# ON THE EXPONENT SEMIGROUPS OF FINITE $p$-GROUPS

PRIMOŽ MORAVEC

Abstract. In this note we describe the exponent semigroups of finite $p$-groups of maximal class and finite $p$-groups of class $\leq 5$. Consequently, sharp bounds for the exponent of the Schur multiplier of a finite $p$-group of class $\leq 4$ are obtained. Our results extend some well-known results of Jones (1974).

## 1. Introduction

A group $G$ is said to be $n$-*abelian* if the map $x \mapsto x^n$ is an endomorphism of $G$. The study of $n$-abelian groups was initiated by Levi in [14], and has been a topic of several other papers, see, e.g., [1, 4, 8, 12, 15]. Given a group $G$, define

$$\mathcal{E}(G) = \{n \in \mathbb{Z} : (xy)^n = x^n y^n \text{ for all } x, y \in G\}.$$

It is clear that $\mathcal{E}(G)$ is a multiplicative subsemigroup of $\mathbb{Z}$ containing 0 and 1. Following Kappe [12], we say that $\mathcal{E}(G)$ is the *exponent semigroup* of $G$. One of the main results of [12] is a number-theoretic characterization of $\mathcal{E}(G)$ for an arbitrary group $G$. We have that $\mathcal{E}(G)$ is either $\{0, 1\}$, $\mathbb{Z}$ or a so-called Levi system [12]. When $G$ is a finite $p$-group, a more refined description of $\mathcal{E}(G)$ can be obtained. It is proved in [16] that for every finite $p$-group $G$ there exists a nonnegative integer $r$ such that $\mathcal{E}(G) = p^{e+r}\mathbb{Z} \cup (p^{e+r}\mathbb{Z} + 1)$, where $\exp G/Z(G) = p^e$. Following [16], we say that $r$ is the *exponential rank* of $G$, and denote it by $\operatorname{exprank}(G)$. The exponential rank of a finite $p$-group $G$, together with $\exp G/Z(G)$, completely determines the endomorphisms of $G$ of the form $x \mapsto x^n$. Moreover, it has been shown in [16] that $\operatorname{exprank}(G)$ can be bounded in terms of $\exp G/Z(G)$. In [16], some classes of finite $p$-groups having small exponential rank have been exhibited. For instance, every finite abelian $p$-group clearly has exponential rank zero, and the same is true for regular $p$-groups [16]. It has also been proved in [16] that powerful $p$-groups have exponential rank at most 1; when $p$ is odd, then the exponential rank is always zero, and when $p = 2$, the exponential rank is 1, if the group in question is nonabelian.

Roughly speaking, the exponential rank of a finite $p$-group $G$ can be calculated once the power-commutator structure of $G$ has been determined. A prominent class of groups for which this structure is well understood is the class of finite $p$-groups of maximal class. It can be seen that if $G$ is a finite $p$-group of maximal class, then its exponential rank is at most one. This bound is best possible, as Example 3 shows. We characterize finite $p$-groups of maximal class with zero exponential rank. More precisely, we prove that a finite $p$-group $G$ of maximal class has zero exponential rank if and only if $\exp G = \exp G/Z(G)$ or $G$ is regular. At the other end of the scale, finite $p$-groups of small nilpotency class also have small exponential rank. If $G$ is nilpotent of class $\leq 4$, then we prove that $\operatorname{exprank}(G) \leq 1$. Furthermore, if the nilpotency class of $G$ is at most 3, then there is a nice criterion for $G$ to have zero exponential rank. For instance, if $G$ is nilpotent of class 2, then $\operatorname{exprank}(G) = 1$ if $p = 2$, and $\operatorname{exprank}(G) = 0$ for odd $p$. We also deal with groups of class 5 and show that their exponential rank is at most 1 if $p$ is odd, whereas $\operatorname{exprank}(G) \leq 2$ for every finite 2-group of class 5. These bounds are shown to be best possible. Note that similar calculations could be done for groups of class $\geq 6$, but they would probably become rather lengthy.

*Date*: November 1, 2007.

2000 *Mathematics Subject Classification.* 20D15.

The author was supported by the Ministry of Higher Education, Science and Technology of Slovenia. He wishes to thank the referee for providing constructive comments and help in improving the contents of this paper, and Robert F. Morse and Gunnar Traustason for their help.

As observed in [16], the exponent of the Schur multiplier $M(G)$ of a finite $p$-group $G$ is closely related to the exponential rank of its covering group $H$. For instance, it follows from [16] that if $H$ is $p^e$-abelian, then $\exp M(G)$ divides $p^e$. Thus the exponential rank of $H$, together with the exponent of $H/Z(H)$, provides an estimate for the exponent of $M(G)$. It was a long-standing problem as to whether the exponent of $M(G)$ divides the exponent of $G$. This question was settled by Bayes et al. [2] who constructed a group of order $2^{21}$ which has exponent 4 and multiplier of exponent 8. This is in a sense best possible, since the exponent of $M(G)$ divides 8 for any group $G$ of exponent 4 [16]. On the other hand, Jones [11] proved that if $G$ is a finite $p$-group of class $\le 2$, then $\exp M(G)$ divides $\exp G$, and if $G$ is of class $\le 3$, then $\exp M(G)$ divides $\exp G$ when $p \neq 3$. These results were extended by Kayvanfar and Sanati [13] who proved that if $G$ is a finite $p$-group of class $\le 6$ and exponent $p^e$, then $\exp M(G)$ divides $p^e$ for appropriately chosen $p$ and $e$. What we show in this note is that most of those restrictions on $p$ and $e$ are redundant if $G$ is nilpotent of class $\le 4$. First we improve the above mentioned result of Jones by proving that if $G$ is a finite $p$-group of class 3, then $\exp M(G)$ divides $\exp G$ for any prime $p$. If $G$ is nilpotent of class 4, then $\exp M(G)$ divides $\exp G$, if $p$ is odd. For 2-groups of class 4 this is no longer true. We construct a group of order $2^{68}$, exponent 4 and class 4, with multiplier of exponent 8. Our example is obtained with the help of GAP [7]. Using the notion of exponential rank, we also prove that, given the nilpotency class, the exponent of the Schur multiplier of a $p$-group $G$ divides $\exp G$ for almost all primes $p$.

## 2. Finite $p$-groups of maximal class

For a finite $p$-group $G$, $\Omega_i(G)$ and $\mho_i(G)$ denote, respectively, the subgroups generated by all elements of order at most $p^i$ and by all $p^i$th powers in $G$.

Let $G$ be a finite $p$-group of order $p^n$. It is well known that the nilpotency class of $G$ is at most $n-1$. Thus $G$ is said to be of *maximal class*, if $n > 3$ and the nilpotency class of $G$ is $n-1$. We refer to Blackburn [3] or Huppert [9] for a comprehensive account on finite $p$-groups of maximal class.

The purpose of this section is to determine the exponential rank of finite $p$-groups of maximal class. We note here that some of our arguments actually work in a more general setting. For $3 < m \le n-1$ denote by $\mathrm{CF}(m,n,p)$ the set of all groups of order $p^n$ and class $m-1$ in which $|\gamma_i(G):\gamma_{i+1}(G)| = p$ for $i = 2,3,\ldots,m-1$, see [3]. Clearly every group of order $p^n$ and class $n-1$, where $n > 3$, belongs to $\mathrm{CF}(n,n,p)$.

**Proposition 1.** *Let $G \in \mathrm{CF}(m,n,p)$. Then* $\operatorname{exprank}(G) \le 1$.

*Proof.* Let $\exp G/Z(G) = p^f$ and let $x$ and $y$ be arbitrary elements of $G$. Then we can write $(xy)^{p^f} = x^{p^f} y^{p^f} c$ for some $c \in \gamma_2(G) \cap Z(G)$. We have that $\gamma_{m-1}(G) \le Z(G)$ and $\gamma_{m-2}(G) \not\le Z(G)$. If $Z(G)\cap\gamma_2(G) > \gamma_{m-1}(G)$, then $\gamma_{m-2}(G)$ is contained in $Z(G)\cap\gamma_2(G)$, since $G \in \mathrm{CF}(m,n,p)$. This contradiction shows that $\gamma_2(G)\cap Z(G)$ is cyclic of order $p$, hence $(xy)^{p^{f+1}} = x^{p^{f+1}} y^{p^{f+1}}$. This proves our claim. □

Let $m$, $n$ and $p$ be as above. Following [3], let $\mathrm{ECF}(m,n,p)$ be the set of all groups of $\mathrm{CF}(m,n,p)$ in which $G/\gamma_2(G)$ is elementary abelian. Again it is easy to see that $\mathrm{ECF}(n,n,p)$ contains all groups of order $p^n$ and maximal class.

**Proposition 2.** *Let $G \in \mathrm{ECF}(m,n,2)$. Then* $\operatorname{exprank}(G) = 1$.

*Proof.* We use a description of groups in $\mathrm{ECF}(m,n,2)$ obtained by James [10]. Define

$$P_1 = C_G(\gamma_2(G)/\gamma_4(G)).$$

Then there exist $s \in G\backslash P_1$, $s_1 \in P_1\backslash Z_{m-2}(G)$ and a subset $T = \{t_1,\ldots,t_{n-m}\}$ of $C_G(s)$ such that $G = \langle s\rangle P_1$, $P_1 = \langle s_1\rangle Z_{m-2}(G)$, $Z_{m-2}(G) = \langle T\rangle\gamma_2(G)$, and we have the following relations:

$$\begin{aligned}
s_i &= [s_{i-1}, s] && \text{for } i = 2, 3, \ldots, m-1,\\
s^2 &= s_{m-1}^{\alpha},\\
s_1^2 s_2 &= s_{m-1}^{\beta},\\
s_i^2 s_{i+1} &= 1 && \text{for } i = 2, 3, \ldots, m-2,\\
t_j^2 &= s_{m-1}^{\gamma_j} && \text{for } j = 1, 2, \ldots, n-m,\\
[s_i, s_j] &= 1 && \text{for } i \geq 2, j \geq 1,\\
[T, \gamma_2(G)] &= 1,\\
[T, s] &= 1,\\
[s_1, t_j] &= s_{m-1}^{\delta_j} && \text{for } j = 1, 2, \ldots, n-m,
\end{aligned}$$

where $\alpha, \beta, \gamma_j, \delta_j \in \{0, 1\}$, and $\alpha$ and $\beta$ are not both 1. There exists an integer $l$ such that $Z(G) = \langle t_{l+1}, \ldots, t_{n-m}, s_{m-1}\rangle$. We obtain from the above relations that $|s_1| = 2^{m-1}$, and it also follows that $\exp G/Z(G) = 2^{m-2}$. Since $ss_1 \notin P_1$, we have that $(ss_1)^4 = 1$, and similarly also $s^4 = 1$. As $m > 3$, we obtain $(ss_1)^{2^{m-2}} = 1$, whereas $s^{2^{m-2}} s_1^{2^{m-2}} = s_1^{2^{m-2}} \neq 1$. This shows that $\operatorname{exprank}(G) > 0$. By Proposition 1 we therefore have that $\operatorname{exprank}(G) = 1$. □

We note here that, for every prime $p$, there exists a $p$-group $G$ of maximal class with $\operatorname{exprank}(G) = 1$, as the following example shows.

*Example* 3. Let $p$ be a prime and let $G = C_p \wr C_p$ be the wreath product of two cyclic groups of order $p$. Then $G$ is a group of order $p^{p+1}$, exponent $p^2$ and class $p$. We have that $\exp G/Z(G) = p$. We can write $G = H\langle y\rangle$, where $H = \langle x_1, x_2, \ldots, x_p\rangle$ is an elementary abelian $p$-group, $y^p = 1$ and $x_i^y = x_{i+1}$ ($i \mod p$). A short calculation gives

$$(x_1 y^{-1})^p = x_1^{1+y+\cdots+y^{p-1}} y^{-p} = x_1 x_2 \cdots x_p,$$

whereas $x_1^p y^{-p} = 1$. This shows that $\operatorname{exprank}(G) = 1$.

The main theorem of this section describes finite $p$-groups of maximal class with zero exponential rank. Before formulating the result, recall that a $p$-group $G$ is said to be *regular* [9, p. 321], if for all $x, y \in G$ we have that $(xy)^p = x^p y^p \omega$ for some $\omega \in \gamma_2(\langle x, y\rangle)^p$. Note that, given a prime $p$, there are only finitely many $p$-groups of maximal class that are regular, and their order is at most $p^{p+1}$ by [3, Lemma 3.3] and [9, Hilfssatz III.14.15].

**Theorem 4.** *Let $G$ be a finite $p$-group of maximal class and suppose that its exponential rank is zero. Then $p$ is odd and*

(a) $\exp G = \exp G/Z(G)$ *or*
(b) $G$ *is regular.*

*Conversely, every $p$-group satisfying (a) or (b) has zero exponential rank.*

*Proof.* By Proposition 2, we may assume that $p$ is odd. Let $|G| = p^n$. Note that if $n \leq p$, then $G$ is regular, hence $\operatorname{exprank}(G) = 0$ by [16]. Also, if $\exp G = \exp G/Z(G)$, then clearly $\operatorname{exprank}(G) = 0$. Thus we may assume that $n > p$ and $\exp G > \exp G/Z(G)$.

Consider first the case when $n = p+1$. By [3, Theorem 3.2] we conclude that $\gamma_2(G)$ and $G/\gamma_{n-1}(G)$ have exponent $p$, whence $\exp G = p^2$ and $\exp G/Z(G) = p$. In this case the exponential rank of $G$ is zero if and only if $G$ is $p$-abelian, which in turn is equivalent to the fact that $G$ is regular.

In what follows, $G$ is a $p$-group of maximal class, $|G| = p^n$ and $n > p+1$. By a remark in the beginning of the section $G$ is not regular. Suppose also that $\operatorname{exprank}(G) = 0$. First note that this yields $\exp G > p^2$. For, if $\exp G = p^2$ and $\exp G/Z(G) = p$, then $\operatorname{exprank}(G) = 0$ would imply that $G$ is regular, which is not the case. Define $P_0 = G$, $P_1 = C_G(\gamma_2(G)/\gamma_4(G))$, and $P_i = \gamma_i(G)$ for $i \geq 2$. Then we have a chief series

$$G = P_0 > P_1 > P_2 > \cdots > P_{n-1} > P_n = 1$$

with factors of order $p$ [3]. Furthermore, we have that $\Omega_1(P_i) = P_{n-p+1}$ and $\mho_1(P_i) = P_{i+p-1}$ for every $i$ with $1 \le i \le n-p+1$. As $P_1$ is regular by [3, p. 69, Corollary 1], we conclude from here that

$$\mho_k(P_i) = \begin{cases} P_{i+k(p-1)} & : \ i+k(p-1) \le n, \\ 1 & : \ \text{otherwise}, \end{cases}$$

for every $i \ge 1$. It follows that $\exp P_i = p^{e_i}$, where

$$e_i = \left\lfloor \frac{n-p+1-i}{p-1} \right\rfloor_+ + 1$$

for every $i \ge 1$; here we use the notation $\alpha_+ = (\alpha + |\alpha|)/2$ for any real number $\alpha$. We claim that $\exp G = \exp P_1$. First note that we can assume that $e_1 \ge 2$. Choose any $x \in G \backslash P_1$. Then $G = \langle x \rangle P_1$. For any $g \in G$ we have $g = x^t y$ for some $y \in P_1$ and $t \in \mathbb{Z}$. If $t$ is divisible by $p$, then $x^t \in P_{n-1}$ by [3], whence we may assume without loss of generality that $\gcd(t,p) = 1$, that is, $x^t \notin P_1$. If $y \in P_2$, then it follows from [3, p. 64, Corollary 2] that $x^t y$ is conjugate to $x^t$, and $(x^t y)^p = x^{tp}$. We conclude that $g^{p^2} = 1$, hence also $g^{p^{e_1}} = 1$. Suppose now that $y \notin P_2$. Then it follows from [3, Lemma 3.3] that $(x^t y)^p \equiv x^{tp} \mod P_{p+1}$. Hence we can write $g^p = x^{tp} z$ for some $z \in P_{p+1}$. As $x^{tp} \in Z(G)$, we obtain $g^{p^{e_1}} = z^{p^{e_1 - 1}}$. On the other hand, consider $e_{p+1}$. From the above argument it follows that if $n \le 2p$, then $e_{p+1} = 1$. For $n > 2p$ we obtain

$$e_{p+1} = \left\lfloor \frac{n-2p}{p-1} \right\rfloor + 1 = \left\lfloor \frac{n-p-1}{p-1} \right\rfloor = e_2 - 1 \le e_1 - 1,$$

hence in both cases $g^{p^{e_1}} = 1$. This proves our claim.

From here on we may therefore assume that $e_1 > 2$. As $G$ is $p^{e_1-1}$-abelian, it follows that for any $x, y \in G$ we get $[x,y]^{p^{e_1-1}} = [x^{p^{e_1-1}}, y] = 1$, since $x^{p^{e_1-1}} \in Z(G)$. We conclude from here that $\exp P_2$ divides $p^{e_1-1}$. Thus we obtain that $e_2 = e_1 - 1$. Choose $x \in G \backslash P_1$ and $y \in P_1 \backslash P_2$ with $|y| = p^{e_1}$. As above, we have that $(xy)^p = x^p z$ for some $z \in P_{p+1}$. As $e_{p+1} \le e_1 - 1$, we obtain $(xy)^{p^{e_1-1}} = 1$. On the other hand, $x^{p^{e_1-1}} y^{p^{e_1-1}} = y^{p^{e_1-1}} \ne 1$, a contradiction. This shows that if $n > p+1$, then $\operatorname{exprank}(G) = 1$, hence Theorem 4 is proved. □

Let $G$ be a finite $p$-group of exponent $p^e$. Then we clearly have that $p^e\mathbb{Z} \cup (p^e\mathbb{Z}+1) \subseteq \mathcal{E}(G)$. Theorem 4 thus shows that the structure of exponent semigroups of finite $p$-groups of maximal class is quite restricted.

**Corollary 5.** *Let $G$ be a $p$-group of maximal class and exponent $p^e$. Then $\mathcal{E}(G) \ne p^e\mathbb{Z} \cup (p^e\mathbb{Z}+1)$ if and only if $G$ is regular and $\exp G > \exp G/Z(G)$.*

*Proof.* Suppose that $\mathcal{E}(G) \ne p^e\mathbb{Z} \cup (p^e\mathbb{Z}+1)$. Then there exists $f < e$ such that $p^f \in \mathcal{E}(G)$, thus $\exp G/Z(G) < \exp G$. As $G$ is of maximal class, we have that $\exp G/Z(G) = p^{e-1}$. It follows that $G$ is regular by Theorem 4. The converse is obvious. □

## 3. Finite $p$-groups of small nilpotency class

In this section we estimate the exponential rank of finite $p$-groups of class $\le 5$. We first prove the following proposition.

**Proposition 6.** *Let $G$ be a finite $p$-group of class at most 4. Then* $\operatorname{exprank}(G) \le 1$.

*Proof.* If $p > 3$, then $G$ is regular [9, Satz III.10.2], hence $\operatorname{exprank}(G) = 0$ by [16]. Thus we only need to consider 2-groups and 3-groups. Suppose first that $p = 2$ and $\exp G/Z(G) = 2^f$. Let $x, y \in G$. As $\langle x, y \rangle$ is metabelian, expansion of the identity $[x, y^{2^f}] = 1$ with the help of [15, Lemma 2.1] yields

$$[x,y]^{2^f} [x,y,y]^{\binom{2^f}{2}} [x,y,y,y]^{\binom{2^f}{3}} = 1. \tag{3.1}$$

Replacing $x$ by $[x,y]$ in (3.1), we get that $[x,y,y]^{2^f} [x,y,y,y]^{\binom{2^f}{2}} = 1$. As $G$ is of class $\le 4$, we also have $1 = [[x,y]^{2^f}, y] = [x,y,y]^{2^f}$, hence

$$[x,y,y,y]^{2^{f-1}} = 1. \tag{3.2}$$

Thus (3.1) can be rewritten as

$$[x,y]^{2^f}[x,y,y]^{\binom{2^f}{2}}=1. \tag{3.3}$$

Replacing $y$ by $xy$ in the equation (3.3) and using (3.3), we obtain

$$[x,y,x]^{\binom{2^f}{2}}[x,y,y,x]^{\binom{2^f}{2}}=1. \tag{3.4}$$

Further replacement of $x$ by $yx$ in (3.4) gives $[x,y,y]^{\binom{2^f}{2}}[x,y,y,x]^{\binom{2^f}{2}}=1$, hence $[x,y,y]^{\binom{2^f}{2}}=[x,y,x]^{\binom{2^f}{2}}$. Commuting this equation with $y$ and using (3.2), we get $[x,y,x,y]^{2^{f-1}}=1$. It follows from here that $G$ satisfies the law $[x,y,y,x]^{2^{f-1}}=1$. Thus the equation (3.4) implies $[x,y,y]^{2^{f-1}}=[x,y,x]^{2^{f-1}}=1$, and hence also $[x,y]^{2^f}=1$ by (3.1). We now have

$$\begin{aligned}(xy^{-1})^{2^f} &= x^{2^f}[x,y]^{\binom{2^f}{2}}[x,y,y]^{\binom{2^f}{3}}[x,y,x]^{\binom{2^f}{3}}[x,y,x,x]^{\binom{2^f}{4}}\\ &\cdot[x,y,y,x]^{\binom{2^f}{4}}[x,y,y,y]^{\binom{2^f}{4}}y^{-2^f}\end{aligned} \tag{3.5}$$

by [15, Lemma 2.1]. It follows from the above equations that $(xy^{-1})^{2^{f+1}}=x^{2^{f+1}}y^{-2^{f+1}}$ for all $x,y\in G$, hence $\operatorname{exprank}(G)\le 1$.

Consider now the case $p=3$. Let $\exp G/Z(G)=3^f$ and $x,y\in G$. Similarly as above we have that $[x,y,y]^{3^f}=1$, and thus expansion of the identity $[x,y^{3^f}]=1$ gives

$$[x,y]^{3^f}[x,y,y,y]^{\binom{3^f}{3}}=1. \tag{3.6}$$

Replacing $y$ by $xy$ in (3.6) and using (3.6), we get $[x,y,y,x]^{2\binom{3^f}{3}}[x,y,x,x]^{\binom{3^f}{3}}=1$, hence $[x,y,y,x]^{\binom{3^f}{3}}=[x,y,x,x]^{\binom{3^f}{3}}$. Replacing $y$ by $xy$ in this equation, we conclude that $[x,y,y,x]^{3^{f-1}}=[x,y,x,x]^{3^{f-1}}=1$. From (3.6) it follows that $[x,y]^{3^f}=1$, and the identity

$$\begin{aligned}(xy^{-1})^{3^f} &= x^{3^f}[x,y]^{\binom{3^f}{2}}[x,y,y]^{\binom{3^f}{3}}[x,y,x]^{\binom{3^f}{3}}[x,y,x,x]^{\binom{3^f}{4}}\\ &\cdot[x,y,y,x]^{\binom{3^f}{4}}[x,y,y,y]^{\binom{3^f}{4}}y^{-3^f}\end{aligned} \tag{3.7}$$

thus implies that $G$ is $3^{f+1}$-abelian, as required. □

When the nilpotency class of the group is 2, even more can be said. We can precisely determine which $p$-groups of class 2 have exponential rank zero.

**Corollary 7.** *Let $G$ be a finite $p$-group of class 2. If $p$ is odd, then* $\operatorname{exprank}(G)=0$*. If $p=2$, then* $\operatorname{exprank}(G)=1$*.*

*Proof.* If $p$ is odd, then $G$ is regular, hence $\operatorname{exprank}(G)=0$ by [16]. Assume that $G$ is a 2-group of class 2. Because of the class restriction, the identity $[x,y]^n=[x^n,y]$ holds in $G$ for all integers $n$ and $x,y\in G$. In particular, $\exp\gamma_2(G)=\exp G/Z(G)=2^f$. If $\operatorname{exprank}(G)=0$, then the identity $x^{2^f}y^{2^f}=(xy)^{2^f}=x^{2^f}y^{2^f}[y,x]^{\binom{2^f}{2}}$ implies that $\exp\gamma_2(G)$ divides $2^{f-1}$, a contradiction. □

Given a group $G$, denote by $E_2(G)$ the subgroup of $G$ generated by all commutators of the form $[x,y,y]$, where $x,y\in G$.

**Corollary 8.** *Let $G$ be a finite $p$-group of class 3. Then* $\operatorname{exprank}(G)=0$ *if and only if one of the following holds.*

(a) $p>3$.
(b) $p=2$ *and* $\exp\gamma_2(G)$ *divides* $(\exp G/Z(G))/2$.
(c) $p=3$ *and* $\exp E_2(G)$ *divides* $(\exp G/Z(G))/3$.

*Proof.* This follows from the proof of Proposition 6, in particular from (3.5) and (3.7), and from [16]. □

For $p\in\{2,3\}$ there exist $p$-groups of class 3 with exponential rank one. For $p=2$, the dihedral group $D_{2\cdot 8}$ of order 16 is an appropriate example. When $p=3$, the group $C_3\wr C_3$ has specified properties, see Example 3.

Next we deal with groups of class 5. To this end, the following technical lemma is needed.

**Lemma 9.** *Let $G$ be a group of nilpotency class 5. Then the following identities hold in $G$.*

(a) $(xy)^n = x^n y^n c_1^{m_1} \cdots c_{12}^{m_{12}}$, *where* $c_1 = [y,x]$, $c_2 = [y,x,x]$, $c_3 = [y,x,y]$, $c_4 = [y,x,x,x]$, $c_5 = [y,x,x,y]$, $c_6 = [y,x,y,y]$, $c_7 = [y,x,x,x,x]$, $c_8 = [y,x,x,x,y]$, $c_9 = [y,x,x,y,x]$, $c_{10} = [y,x,x,y,y]$, $c_{11} = [y,x,y,y,x]$, $c_{12} = [y,x,y,y,y]$, *and each $m_i$ is a $\mathbb{Z}$-linear combination of* $\binom{n}{1}, \ldots, \binom{n}{w_i}$, *where $w_i$ is the weight of $c_i$.*

(b) $[x^n, y] = [x,y]^n [x,y,x]^{\binom{n}{2}} [x,y,x,x]^{\binom{n}{3}} [x,y,x,x,x]^{\binom{n}{4}} [x,y,x,[x,y]]^{\alpha(n)}$, *where* $\alpha(n) = n(n-1)(2n-1)/6$.

*Proof.* Part (a) can be proved using Hall's Collection Process, see for instance [9, p. 315–321]. The identity (b) is proved by induction on $n$, the case $n = 1$ being obvious. Suppose that the identity holds true for some positive integer $n$. Then

$$\begin{aligned}[x^{n+1}, y] &= [x,y][x,y,x^n][x^n,y] \\ &= [x,y][x^n,y][x,y,x^n][x,y,x,[x,y]]^{n^2} \\ &= [x,y]^{n+1}[x,y,x]^{\binom{n}{2}+n}[x,y,x,x]^{\binom{n}{3}+\binom{n}{2}}[x,y,x,x,x]^{\binom{n}{4}+\binom{n}{3}}[x,y,x,[x,y]]^{\alpha(n)+n^2},\end{aligned}$$

and our conclusion easily follows. □

**Proposition 10.** *Let $G$ be a finite $p$-group of class 5.*

(a) *If $p > 5$, then* $\operatorname{exprank}(G) = 0$.
(b) *If $p \in \{3,5\}$, then* $\operatorname{exprank}(G) \le 1$.
(c) *If $p = 2$, then* $\operatorname{exprank}(G) \le 2$.

*Proof.* If $p > 5$, then $G$ is regular and so it has zero exponential rank by [16]. Thus it suffices to prove the result for $p \in \{2,3,5\}$.

First consider the case $p = 2$. Let $\exp G/Z(G) = 2^f$ and $x,y,z \in G$. By Lemma 9 we have $1 = [x^{2^f}, y] = [x,y]^{2^f}[x,y,x]^{\binom{2^f}{2}}[x,y,x,x]^{\binom{2^f}{3}}[x,y,x,x,x]^{\binom{2^f}{4}}[x,y,x,[x,y]]^{\alpha(2^f)}$. Commuting this identity with $z$, we get $[x,y,x,z]^{2^{f-1}} = 1$, hence $[x,y]^{2^{f+1}}[x,y,x]^{2\binom{2^f}{2}} = 1$. Replacing $x$ by $yx$ in this equation, we obtain $([x,y,y][x,y,y,x])^{2^f} = 1$. As $[x,y,y,x]^{2^f} = [y,x,y,x]^{-2^f} = 1$, we immediately get $[x,y,x]^{2^f} = 1$, hence also $[x,y]^{2^{f+1}} = 1$. Using Lemma 9, we get that $(xy)^{2^{f+2}} = x^{2^{f+2}}y^{2^{f+2}}$, hence $\operatorname{exprank}(G) \le 2$.

Now let $G$ be a 3-group and $\exp G/Z(G) = 3^f$. Expanding $[[x,z]^{3^f}, y] = 1$ with the help of Lemma 9, we obtain $[x,z,y]^{3^f}[x,z,y,[x,z]]^{\binom{3^f}{2}} = 1$. As $1 = [[x,y,z]^{3^f}, w] = [x,y,z,w]^{3^f}$, we obtain $[x,z,y]^{3^f} = 1$. Thus the identity $[x^{3^f}, y] = 1$ can be written as

$$[x,y]^{3^f}[x,y,x,x]^{\binom{3^f}{3}}[x,y,x,[x,y]]^{\alpha(3^f)} = 1. \tag{3.8}$$

Commuting (3.8) with $z$, we get $[x,y,x,x,z]^{3^{f-1}} = 1$. Expanding $[y,yx,yx,yx,y]^{3^{f-1}} = 1$, we obtain $([y,x,y,x,y][y,x,x,y,y])^{3^{f-1}} = 1$. We expand the commutator $[yx,xy]$ in two ways to obtain

$$[yx,xy] = [yx,y][yx,x][yx,x,y] = [y,x,x][y,x,y][y,x,x,y]$$

and

$$[yx,xy] = [y,xy][y,xy,x][x,xy] = [y,x][y,x,y][y,x,x][y,x,y,x][x,y].$$

As $[y,x,y]^{-1} = [x,y,y]^{[y,x]}$ and $[y,x,x]^{-1} = [x,y,x]^{[y,x]}$, this further gives

$$[y,x,x,y] = [x,y,y]^{[y,x]}[x,y,x]^{[y,x]}[x,y,y]^{-1}[x,y,x]^{-1}[y,x,y,x].$$

Because of the class restriction $[x,y,y]$ commutes with $[x,y,x]$. Since we have $[x,y,y]^{[y,x]} = [x,y,y][x,y,y,[y,x]]$ and $[x,y,x]^{[y,x]} = [x,y,x][x,y,x,[y,x]]$, we get

$$[y,x,x,y] = [y,x,y,x][x,y,y,[y,x]][x,y,x,[y,x]], \tag{3.9}$$

hence $[y,x,y,x,y] = [y,x,x,y,y]$. It follows that $[y,x,y,x,y]^{2\cdot 3^{f-1}} = 1$, and consequently $[y,x,y,x,y]^{3^{f-1}} = [y,x,x,y,y]^{3^{f-1}} = 1$. We conclude from here that $\gamma_5(\langle x,y\rangle)^{3^{f-1}} = 1$. The equation (3.8) now implies

$$[x,y]^{3^f}[x,y,x,x]^{\binom{3^f}{3}} = 1. \tag{3.10}$$

Replacing $x$ by $yx$ in (3.10) and using (3.10), we obtain $([x,y,y,y][x,y,x,y][x,y,y,x])^{3^{f-1}} = 1$. By (3.9) this can be rewritten as $[x,y,y,y]^{3^{f-1}} = [x,y,y,x]^{3^{f-1}}$. Replacing $x$ by $yx$ in this equation, we get $[x,y,y,y]^{3^{f-1}} = 1$, hence also $[x,y,y,x]^{3^{f-1}} = 1$ and $[x,y]^{3^f} = 1$. By Lemma 9 we now conclude that $(xy)^{3^{f+1}} = x^{3^{f+1}} y^{3^{f+1}}$.

Finally let $p = 5$ and $\exp G/Z(G) = 5^f$. Then $[x,y,z,w]^{5^f} = 1$ and thus Lemma 9 yields $1 = [x^{5^f}, y] = [x,y]^{5^f}[x,y,x]^{\binom{5^f}{2}}$. Replacing $x$ by $yx$ in this equation, we get $[x,y,x]^{5^f} = 1$, and thus $[x,y]^{5^f} = 1$. Using Lemma 9 again, we get $(xy)^{5^{f+1}} = x^{5^{f+1}} y^{5^{f+1}}$. □

The bounds given by Proposition 10 are tight. There exists a 5-group $G$ of class 5 with $\operatorname{exprank}(G) = 1$, see, for instance, Example 3. An example of a 3-group $G$ of class 5 with $\operatorname{exprank}(G) = 1$ can be constructed as follows. Let $N = \langle a \rangle \times \langle b \rangle \cong C_9 \times C_{27}$, and let $c$ be an automorphism of $N$ of order 3 acting on $N$ by the rules $a^c = a^7 b^3$ and $b^c = a^8 b$. Let $G = N \rtimes \langle c \rangle$. Then $G$ is nilpotent of class 5 and $|G| = 3^6$. We have that $\exp G/Z(G) = 9$ and $(bc)^9 = 1 \neq b^9 c^9$, hence $\operatorname{exprank}(G) = 1$.

Let $G$ be the group constructed in Example 3.3 of [15]. $G$ is a metabelian group of order $2^{19}$, exponent 16 and class 5. We have that $\exp G/Z(G) = 4$, and $G$ is not $k$-abelian for any $1 < k < 16$. Thus $\operatorname{exprank}(G) = 2$.

## 4. The Schur conjecture for groups of small nilpotency class

Let $G$ be a group presented as the quotient of a free group $F$ by a normal subgroup $R$. The abelian group

$$M(G) = (R \cap \gamma_2(F))/[R, F]$$

is said to be the *Schur multiplier* of $G$. It is well known that $M(G)$ is isomorphic to the second integral homology group $H_2(G, \mathbb{Z})$ of $G$. A crucial role in studying the Schur multipliers of finite groups is played by the covering groups. Recall that a group $H$ is said to be a *covering group* of a group $G$, if there exists $M \leq H$ isomorphic to $M(G)$ such that $M \leq H' \cap Z(H)$ and $H/M \cong G$. Schur (1904) proved that covering groups of finite groups always exist, although they need not be unique. For further account on the theory of the Schur multipliers see e.g. [9, Kapitel V].

The Schur conjecture states that, for every finite group $G$, the exponent of $M(G)$ divides $\exp G$. It is now known that this conjecture is false [2, 16]. Thus the question arises as to whether the conjecture holds true for certain classes of groups. In [16], we discovered that there is a close relationship between the exponent of the Schur multiplier and exponential rank. It is proved for instance that the exponential rank of a finite $p$-group $G$ does not exceed $\log_p \exp M(G/Z(G))$. On the other hand, if $H$ is a covering group of $G$ and if $H$ is $p^e$-abelian, then $\exp M(G)$ divides $p^e$. Therefore it is possible to estimate the exponent of $M(G)$ by determining the exponential rank of $H$, together with the exponent of $H/Z(H)$.

There are known estimates for $\exp M(G)$ in case $G$ is a finite $p$-group of small class, given by Jones [11] and Kayvanfar and Sanati [13]. Most of those results are dealing with the question whether $\exp M(G)$ divides $\exp G$. An easy argument shows that, given the nilpotency class, only finitely many primes need to be considered.

**Proposition 11.** *Let $G$ be a finite $p$-group of class less than $p-1$. Then* $\exp M(G)$ *divides* $\exp G$.

*Proof.* Denote $\exp G = p^f$ and let $H$ be a covering group of $H$. Then $H$ is nilpotent of class less than $p$, hence it is regular [9, Satz III.10.2]. Therefore $\operatorname{exprank}(H) = 0$ by [16]. As $\exp H/Z(H)$ divides $p^f$, it follows that $H$ is $p^f$-abelian, whence $\exp \gamma_2(H)$ divides $p^f$ (see Section 2). We conclude that $\exp M(G)$ divides $p^f$, as required. □

Jones [11] proved that if $G$ is a finite $p$-group of class 3 and $p \neq 3$, then $\exp M(G)$ divides $\exp G$. Note that when $p > 3$, this is a direct consequence of Proposition 11. We extend Jones's result as follows.

**Theorem 12.** *Let $G$ be a finite $p$-group of class $\leq 3$. Then* $\exp M(G)$ *divides* $\exp G$.

*Proof.* If $p \neq 3$ then our assertion follows from [11]. Now let $p = 3$ and let $\exp G = 3^f$. Let $H$ be a covering group of $G$. Then $H$ is nilpotent of class $\leq 4$, and $H/Z(H)$ has exponent dividing $3^f$. Let $x, y \in H$ and $\omega \in \gamma_2(H)$. Then $[x,y]^{3^f} = 1$ by the proof of Proposition 6. As $\gamma_2(H)$ is nilpotent of class $\leq 2$, we have $(\omega[x,y])^{3^f} = \omega^{3^f}[x,y]^{3^f}[x,y,\omega]^{\binom{3^f}{2}} = \omega^{3^f}$, hence $\gamma_2(H)$ has exponent dividing $3^f$. This concludes the proof. □

Finite $p$-groups of class 4 were considered by Kayvanfar and Sanati [13]. They proved that if $G$ is a finite $p$-group of class 4 and exponent $p^f$ with $p > 3$ and $f$ odd, then $\exp M(G)$ divides $p^f$. We prove the following result.

**Theorem 13.** *Let $G$ be a finite $p$-group of class 4. If $p$ is odd, then $\exp M(G)$ divides $\exp G$. If $p = 2$, then $\exp M(G)$ divides $2 \cdot \exp G$.*

*Proof.* Let $\exp G = p^f$ and let $H$ be a covering group of $G$. Then $\exp H/Z(H)$ divides $p^f$, and $H$ is nilpotent of class $\leq 5$. If $p > 5$, then $\exp M(G)$ divides $\exp G$ by Proposition 11. If $p \in \{3, 5\}$, then $H$ satisfies the law $[x,y]^{p^f} = 1$, see the proof of Proposition 10. As $\gamma_2(H)$ has class $\leq 2$, we have $([x,y]\omega)^{p^f} = [x,y]^{p^f}\omega^{p^f}[\omega,[x,y]]^{\binom{p^f}{2}} = \omega^{p^f}$ for all $x, y \in H$ and $\omega \in \gamma_2(H)$, hence $\exp \gamma_2(H)$ divides $p^f$.

Consider the case $p = 2$. From the proof of Proposition 10 we know that $H$ satisfies the law $[x,y]^{2^{f+1}} = 1$. It also follows from there that $H$ satisfies the identity $[x,y]^{2^f}[x,y,x]^{\binom{2^f}{2}}[x,y,x,x,x]^{\binom{2^f}{4}} = 1$. Replacing $y$ by $[y,z]$ in the latter equation, we get

$$[y,z,x]^{2^f}[y,z,x,x]^{\binom{2^f}{2}} = 1. \tag{4.1}$$

Next we prove that $[y,z,x,x]^{2^{f-1}} = 1$. To this end, consider first the commutator $[y^2,x,x]$. As $H$ is of class at most 5, we can expand this commutator as $[y^2,x,x] = [y,x,x]^2[y,x,x,y]$. From the proof of Proposition 10 we get that $[y,x,x]^{2^f} = 1$ and $[y,x,x,y]^{2^{f-1}} = 1$, hence the order of $[y^2,x,x]$ divides $2^{f-1}$. Next consider the commutator $[y^2z^2,x,x]$. We have $[y^2z^2,x,x] = [y^2,x,x][z^2,x,x][y^2,x,z^2,x]$. Since $H$ is nilpotent of class $\leq 5$, the commutator $[y^2,x,z^2,x]$ can be written as a product of squares of certain commutators of weight at least 4. As $H$ satisfies the identity $1 = [[x,y,z]^{2^f},w] = [x,y,z,w]^{2^f}$, it follows from here that $[y^2z^2,x,x]^{2^{f-1}} = 1$. It is not difficult to see that this yields $[H^2,x,x]^{2^{f-1}} = 1$. Since $\gamma_2(H) \leq H^2$, we get that $[y,z,x,x]^{2^{f-1}} = 1$. Consequently, we obtain from (4.1) that $\exp \gamma_3(H)$ divides $2^f$. Now let $x, y \in H$ and $\omega \in \gamma_2(H)$. As $\gamma_2(H)$ is nilpotent of class $\leq 2$, we have $([x,y]\omega)^{2^{f+1}} = [x,y]^{2^{f+1}}\omega^{2^{f+1}}[\omega,[x,y]]^{\binom{2^{f+1}}{2}} = \omega^{2^{f+1}}$, hence $\exp \gamma_2(H)$ divides $2^{f+1}$. This concludes the proof. □

Theorem 13 provides best possible bounds for $\exp M(G)$, as the following example shows. We acknowledge the help of R. F. Morse who greatly simplified the presentation of the example.

*Example* 14. There exists a group $G$ of exponent 4 and nilpotency class 4 with $M(G)$ of exponent 8. An example of such a group is for instance the largest nilpotent-of-class-4 quotient of the group $G = \langle a,b,c,d \mid [a,b][c,d] \rangle$ satisfying the law $x^4 = 1$. This group can be easily constructed with the help of the Nilpotent Quotient Algorithm [17]. The order of $G$ is $2^{68}$. Using GAP [7], in particular the function `SchurMultiplicator` in the Polycyclic Quotient Algorithm [5], one can compute that $M(G) \cong \mathbb{Z}_2^{121} \oplus \mathbb{Z}_4^5 \oplus \mathbb{Z}_8$. We include the appropriate GAP commands for the reader's convenience:

```
gap> F := FreeGroup("a", "b", "c", "d", "x");;
gap> AssignGeneratorVariables(F);;
gap> R := [x^4, Comm(a, b) * Comm(c, d)];;
gap> G := NilpotentQuotient(F/R, 4: idgens := [x]);;
gap> SchurMultiplicator(G);
[ [2, 121 ], [4, 5], [8, 1] ]
```

Kayvanfar and Sanati [13] have given some sufficient conditions under which $\exp M(G)$ divides $\exp G$ for a $p$-group $G$ of class $\leq 6$. These conditions involve some arithmetic restrictions on $\exp G$. Our results, in particular Proposition 11, show that most of those restrictions are redundant. The exact bounds for $\exp M(G)$ in these cases are still not known.

Another open question is whether there exists a $p$-group $G$, $p$ odd, for which the exponent of $M(G)$ does not divide $\exp G$. We note here that the immediate analogue of Example 14 for 3-groups of class 5 did not yield an example of such a group. This will be the topic of future investigations.

## REFERENCES


[1] J. L. Alperin, *A classification of $n$-abelian groups*, Canad. J. Math. **21** (1969), 1238–1244.

[2] A. J. Bayes, J. Kautsky, J. W. Wamsley, *Computation in nilpotent groups (application)*, Proc. 2nd Internat. Conf. Theory of Groups, Lecture Notes in Mathematics No. 372, Springer Verlag 1973, 82–89.

[3] N. Blackburn, *On a special class of $p$-groups*, Acta Math. **100** (1958), 45–92.

[4] C. Delizia, P. Moravec and C. Nicotera, *Locally graded Bell groups*, Publ. Math. Debrecen **71** (2007), no. 1–2, 1–9.

[5] B. Eick and W. Nickel, *Polycyclic - computing with polycyclic groups*, a refereed $G$AP 4 package, Version 2.1, 2006.

[6] G. Ellis, *On the relation between upper central quotients and lower central series of a group*, Trans. Amer. Math. Soc. **353** (2001), no. 10, 4219–4234.

[7] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.4*; 2005, (http://www.gap-system.org).

[8] N. D. Gupta and A. H. Rhemtulla, *A note on centre-by-finite-exponent varieties of groups*, J. Austral. Math. Soc. **11** (1970), 33–36.

[9] B. Huppert, *Endliche Gruppen*, Springer-Verlag, Berlin, 1967.

[10] R. James, 2-*groups of almost maximal class*, J. Austral. Math. Soc. Ser. A **19** (1975), 343–357.

[11] M. R. Jones, *Some inequalities for the multiplicator of a finite group. II*, Proc. Amer. Math. Soc. **45** (1974), no. 2, 167–172.

[12] L.-C. Kappe, *On $n$-Levi groups*, Arch. Math. **47** (1986), 198–210.

[13] S. Kayvanfar and M. A. Sanati, *A bound for the exponent of the Schur multiplier of some finite $p$-groups*, Bull. Iranian Math. Soc. **26** (2000), no. 2, 89–96.

[14] F. W. Levi, *Notes on group theory I*, J. Indian Math. Soc. **8** (1944), 1–7.

[15] P. Moravec, *On power endomorphisms of $n$-central groups*, J. Group Theory **9** (2006), no. 4, 519–536.

[16] P. Moravec, *Schur multipliers and power endomorphisms of groups*, J. Algebra **308** (2007), no. 1, 12–25.

[17] W. Nickel, *NQ - Nilpotent quotients of finitely presented groups*, a refereed $G$AP 4 package, Version 2.2, 2007.



Primož Moravec, Oddelek za matematiko, Inštitut za matematiko, fiziko in mehaniko, Jadranska 19, 1000 Ljubljana, Slovenia

*E-mail address*: primoz.moravec@fmf.uni-lj.si