}, g_t p_{\lambda i}, g_{t+1}, \dots, g_n) \; \rho \; w(g_1, \dots, g_{t-1}, g_t, g_{t+1}, \dots, g_n)$$

for all $i \in I$, $\lambda \in \Lambda$, $g_1, \dots, g_n \in G$, $t = 1, \dots, n$. This shows that S/ρ is w-marginal. If $\tau = \rho_{(r, M, \pi)}$ is an arbitrary w-marginal congruence on S, then a similar argument as in the proof of Proposition 3.3 shows that $N \leq M$, hence ρ is the least w-marginal congruence on S. □

4. Nilpotency

Let M be a free monoid of countable rank. For $x, y, z_0, z_1, \dots$ in M define a sequence of words $q_n(x, y, z_0, \dots, z_{n-1})$ by $q_0(x, y) = x$ and

$$q_{n+1}(x, y, z_0, \dots, z_n) = q_n(x, y, z_0, \dots, z_{n-1}) z_n q_n(y, x, z_0, \dots, z_{n-1})$$

for $n \geq 0$. A semigroup S is said to be *nilpotent of class* c if it satisfies the identity $q_c(x, y, z_0, \ldots, z_{c-1}) = q_c(y, x, z_0, \ldots, z_{c-1})$ for all $x, y \in S$, $z_i \in S^1$, and c is the least positive integer with this property. The notion of a nilpotent semigroup was introduced by Neumann and Taylor [3]. They showed that a group is nilpotent of class $\leq c$ in the classical sense [8, Part 1, p. 49] if and only if it satisfies the above identity. Thus Proposition 3.2 provides a law characterizing completely simple over-$\mathscr{N}_c$ semigroups. Furthermore, that law can be replaced by a simpler one, as the following result shows.

Proposition 4.1. *Let S be a completely simple semigroup. Then S is over-$\mathscr{N}_c$ if and only if $q_c(a^0, a, x_0, \ldots, x_{c-1}) = q_c(a, a^0, x_0, \ldots, x_{c-1})$ for all $a, x_0, \ldots, x_{c-1} \in S$.*

Proof. Suppose first that the $\mathscr{H}$-classes of S are nilpotent of class $\leq c$. Let a and $x_0, x_1, \ldots, x_{c-1}$ be arbitrary elements of S. Then a, a^0 and $a^0 x_i a^0$, where $i = 0, \ldots, c-1$, belong to the same $\mathscr{H}$-class. Thus we have that $q_c(a, a^0, a^0x_0a^0, \ldots, a^0x_{c-1}a^0) = q_c(a^0, a, a^0x_0a^0, \ldots, a^0x_{c-1}a^0)$. Note that $q_1(a, a^0, a^0x_0a^0) = ax_0a^0 = q_1(a, a^0, x_0)$ and $q_1(a^0, a, a^0x_0a^0) = a^0x_0a = q_1(a^0, a, x_0)$. An induction argument now yields that $q_c(a, a^0, a^0x_0a^0, \ldots, a^0x_{c-1}a^0) = q_c(a^0, a, x_0, \ldots, x_{c-1})$, and similarly we also have $q_c(a^0, a, a^0x_0a^0, \ldots, a^0x_{c-1}a^0) = q_c(a^0, a, x_0, \ldots, x_{c-1})$ for all $a, x_0, \ldots, x_{c-1} \in S$.

Conversely, suppose that a semigroup S satisfies the law $q_c(a^0, a, x_0, \ldots, x_{c-1}) = q_c(a, a^0, x_0, \ldots, x_{c-1})$. Let $S = \mathcal{M}(I, G, \Lambda; P)$ with P normalized. Choosing $a = (1, g, 1)$ and $x_i = (1, g_i, 1)$ for $i = 0, \ldots, c-1$, we see that G satisfies the law $q_c(e, g, g_0, \ldots, g_{c-1}) = q_c(g, e, g_0, \ldots, g_{c-1})$. We show by induction on c that G is nilpotent of class $\leq c$. For $c = 1$ this is obvious, so assume that our claim holds true for some $c \geq 1$. Let now G be a group satisfying the law $q_{c+1}(e, g, g_0, \ldots, g_{c-1}, g_c) = q_{c+1}(g, e, g_0, \ldots, g_{c-1}, g_c)$. this can be rewritten as

$$q_c(e, g, g_0, \ldots, g_{c-1})g_c q_c(g, e, g_0, \ldots, g_{c-1}) = q_c(g, e, g_0, \ldots, g_{c-1})g_c q_c(e, g, g_0, \ldots, g_{c-1}).$$

Replacing the variable g_c by the expression $g_c q_c(g, e, g_0, \ldots, g_{c-1})^{-1}$, we conclude that the latter law is equivalent to the law $q_c(g, e, g_0, \ldots, g_{c-1})^{-1} q_c(e, g, g_0, \ldots, g_{c-1}) g_c = g_c q_c(g, e, g_0, \ldots, g_{c-1})^{-1} q_c(e, g, g_0, \ldots, g_{c-1})$. From here it follows directly that the element $q_c(g, e, g_0, \ldots, g_{c-1})^{-1} q_c(e, g, g_0, \ldots, g_{c-1})$ belongs to $Z(G)$, whence the factor group $G/Z(G)$ satisfies the law $q_c(e, g, g_0, \ldots, g_{c-1}) = q_c(g, e, g_0, \ldots, g_{c-1})$. By induction assumption, $G/Z(G)$ is nilpotent of class $\leq c$, therefore G is nilpotent of class $\leq c + 1$, as required. □

Note that it is proved in [4] that the variety of completely simple over-abelian semigroups is characterized by the law $a^0xa = axa^0$, thus Proposition 4.1 is a generalization of this result. It is also shown in [4] that the class of completely simple semigroups with over-abelian core is characterized by the law $ax^0a^0y^0a = ay^0a^0x^0a$. Our next result generalizes this to the nilpotent case.

Theorem 4.2. *Let S be a completely simple semigroup. Then $C(S)$ is over-$\mathscr{N}_c$ if and only if $aq_c(x^0, y^0, a^0x_0^0a^0, \ldots, a^0x_{c-1}^0a^0)a = aq_c(y^0, x^0, a^0x_0^0a^0, \ldots, a^0x_{c-1}^0a^0)a$ for all $a, x, y, x_0 \ldots, x_{c-1} \in S$.*

Proof. Suppose that the $\mathscr{H}$-classes of $C(S)$ are nilpotent of class $\leq c$. Let a, x, y and $x_0, \ldots, x_{c-1}$ be elements of S. Then $a^0x^0a^0$, $a^0y^0a^0$ and $a^0x_i^0a^0$, where $i = 0, \ldots, c-1$ are all elements of $C(S)$ and belong to the same $\mathscr{H}$-class. Thus it follows that $q_c(a^0x^0a^0, a^0y^0a^0, a^0x_0^0a^0, \ldots, a^0x_{c-1}^0a^0) = q_c(a^0y^0a^0, a^0x^0a^0, a^0x_0^0a^0, \ldots, a^0x_{c-1}^0a^0)$. It is straightforward to see that this is equivalent to $aq_c(x^0, y^0, a^0x_0^0a^0, \ldots, a^0x_{c-1}^0a^0)a = aq_c(y^0, x^0, a^0x_0^0a^0, \ldots, a^0x_{c-1}^0a^0)a$.

Conversely assume that S satisfies the law

$$aq_c(x^0, y^0, a^0x_0^0a^0, \ldots, a^0x_{c-1}^0a^0)a = aq_c(y^0, x^0, a^0x_0^0a^0, \ldots, a^0x_{c-1}^0a^0)a.$$

As we have seen in the first part of the proof, this law is equivalent to the law $q_c(a^0x^0a^0, a^0y^0a^0, a^0x_0^0a^0, \ldots, a^0x_{c-1}^0a^0) = q_c(a^0y^0a^0, a^0x^0a^0, a^0x_0^0a^0, \ldots, a^0x_{c-1}^0a^0)$. Suppose $S = \mathcal{M}(I, G, \Lambda; P)$ with P normalized and choose $a = (1, e, 1)$, $x = (i, g, \lambda)$,

$y = (j, h, \mu)$ and $x_k = (i_k, g_k, \lambda_k)$ for $k = 0, \ldots, c-1$. Using Lemma 3.1, we get $q_c(p_{\lambda i}^{-1}, p_{\mu j}^{-1}, p_{\lambda_0 i_0}^{-1}, \ldots, p_{\lambda_{c-1} i_{c-1}}^{-1}) = q_c(p_{\mu j}^{-1}, p_{\lambda i}^{-1}, p_{\lambda_0 i_0}^{-1}, \ldots, p_{\lambda_{c-1} i_{c-1}}^{-1})$. It follows that a generating set $\{p_{\lambda i}^{-1} : i \in I, \lambda \in \Lambda\}$ of $\langle P \rangle$ satisfies the Neumann-Taylor nilpotency condition. By [3, Corollary 3] we conclude that $\langle P \rangle$ is nilpotent of class $\leq c$. □

Lemma 4.3. *Let G be a group and e its identity element. Let c be a positive integer and $a \in G$. Then $a \in Z_c(G)$ if and only if $q_c(a, e, g_0, \ldots, g_{c-1}) = q_c(e, a, g_0, \ldots, g_{c-1})$ for all $g_0, \ldots, g_{c-1} \in G$.*

Proof. We prove this by induction on c. The case $c = 1$ is clear, thus we assume that the assertion holds true for some $c \geq 1$. Suppose that $a \in G$ is such that $q_{c+1}(a, e, g_0, \ldots, g_c) = q_{c+1}(e, a, g_0, \ldots, g_c)$ for all $g_0, \ldots, g_c \in G$. Similarly as in the proof of Proposition 4.1 we conclude that this equation is equivalent to the fact that $q_c(a, e, g_0, \ldots, g_{c-1})^{-1} q_c(e, a, g_0, \ldots, g_{c-1})$ belongs to $Z(G)$. For arbitrary $h \in G$ denote by $\bar{h}$ its canonical image in $G/Z(G)$. We get that $q_c(\bar{a}, \bar{e}, \bar{g}_0, \ldots, \bar{g}_{c-1}) = q_c(\bar{e}, \bar{a}, \bar{g}_0, \ldots, \bar{g}_{c-1})$ for all $g_0, \ldots, g_{c-1} \in G$. By induction hypothesis it follows that this is equivalent to $\bar{a} \in Z_c(G/Z(G)) = Z_{c+1}(G)/Z(G)$, which is further equivalent to $a \in Z_{c+1}(G)$. □

The class of completely simple central semigroups can be characterized by the law $a^0 x^0 a = a x^0 a^0$ [4]. This can be generalized as follows.

Theorem 4.4. *Let S be a completely simple semigroup. Then S is ω_c-marginal if and only if it satisfies the identity $q_c(a^0, a, x^0, x_1, \ldots, x_{c-1}) = q_c(a, a^0, x^0, x_1, \ldots, x_{c-1})$ for all $a, x, x_1, \ldots, x_{c-1} \in S$.*

Proof. Let $S = \mathcal{M}(I, G, \Lambda; P)$ with P normalized. Suppose first that S is ω_c-marginal. We prove that S satisfies the above identity by induction on c. When $c = 1$, this follows from [4]. Suppose that the assertion holds true for some $c \geq 1$. Let now S be ω_{c+1}-marginal. Let $\rho = \rho_{(\epsilon, Z(G), \epsilon)}$. By [6, Theorem III.4.6], $S/\rho \cong \mathcal{M}(I, G/Z(G), \Lambda; \bar{P})$, where $\bar{P}$ is the $\Lambda \times I$-matrix with (λ, i)-entry equal to $p_{\lambda i} Z(G)$. By Theorem 3.6 we have that $\langle P \rangle \leq Z_{c+1}(G)$. Factoring over $Z(G)$, we get that $\langle P \rangle Z(G)/Z(G) \leq Z_c(G/Z(G))$. This shows that S/ρ is ω_c-marginal. By induction assumption we get that

$$q_c(a^0, a, x^0, x_1, \ldots, x_{c-1}) \;\rho\; q_c(a, a^0, x^0, x_1, \ldots, x_{c-1})$$

for all $a, x, x_1, \ldots, x_{c-1} \in S$. It is not difficult to observe that $q_c(a^0, a, x^0, x_1, \ldots, x_{c-1})$ and $q_c(a, a^0, x^0, x_1, \ldots, x_{c-1})$ belong to the same $\mathscr{H}$-class. It follows from here that we can write $q_c(a^0, a, x^0, x_1, \ldots, x_{c-1}) = (i, u, \lambda)$ and $q_c(a, a^0, x^0, x_1, \ldots, x_{c-1}) = (i, v, \lambda)$ for some $i \in I$, $\lambda \in \Lambda$ and $u, v \in G$ with $u \equiv v \mod Z(G)$. We thus have $u = vz$ for some $z \in Z(G)$. Let $x_c = (j, g_c, \mu)$ be an arbitrary element of S. Then

$$\begin{aligned} q_{c+1}(a^0, a, x^0, x_1, \ldots, x_c) &= q_c(a^0, a, x^0, x_1, \ldots, x_{c-1}) x_c q_c(a, a^0, x^0, x_1, \ldots, x_{c-1}) \\ &= (i, u, \lambda)(j, g_c, \mu)(i, v, \lambda) \\ &= (i, u p_{\lambda j} g_c p_{\mu i} u z, \lambda) \\ &= (i, u z p_{\lambda j} g_c p_{\mu i} u, \lambda) \\ &= q_c(a, a^0, x^0, x_1, \ldots, x_{c-1}) x_c q_c(a^0, a, x^0, x_1, \ldots, x_{c-1}) \\ &= q_{c+1}(a, a^0, x^0, x_1, \ldots, x_c). \end{aligned}$$

Suppose now that S satisfies the identity

$$q_c(a^0, a, x^0, x_1, \ldots, x_{c-1}) = q_c(a, a^0, x^0, x_1, \ldots, x_{c-1})$$

for all $a, x, x_1, \ldots, x_{c-1} \in S$. Let $a = (i, g, \lambda)$, $x = (1, h, 1)$ and $x_l = (i, g_l, \lambda)$ be elements of S. We claim that $q_n(a^0, a, x^0, x_1, \ldots, x_{n-1}) = (i, p_{\lambda i}^{-1} q_n(e, p_{\lambda i}, g, g_1, \ldots, g_{n-1}) p_{\lambda i}^{-1}, \lambda)$ and $q_n(a, a^0, x^0, x_1, \ldots, x_{n-1}) = (i, p_{\lambda i}^{-1} q_n(p_{\lambda i}, e, g, g_1, \ldots, g_{n-1}) p_{\lambda i}^{-1}, \lambda)$ for all positive integers n. For $n = 1$ this is clearly true. For the induction step note that

$$\begin{aligned} q_{n+1}(a^0, a, x^0, x_1, \ldots, x_n) &= q_n(a^0, a, x^0, x_1, \ldots, x_{n-1}) x_n q_n(a, a^0, x^0, x_1, \ldots, x_{n-1}) \\ &= (i, p_{\lambda i}^{-1} q_n(e, p_{\lambda i}, g, g_1, \ldots, g_{n-1}) p_{\lambda i}^{-1}, \lambda)(i, g_n, \lambda) \end{aligned}$$

$$
\begin{aligned}
& \cdot(i, p_{\lambda i}^{-1} q_n(p_{\lambda i}, e, g, g_1, \ldots, g_{n-1}) p_{\lambda i}^{-1}, \lambda) \\
= \quad & (i, p_{\lambda i}^{-1} q_n(e, p_{\lambda i}, g, g_1, \ldots, g_{n-1}) g_n \\
& \cdot q_n(p_{\lambda i}, e, g, g_1, \ldots, g_{n-1}) p_{\lambda i}^{-1}, \lambda) \\
= \quad & (i, p_{\lambda i}^{-1} q_{n+1}(e, p_{\lambda i}, g, g_1, \ldots, g_n) p_{\lambda i}^{-1}, \lambda).
\end{aligned}
$$

Similarly we get that $q_{n+1}(a, a^0, x^0, x_1, \ldots, x_n) = (i, p_{\lambda i}^{-1} q_{n+1}(p_{\lambda i}, e, g, g_1, \ldots, g_n) p_{\lambda i}^{-1}, \lambda)$. It follows that the identity $q_c(e, p_{\lambda i}, g, g_1, \ldots, g_{c-1}) = q_c(p_{\lambda i}, e, g, g_1, \ldots, g_{c-1})$ holds true for all $g, g_1, \ldots, g_{c-1} \in G$ and $i \in I$, $\lambda \in \Lambda$. By Lemma 4.3 we get that $p_{\lambda i} \in Z_c(G)$, thus also $\langle P \rangle \leq Z_c(G)$. This shows that S is ω_c-marginal. □

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