Noname manuscript No.
(will be inserted by the editor)

On the centralizer and the commutator subgroup of an automorphism

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the date of receipt and acceptance should be inserted later

Abstract Let φ be an automorphism of a group G. In this paper, we study the influence of its centralizer $C_G(\varphi)$ on its commutator subgroup $[G,\varphi]$ when G is polycyclic or metabelian. For instance, when G is metabelian and φ fixed-point-free of prime order p, we prove that $[G,\varphi]$ is nilpotent of class $\leq p$. Also, when G is polycyclic and φ of order 2, we show that if $C_G(\varphi)$ is finite, then so are $G/[G,\varphi]$ and $[G,\varphi]'$.

Keywords Polycyclic group · Metabelian group · Automorphism · Fixed point

Mathematics Subject Classification (2010) 20E 36, 20F28

1 Introduction and main results

Let φ be an automorphism of a group G. Denote by $C_G(\varphi)$ the centralizer of φ in G, i.e., the subgroup of G consisting of all elements fixed by φ. Let $[G,\varphi] = \langle x^{-1}\varphi(x) \mid x \in G\rangle$ be the commutator subgroup of φ. Notice that $[G,\varphi]$ is normal in G since for all $x,y \in G$, if $z = \varphi^{-1}(y)$, we can write

$$\begin{aligned} y^{-1}\{x^{-1}\varphi(x)\}y &= \varphi(z^{-1})x^{-1}\varphi(x)\varphi(z) \\ &= \varphi(z^{-1})z(xz)^{-1}\varphi(xz). \end{aligned}$$

It is well known that, under suitable hypotheses, the fact that $C_G(\varphi)$ is "small" in some sense has consequences on G, and in particular on $G/[G,\varphi]$ and $[G,\varphi]'$. For example, Belyaev and Sesekin proved that if G is locally finite and if φ is of order 2,

The authors were partially supported by the bilateral grant BI-FR/10-11-PROTEUS-009.

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the fact that $C_G(\varphi)$ is finite implies that $G/[G,\varphi]$ and $[G,\varphi]'$ are finite too [1]. Further results on the influence of $C_G(\varphi)$ on G, $G/[G,\varphi]$ and $[G,\varphi]'$ can be found in a survey by Shumyatsky [8].
In this paper, we are interested in the case where G is polycyclic or metabelian. In particular, as a consequence of our first two results, we shall see that the result of Belyaev and Sesekin cited above remains valid when G is polycyclic (see Theorem 1 and 2 below).

Theorem 1 *Let φ be an automorphism of order 2 of a polycyclic group G. If $C_G(\varphi)$ is finite, then so is $[G,\varphi]'$.*

In a first version of this paper, we conjectured that when G is polycyclic and φ of prime order p, if $C_G(\varphi)$ is finite, then $[G,\varphi]$ is finite-by-nilpotent. According to Theorem 1, that is true when $p=2$, and also for an arbitrary prime p when G is metabelian (see Theorem 5 below). In fact, as Khukhro pointed out to the authors, this conjecture is false (see Example 1 in Section 3). In our next result, we have no restriction on the order of φ.

Theorem 2 *Let φ be an automorphism of a polycyclic group G. If $C_G(\varphi)$ is finite, then so is $G/[G,\varphi]$.*

In the particular case where φ is of finite order, the fact that $G/[G,\varphi]$ is finite is an easy consequence of the following result.

Theorem 3 *If φ is an automorphism of finite order of a polycyclic group G, we have $|G{:}C_G(\varphi)[G,\varphi]| < \infty$.*

However, we cannot deduce Theorem 2 from Theorem 3 in the general case. When φ is of infinite order, it can easily happen that $|G{:}C_G(\varphi)[G,\varphi]|$ is not finite. For example, if φ is the automorphism of $G=\mathbb{Z}^2$ defined by $\varphi(x,y)=(x+y,y)$, we have $C_G(\varphi)=[G,\varphi]=\{(x,y)\in\mathbb{Z}^2 \mid y=0\}$ and so the index of $C_G(\varphi)[G,\varphi]$ in G is infinite. If we only assume that G is soluble, then Theorems 2 and 3 fail, even if G is a finitely generated metabelian group (see Example 2 in Section 3). Our next results treat the case where G is metabelian and $C_G(\varphi)$ periodic. At first we introduce some notations. If π is a (possibly empty) set of primes, the class of π-groups will be denoted by $\mathscr{T}_\pi$ (recall that a π-group is a periodic group in which the order of each element is a π-integer, namely an integer such that π contains the set of primes dividing it). In particular, $\mathscr{T}_\pi$ is the class of trivial groups when π is empty and the class of periodic groups when π is the set of all prime numbers. Furthermore, we shall write $\mathscr{A}$ for the class of abelian groups and $\mathscr{E}_n$ for the class of groups satisfying the law $x^n=1$ (n being a positive integer). If $\mathscr{X}_1,\dots,\mathscr{X}_k$ are classes of groups, we say that a group G belongs to the class $\mathscr{X}_1\cdots\mathscr{X}_k$ if it has a series

$$1=H_0\trianglelefteq H_1\trianglelefteq\cdots\trianglelefteq H_{k-1}\trianglelefteq H_k=G$$

such that H_i/H_{i-1} belongs to $\mathscr{X}_i$ for $i=1,\dots,k$.

Theorem 4 *Let G be a metabelian group and φ an automorphism of G of finite order n. Suppose that $C_G(\varphi)$ is a periodic group which belongs to $\mathscr{T}_\pi$ (where π is a set of primes). Then $G/[G,\varphi]$ belongs to $\mathscr{T}_\pi\mathscr{E}_n\mathscr{A}$.*

Under the hypothesis of this theorem, when G is finite and n is a prime power p^{λ}, it is easy to see that $G/[G,\varphi]$ actually belongs to $\mathscr{T}_{\pi}\mathscr{A}$ (observe that if $p \notin \pi$, then p does not divide the order of G since p divides the cardinality of the set $G \setminus C_G(\varphi)$). However, in general, the quotient $G/[G,\varphi]$ need not be in $\mathscr{T}_{\pi}\mathscr{E}_n$ or in $\mathscr{T}_{\pi}\mathscr{A}$ (see Examples 2 and 3 respectively). Also, trivially, $G/[G,\varphi]$ need not be in $\mathscr{E}_n\mathscr{A}$: for instance, if G is a finite non-abelian metabelian group and if φ is the identity automorphism, $G/[G,\varphi] \simeq G$ does not belong to $\mathscr{E}_1\mathscr{A} = \mathscr{A}$.
Denote by $\mathscr{N}_c$ the class of all nilpotent groups of class at most c. The next result is about the subgroup $[G,\varphi]$:

Theorem 5 *Let G be a metabelian group, and let φ be an automorphism of G of prime order p. Let π be a set of primes, and suppose that $C_G(\varphi)$ is a periodic group which belongs to $\mathscr{T}_{\pi}$. Then $[G,\varphi]$ belongs to $\mathscr{T}_{\pi}\mathscr{N}_p$ (and even to $\mathscr{T}_{\pi}\mathscr{A}$ when $p=2$). In particular, if φ is fixed-point-free, then $[G,\varphi]$ is nilpotent of class at most p.*

This theorem fails to hold when G is soluble of derived length 3. For instance, if $G = S_4$ is the permutation group on $\{a,b,c,d\}$ and if φ is the inner automorphism $\sigma \mapsto \begin{pmatrix} a & b & c \end{pmatrix}^{-1} \sigma \begin{pmatrix} a & b & c \end{pmatrix}$, then $C_G(\varphi)$ is the subgroup generated by $\begin{pmatrix} a & b & c \end{pmatrix}$ and $[G,\varphi]$ the alternating subgroup A_4; but A_4 is not an extension of a 3-group by a nilpotent group. Also, in Theorem 5, we cannot conclude that $[G,\varphi]$ is nilpotent: if H is a finite non-nilpotent metabelian group and if φ is the automorphism of $G = H \times H$ defined by $\varphi(x,y) = (y,x)$, it is easy to see that $[G,\varphi]$ is not nilpotent. In the same way, Example 2 (or 3) shows that $[G,\varphi]$ need not be periodic. These examples also show that under the hypothesis of Theorem 5, unlike $[G,\varphi]$, the group G itself need not be periodic-by-nilpotent. Notice that if φ is an automorphism of prime order p of a *polycyclic* group G such that $C_G(\varphi)$ is periodic (and so finite), then G is an extension of a nilpotent group (of class bounded by a function of p) by a finite group [3].
It follows from Theorem 5 that if φ is a fixed-point-free automorphism of order 2, then $[G,\varphi]$ is abelian when G is metabelian. But in fact, this result is true for any soluble group:

Theorem 6 *If φ is a fixed-point-free automorphism of order 2 of a soluble group G, then $[G,\varphi]$ is abelian.*

2 Proofs

Notations used in this paper are standard. In particular, we shall denote by $G^{(r)}$ the r-th term of the derived series: thus $G^{(r+1)}$ is the derived subgroup of $G^{(r)}$, with $G^{(0)} = G$. We also write $G' = G^{(1)}$ and $G'' = G^{(2)}$. To prove our first theorem, we need a preliminary result:

Lemma 1 *Let φ be an automorphism of order 2 of a group G and let A be a normal φ-invariant abelian subgroup of G. If $A \cap C_G(\varphi) = 1$, then $[[G,\varphi],A] = 1$.*

Proof For all $a \in A$, the element $a\varphi(a)$ belongs to $A \cap C_G(\varphi)$ whence the relation $\varphi(a) = a^{-1}$. Therefore, for all $x \in G$, we may write

$$\varphi(x^{-1}ax) = \varphi(x^{-1})a^{-1}\varphi(x)$$

and also

$$\varphi(x^{-1}ax) = x^{-1}a^{-1}x$$

since $x^{-1}ax$ belongs to A. It follows $\varphi(x)x^{-1}a^{-1}x\varphi(x^{-1}) = a^{-1}$ and the lemma is proved. $\square$

Proof of Theorem 1 We proceed by induction on the derived length r of $[G,\varphi]$. If $r = 0$ or 1 (that is, $[G,\varphi] = 1$ or $[G,\varphi]' = 1$), the result is obvious. Now suppose $r \geq 2$. The subgroup $[G,\varphi]^{(r-1)}$ is then abelian; denote by T its torsion subgroup and by $\overline{\varphi}$ the automorphism induced by φ on G/T. Since T is finite, the subgroup $C_{G/T}(\overline{\varphi})$ is finite [2, Lemma 2.4(i)] and so the intersection of $C_{G/T}(\overline{\varphi})$ and $[G,\varphi]^{(r-1)}/T$ is trivial for $[G,\varphi]^{(r-1)}/T$ is torsion-free. Consequently, by Lemma 1, $[G,\varphi]^{(r-1)}/T$ is included in the centre of $[G,\varphi]/T$ and it follows that $[G,\varphi]^{(r-2)}/T$ is nilpotent (of class at most 2). Applying Corollary 2.1 of [2], we may then deduce that $[G,\varphi]^{(r-2)}/T$ is finite-by-abelian and so that $[G,\varphi]^{(r-1)}/T$ is finite. Since $[G,\varphi]^{(r-1)}/T$ is torsion-free, we have then $[G,\varphi]^{(r-1)} = T$, thus $[G/T,\overline{\varphi}]^{(r-1)} = 1$. By induction, it follows that $[G/T,\overline{\varphi}]'$ is finite. But $[G/T,\overline{\varphi}]' = [G,\varphi]'/T$, hence $[G,\varphi]'$ is finite and the proof is complete. $\square$

Proof of Theorem 2 We proceed again by induction on the derived length r of $[G,\varphi]$. If $r = 0$, we have $[G,\varphi] = 1$ and so $C_G(\varphi) = G$. Thus G is finite and the result follows. If $r > 0$, put $A = [G,\varphi]^{(r-1)}$ and consider the automorphism $\overline{\varphi}$ induced by φ on G/A. Since A is finitely generated and abelian, the subgroup $C_{G/A}(\overline{\varphi})$ is finite [2, Lemma 2.4(ii)]. By induction, we deduce that the index of $[G/A,\overline{\varphi}]$ in G/A is finite. But $[G/A,\overline{\varphi}] = [G,\varphi]/A$ and $|G/A\colon[G,\varphi]/A| = |G\colon[G,\varphi]|$, hence $|G\colon[G,\varphi]|$ is finite, as required. $\square$

For convenience, we recall the following well-known result:

Lemma 2 *Let G be an infinite polycyclic group. Then G contains a characteristic abelian subgroup A which is torsion-free infinite.*

Proof The group G contains a normal torsion-free subgroup K of finite index [6, 1.3.4] and K can be chosen characteristic [6, 1.3.7]. We can then take A to be the last non-trivial term of the derived series of K. $\square$

Lemma 3 *Let φ be an automorphism of a group G such that the subgroup $[G,\varphi]$ is finite. Then the index of $C_G(\varphi)$ in G is finite.*

Proof Denote by m the order of $[G,\varphi]$ and consider $m+1$ elements $x_1,\dots,x_{m+1}$ in G. Therefore, among the elements

$$x_1^{-1}\varphi(x_1),\dots,x_{m+1}^{-1}\varphi(x_{m+1}),$$

at least two coincide. If $x_i^{-1}\varphi(x_i) = x_j^{-1}\varphi(x_j)$ ($i,j \in \{1,\dots,m+1\}$, $i \neq j$), it follows $\varphi(x_ix_j^{-1}) = x_ix_j^{-1}$ and so $x_ix_j^{-1} \in C_G(\varphi)$. Hence we deduce that $|G\colon C_G(\varphi)| \leq m$, and so the lemma is proved. $\square$

Lemma 4 *Let φ be an automorphism of an abelian group G. If φ is of finite order n, for any $x \in G$, the element x^n can be written in the form $x^n = vw^{-1}\varphi(w)$, with $v \in C_G(\varphi)$ and $w \in G$. In particular, if G is finitely generated, then the quotient $G/C_G(\varphi)[G,\varphi]$ is finite.*

Proof Consider an element $x \in G$ and put $v = x\varphi(x)\cdots\varphi^{n-1}(x)$. First notice that v clearly belongs to $C_G(\varphi)$. Now observe that

$$\varphi(x) \equiv x \bmod [G,\varphi]$$

and so more generally

$$\varphi^k(x) \equiv x \bmod d[G,\varphi]$$

for any positive integer k. That implies

$$v \equiv x^n \bmod [G,\varphi],$$

thus the element $u = v^{-1}x^n$ belongs to $[G,\varphi]$. It follows that $x^n = vu$, with $v \in C_G(\varphi)$ and $u \in [G,\varphi]$. Since G is abelian, each element of $[G,\varphi]$ is of the form $w^{-1}\varphi(w)$ ($w \in G$), hence the proof is complete. □

Lemma 5 *Let φ be an automorphism of order n of a group G. Let A be a torsion-free normal φ-invariant abelian subgroup of G. Put $B = A^n$ and consider an element $x \in G$ such that $\varphi(x) \equiv x \bmod B$ (in other words, xB is a fixed point of the automorphism induced by φ in G/B). Then x belongs to $C_G(\varphi)A$.*

Proof We have $\varphi(x) = xy$ for some $y \in B$. Applying Lemma 4 to the restriction φ_A of φ to A, we conclude that y can be written in the form $y = vw^{-1}\varphi(w)$, where $v \in C_A(\varphi_A)$, $w \in A$, and so $\varphi(x) = xvw^{-1}\varphi(w)$. Induction now shows that $\varphi^k(x) = xv^k w^{-1}\varphi^k(w)$ for all positive integers k. For $k = n$, we obtain $v^n = 1$ whence $v = 1$ since A is torsion-free. Consequently, we have $\varphi(x) = xw^{-1}\varphi(w)$ and so $\varphi(xw^{-1}) = xw^{-1}$. In other words, the element $u = xw^{-1}$ belongs to $C_G(\varphi)$, thus $x = uw \in C_G(\varphi)A$. □

Proof of Theorem 3 We proceed by induction on the Hirsch length λ of G. The result is trivial when $\lambda = 0$, so suppose that $\lambda > 0$. By Lemma 3, if $[G,\varphi]$ is finite, then the index of $C_G(\varphi)$ in G is finite and so the result follows. Therefore, we can assume that $[G,\varphi]$ is infinite. By Lemma 2, $[G,\varphi]$ contains a characteristic abelian subgroup A which is torsion-free infinite (notice that A is φ-invariant). Put $B = A^n$ and denote by $\overline{\varphi}$ the automorphism induced by φ on G/B. It follows from Lemma 5 that $C_{G/B}(\overline{\varphi}) \leq AC_G(\varphi)/B$. Since the Hirsch length of G/B is $< \lambda$, we deduce from the inductive hypothesis that the index of $C_{G/B}(\overline{\varphi})[G/B,\overline{\varphi}]$ in G/B is finite. But we have clearly

$$C_{G/B}(\overline{\varphi})[G/B,\overline{\varphi}] \leq AC_G(\varphi)[G,\varphi]/B = C_G(\varphi)[G,\varphi]/B,$$

thus $|G/B : C_G(\varphi)[G,\varphi]/B| < \infty$. Consequently, $|G : C_G(\varphi)[G,\varphi]| < \infty$, as required. □

We state here a consequence of Theorem 3:

Proposition 1 *Let $\mathscr{X}$ be a class of groups which is closed with respect to formation of subgroups and homomorphic images of its members. Let φ be an automorphism of finite order of a polycyclic group G and suppose that $C_G(\varphi)$ belongs to $\mathscr{X}$. Then $G/[G,\varphi]$ belongs to $\mathscr{X}\mathscr{F}$, where $\mathscr{F}$ is the class of finite groups.*

Proof First observe that the group $C_G(\varphi)[G,\varphi]/[G,\varphi]$ belongs to $\mathscr{X}$ since it is isomorphic to $C_G(\varphi)/C_G(\varphi)\cap[G,\varphi]$. Now consider the core

$$N = \bigcap_{x\in G} x^{-1}C_G(\varphi)[G,\varphi]x$$

of $C_G(\varphi)[G,\varphi]$ in G. Since $|G{:}C_G(\varphi)[G,\varphi]| < \infty$ by Theorem 3, $C_G(\varphi)[G,\varphi]$ has only a finite number of conjugates and so G/N is finite. Furthermore $N/[G,\varphi]$ is a subgroup of $C_G(\varphi)[G,\varphi]/[G,\varphi]$, hence $N/[G,\varphi]$ belongs to $\mathscr{X}$ and the proof is complete. □

Proof of Theorem 4 Clearly it suffices to show that $G'/[G,\varphi]\cap G'$ belongs to $\mathscr{T}_\pi\mathscr{E}_n$. In other words, we must prove that for all $x\in G'$, x^n is a π-element modulo $[G,\varphi]\cap G'$. For that, consider the restriction $\varphi'{:}G'\to G'$ of φ to G'. By Lemma 4, for all $x\in G'$, we have $x^n = vw^{-1}\varphi(w)$, with $v\in C_{G'}(\varphi')$ and $w\in G'$. Since $w^{-1}\varphi(w)$ belongs to $[G,\varphi]\cap G'$ and v is a π-element (it belongs to $C_G(\varphi)$), the proof is complete. □

Let n be a positive integer. Recall that an automorphism φ of a group G is said to be *splitting of order n* if φ^n is the identity automorphism and if $x\varphi(x)\cdots\varphi^{n-1}(x) = 1$ for all $x\in G$. To prove Theorem 5, we shall use the following result due to Khukhro:

Proposition 2 ([5]) *If a soluble group of derived length r admits a splitting automorphism of prime order p, then it is nilpotent, and its nilpotency class is bounded by a function $g = g(p,r)$ depending only on p and r.*

A splitting automorphism of order 2 inverts every element. Therefore, in the proposition above, we can take $g(2,r) = 1$ and so the bound is independent of r when $p = 2$. For an arbitrary prime p, we can take $g(p,2) = p$ (see [5, p. 78]).

Lemma 6 *Let φ be an automorphism of finite order n of a metabelian group G such that $C_G(\varphi)$ is a π-group. Put $H = \prod_{q\in\pi} H_q$, where H_q is the q-primary component of $[G,\varphi]\cap G'$. Then:*

(i) For all $t\in[G,\varphi]\cap G'$, the product $t\varphi(t)\cdots\varphi^{n-1}(t)$ belongs to H;
(ii) φ induces a splitting automorphism of order n on $[G,\varphi]/H$.

Proof (i) Clearly, the automorphism φ fixes $t\varphi(t)\cdots\varphi^{n-1}(t)$. It follows that this product is a π-element and so belongs to H.
(ii) Consider an element $x\in[G,\varphi]$ and put $y = x\varphi(x)\cdots\varphi^{n-1}(x)$. We must prove that y belongs to H. For that, first notice that x can be written in the form $x = w^{-1}\varphi(w)t$, with $w\in G$ and $t\in G'$. Observe that $t\in[G,\varphi]\cap G'$. Then we have:

$$y = \prod_{k=1}^{n} \varphi^{k-1}(w^{-1})\varphi^k(w)\varphi^{k-1}(t)$$

$$
\begin{aligned}
&= w^{-1}\{\prod_{k=1}^{n}\varphi^k(w)\varphi^{k-1}(t)\varphi^k(w^{-1})\}w\\
&= w^{-1}\{\prod_{k=1}^{n}\varphi^{k-1}(t)[\varphi^{k-1}(t),\varphi^k(w^{-1})]\}w\\
&= w^{-1}\{\prod_{k=1}^{n}\varphi^{k-1}(t)\varphi^{k-1}([t,\varphi(w^{-1})])\}w.
\end{aligned}
$$

Since the factors $\varphi^{k-1}(t)$ and $\varphi^{k-1}([t,\varphi(w^{-1})])$ are in G', we obtain

$$y = w^{-1}\{\prod_{k=1}^{n}\varphi^{k-1}(t)\}\{\prod_{k=1}^{n}\varphi^{k-1}([t,\varphi(w^{-1})])\}w.$$

By the first part of our lemma, $\prod_{k=1}^{n}\varphi^{k-1}(t)$ and $\prod_{k=1}^{n}\varphi^{k-1}([t,\varphi(w^{-1})])$ belong to H and so does y, as required. □

Proof of Theorem 5 The proof is now immediate: with the notation of Lemma 6 and Proposition 2, and as a consequence of these results, we may assert that $[G,\varphi]/H$ is nilpotent of class at most $g(p,2) = p$ (and of class at most 1 when $p = 2$). Since H is a π-group, we obtain the desired conclusion. □

Proof of Theorem 6 Let $A \trianglelefteq G$ be a maximal abelian normal φ-invariant subgroup contained in $[G,\varphi]$. In order to obtain a contradiction, suppose that A and $[G,\varphi]$ are distinct. Let B/A be the smallest non-trivial term of the derived series of $[G,\varphi]/A$. Thus A is a proper subgroup of B. By Lemma 1, A is contained in the centre of $[G,\varphi]$, hence B is nilpotent of class at most 2. But a nilpotent group admitting a fixed-point-free automorphism of order 2 is abelian [4, Theorem 3]. It follows that B is abelian, a contradiction by maximality of A. □

3 Examples

Example 1 Let R denote the subgroup of $GL(2,\mathbb{C})$ generated by the matrices

$$a = \begin{pmatrix} i & 0\\ 0 & -i\end{pmatrix},\ b = \begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix},\ c = \begin{pmatrix} 0 & i\\ i & 0\end{pmatrix},$$

$$d = 2^{-1}(i-1)\begin{pmatrix} 1 & -i\\ 1 & i\end{pmatrix},\ \text{with } i^2 = -1.$$

The matrix d has order 3 and the subgroup $Q := \langle a,b,c\rangle$ is a normal subgroup of R isomorphic to the quaternion group of order 8 (see [7, p. 245]). Thus R is a finite group of order 24. Regard the elements of $\mathbb{C}^2$ as column matrices with two lines and coefficients in $\mathbb{C}$. The group R acting in a natural way on $\mathbb{C}^2$, choose a non-zero element $v_0 \in \mathbb{C}^2$ and denote by V the (additive) subgroup of $\mathbb{C}^2$ generated by the R-orbit of v_0. In the semidirect product $V \rtimes R$, consider the normal subgroup $G := V \rtimes Q$. Note that $V \rtimes R$ is polycyclic (and so G too). The element $(0,d)$ induces by conjugation an automorphism $\varphi\colon G \to G$. Thus φ has order 3 and we have $\varphi(v,m) = (d^{-1}v, d^{-1}md)$

for all $v \in V$ and $m \in Q$. It is easy to verify that $C_G(\varphi) = \{(0,I), (0,-I)\}$, where I denotes the identity matrix. Now consider the elements $x, y \in [G,\varphi]$ defined by

$$x = (v_0, I)^{-1}\varphi(v_0, I) = ((d^{-1} - I)v_0, I)$$

and

$$y = (0,a)^{-1}\varphi(0,a) = (0, a^{-1}d^{-1}ad) = (0, c^{-1}).$$

For any $k \in \mathbb{N}$, define $[x,_k y]$ by $[x,_0 y] = x$ and $[x,_{k+1} y] = [x,_k y]^{-1} y^{-1} [x,_k y] x$. An easy induction leads then to the relation

$$[x,_k y] = ((c-I)^k (d^{-1} - I)v_0, I) \text{ for all } k \in \mathbb{N}.$$

This element is of infinite order since the matrix $(c-I)^k(d^{-1}-I)$ is nonsingular and so $[G,\varphi]$ is not finite-by-nilpotent.

Example 2 This example is given in [2]. For convenience, we summarize here its main properties and we refer to the paper cited above for more details. Let $\mathbb{Z}[x^{\pm 1}]$ be the ring of Laurent polynomials in one indeterminate with coefficients in the ring of integers $\mathbb{Z}$. Let G be the group of matrices of the form

$$m(i,f) = \begin{pmatrix} x^i & f \\ 0 & 1 \end{pmatrix} \quad (i \in \mathbb{Z},\ f \in \mathbb{Z}[x^{\pm 1}])$$

with the usual multiplication. This group is metabelian and generated by $u = m(1,0)$ and $v = m(0,1)$. Consider the subgroup A formed by the elements

$$m(0,f) = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \quad (f \in \mathbb{Z}[x^{\pm 1}]).$$

Each element of G can be uniquely written in the form $u^i a$, where $i \in \mathbb{Z}$ and $a \in A$, and the function $\varphi\colon G \to G$ defined by $\varphi(u^i a) = (uv)^i a^{-1}$ is an automorphism of order 2 such that $C_G(\varphi) = 1$. It is then easy to see that $[G,\varphi] = A$. Consequently, $[G,\varphi]$ is torsion-free infinite and $G/[G,\varphi]$ is infinite cyclic. Thus Theorem 2 cannot be extended to soluble groups. Likewise, in Theorem 4 (resp. in Theorem 5), we cannot conclude that $G/[G,\varphi]$ (resp. $[G,\varphi]$) is periodic.

Example 3 Let q be an odd prime and let ω be a primitive q-th root of unity. Denote by $\mathbb{Z}[\omega]$ the subring of $\mathbb{C}$ generated by ω. Here G is defined as the group of matrices of the form

$$\begin{pmatrix} \omega^i & z \\ 0 & 1 \end{pmatrix} \quad (i \in \mathbb{Z},\ z \in \mathbb{Z}[\omega])$$

with the usual multiplication (notice that this group is a homomorphic image of the group defined in the preceding example). The group G is metabelian and polycyclic since it is an extension of the group

$$A = \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \in G \mid z \in \mathbb{Z}[\omega] \right\}$$

(isomorphic to $\mathbb{Z}^{q-1}$) by a cyclic group of order q. Now consider the automorphism $\varphi\colon G \to G$ of order 2 defined by

$$\varphi\colon \begin{pmatrix} \omega^i & z \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} \omega^i & -z \\ 0 & 1 \end{pmatrix}.$$

Obviously, $C_G(\varphi)$ is cyclic of order q. Straightforward calculation shows that

$$[G,\varphi] = \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \in G \mid z \in 2\mathbb{Z}[\omega] \right\}$$

and

$$G' = \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \in G \mid z \in (1-\omega)\mathbb{Z}[\omega] \right\},$$

where $(1-\omega)\mathbb{Z}[\omega]$ (resp. $2\mathbb{Z}[\omega]$) denotes the ideal of $\mathbb{Z}[\omega]$ generated by $1-\omega$ (resp. by 2). It follows that the quotient $G/[G,\varphi]$ is not abelian. Furthermore, this quotient is an extension of an elementary abelian 2-group of order 2^{q-1} by a cyclic group of order q. Thus $G/[G,\varphi]$ is not in $\mathscr{T}_\pi\mathscr{A}$, where $\pi = \{q\}$. This shows that in the statement of Theorem 4, we cannot replace $\mathscr{T}_\pi\mathscr{E}_n\mathscr{A}$ by $\mathscr{T}_\pi\mathscr{A}$.

Acknowledgements The authors are grateful to E.I. Khukhro for useful comments on a first version of this paper. In particular, he suggested Theorem 6 and Example 1.

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