ON $*$–ORDERABLE GROUPS

Igor Klep and Primož Moravec

12.5.2004

Abstract. We introduce the class of $*$–orderable groups as those groups G for which the complex group ring $\mathbb{C}G$ with the standard involution is $*$–orderable. In [CKM] the authors proved that residually torsion–free nilpotent groups are $*$–orderable and that every $*$–orderable group is orderable. We prove that being a $*$–orderable group is a local and residual property and deduce that the class of residually torsion–free nilpotent groups is strictly contained in the class of $*$–orderable groups. Further, it is proved that $*$–orderable groups are elementary in the language of groups and contain certain normal series. In the last section we give a countable family of orderable metabelian groups that are not $*$–orderable.

1. Introduction and basic definitions

The group ring kG can be totally ordered iff k is formally real and G is orderable. The case of $*$–orderability of kG is far more complex. In [CKM] the authors studied complex group rings and their $*$–orderability. They proved that $\mathbb{C}G$ can be $*$–ordered if G is a residually torsion–free nilpotent group. Moreover, they showed that $\mathbb{C}G$ is $*$–orderable implies G is orderable. The converse is false; there exists a metabelian orderable group for which the complex group ring is not $*$–orderable. This raises several questions. For example: If the complex group ring $\mathbb{C}G$ is $*$–orderable, is G necessarily residually torsion–free nilpotent? And more generally, what can be said about a group G if the corresponding group ring $\mathbb{C}G$ is $*$–orderable?

In order to study these problems, we introduce the class of $*$–orderable groups as those groups G for which the complex group ring $\mathbb{C}G$ (endowed with the standard involution sending $g \mapsto g^{-1}$ for $g \in G$ and $\lambda \mapsto \overline{\lambda}$ for $\lambda \in \mathbb{C}$) can be $*$–ordered.

2000 *Mathematics Subject Classification.* 14P10, 16W10, 06F15.

Key words and phrases. $*$–orderings, ordered groups, group rings, rings with involution.

This work was supported by the Ministry of Education, Science and Sport of Slovenia. The first author would like to thank Murray Marshall and Jaka Cimprič for discussions during the preparation of this paper.

We show that this class is strictly larger than the class of all residually torsion–free nilpotent groups. This will follow from the fact that being a $*$–orderable group is a local property. We also show that the class of $*$–orderable groups is elementary in the language of groups. This is used to deduce that being a $*$–orderable group is a residual property. In Section 4 we construct exhaustive and separating normal series of a $*$–orderable group G with additional properties. We use this to construct orderings of G. Furthemore, we show that the example of an orderable metabelian group given in [CKM] which is not $*$–orderable, is part of a countable family of groups with similar properties.

Let $(A,*)$ be a ring with involution. A subset $P \subseteq \text{ Sym } A$ is called a $*$*–ordering* provided the following hold:

(1) $P+P \subseteq P$,	(2) $a,b \in P \;\Rightarrow\; ab+ba \in P$,
(3) $rPr^* \subseteq P$ for all $r \in A$,	(4) $P \cup -P = \text{ Sym } A$,
(5) $P \cap -P = \{0\}$,	(6) $aba = 0 \;\Rightarrow\; a=0$ or $b=0$ for all $a,b \in \text{ Sym } A$.

If Γ is an ordered abelian semigroup, then an onto mapping $v : A \to \Gamma \cup \{\infty\}$ is a $*$*–valuation* if:

(1) $v(x) = \infty$ iff $x = 0$.
(2) $v(xy) = v(x) + v(y)$ for all $x,y \in A^\times$.
(3) $v(x+y) \geqslant \min\{v(x), v(y)\}$ for all $x,y \in A^\times$.
(4) $v(x) = v(x^*)$ for all $x \in A$.

Here $A^\times := A \setminus \{0\}$. The *corresponding graded* $*$*–ring* $\text{gr}(A,v)$ is constructed as follows. We form $A_\alpha := \{x \in A \mid v(x) \geqslant \alpha\}$, $A_\alpha^+ := \{x \in A \mid v(x) > \alpha\}$ and $\overline{A_\alpha} := A_\alpha / A_\alpha^+$. Then $\text{gr}(A,v) := \oplus_{\alpha\in\Gamma}\overline{A_\alpha}$ is given the componentwise addition and the multiplication induced by $(\overline{a}, \overline{b}) \mapsto \overline{ab}$ for $a \in A_\alpha$ and $b \in A_\beta$. Moreover, there is a canonical involution on $\text{gr}(A,v)$. We call v *quasi–commutative* if $\text{gr}(A,v)$ is commutative. It is easy to see that $\text{gr}(A,v)$ is commutative iff

$$\forall a,b \in A \setminus \{0\} : \; v(ab-ba) > v(ab) = v(ba).$$

If $v : A \to \Gamma \cup \{\infty\}$ is a $*$–valuation, write $\mathcal{O}_v := A_0$ and $\mathfrak{m}_v := A_0^+$. If A is a division $*$–ring, then $\mathcal{O}_v$ is an *invariant valuation* $*$*–ring* and $\mathfrak{m}_v$ is its maximal $*$–ideal. In general, $\mathfrak{m}_v$ is only a completely prime $*$–ideal of $\mathcal{O}_v$. Hence the *residue ring* $k_v := \mathcal{O}_v/\mathfrak{m}_v$ is only a $*$–domain and not necessarily a division $*$–ring.

Let $P \subseteq A$ be a $*$–ordering. The $*$–valuation v is said to be *compatible* with P if for all $x,y \in A^\times$ with $y \in P$ and $x \in \text{ Sym } A$, $v(x) < v(x-y)$ implies $x \in P$. To each $*$–ordering $P \subseteq A$ a *natural* (*order–compatible*) $*$*–valuation* v_P can be associated as follows. The $*$–ordering P gives an order relation $\leqslant$ on $\text{ Sym } A$, which induces the archimedean equivalence $\approx$ on $\text{ Sym } A$. We extend the latter to the whole A by declaring, for all $0 \neq a,b \in A$, that $a \prec b$ if $aa^* \leqslant nbb^*$ for some integer n, and $a \approx b$ if $a \prec b$ and $b \prec a$. Denote by $v_P(a)$ the equivalence class of $0 \neq a \in A$ (and

$v_P(0) := \infty$). Then the relation $\prec$ induces an ordering of the set $\Gamma_P = v_P(R \setminus \{0\})$. By Theorem 3.3 in [Ma1], the binary operation $v_P(a) + v_P(b) := v_P(ab)$ is well–defined on Γ_P, so Γ_P becomes an ordered commutative cancellation semigroup. It is known that v_P is a $*$–valuation and

$$v_P(ab - ba) > v_P(a) + v_P(b) \text{ for all } 0 \neq a, b \in \text{ Sym } A.$$

This result can be somewhat extended if A contains a central skew element i satisfying $i^2 = -1$. Then $v_P(ab - ba) > v_P(ab)$ holds for all $0 \neq a, b \in A$. In particular, the corresponding graded $*$–ring $\mathrm{gr}(A, v_P)$ is commutative in this case. Furthermore, k_{v_P} is a $*$–ordered subring of $\mathbb{C}$. For details, consult [Kl].

To every $*$–valuation $v : A \to \Gamma \cup \{\infty\}$ a *weak $*$–place* $\wp : A \to k_v \cup \{\infty\}$ can be associated as follows. For $x \in \mathcal{O}_v$, $\wp(x) := x + \mathfrak{m}_v \in k_v$. If $x \notin \mathcal{O}_v$, then $\wp(x) := \infty$. $*$–places on division rings have been studied by Craven [Cr]; for more on (weak) $*$–places on domains the reader is referred to [Kl].

1.1. Definition. A group G is called $*$–orderable if the complex group ring $\mathbb{C}G$ endowed with the standard involution can be $*$–ordered.

1.2. *Remark.* The following statements are equivalent:
(i) $\mathbb{C}G$ is $*$–orderable,
(ii) $\mathbb{Q}[i]G$ is $*$–orderable,
(iii) $\mathbb{Z}[i]G$ is $*$–orderable.
The implication (i) $\Rightarrow$ (iii) is trivial and (iii) $\Rightarrow$ (ii) follows by going to the central localization $\mathbb{Q}[i]G = (\mathbb{Z}[i]G)_{\mathbb{N}}$. Finally, a density argument proves (ii) $\Rightarrow$ (i). This remark was provided to us by Murray Marshall; it will not be used in the sequel.

Let us briefly recall two notions from group theory. If $\mathfrak{P}$ is a property of groups, we say that a group G is *residually–*$\mathfrak{P}$ if

$$\bigcap \{N \lhd G \mid G/N \in \mathfrak{P}\} = \{1\}.$$

Equivalently, for every $x \in G \setminus \{1\}$ there exists a normal subgroup $N \subseteq G$ avoiding x such that $G/N \in \mathfrak{P}$. A group G is *locally–*$\mathfrak{P}$ if every finitely generated subgroup of G is in $\mathfrak{P}$.

2. $*$–orderable groups need not be residually torsion–free nilpotent

2.1. Theorem. *Being a $*$–orderable group is a local property, i.e., if every finitely generated subgroup of a group G is $*$–orderable, then so is G.*

Proof. By the Artin–Schreier theory developed by Marshall in [Ma1], it follows that a ring A is $*$–orderable iff $T_0 \cap -T_0 = \{0\}$, where T_0 denotes the smallest extended

$*$–preordering of A. This is the set of all finite sums of permuted products of elements of the form

$$a_1, a_1, a_2, a_2, \ldots, a_m, a_m, r_1, r_1^*, \ldots, r_n, r_n^*, \quad b_j \in \text{ Sym } A,\ r_k \in A,\ m, n \in \mathbb{N}_0$$

which are nested w.r.t. $r_1, r_1^*, \ldots, r_n, r_n^*$.

Now assume G is not $*$–orderable. By definition, this means that $A := \mathbb{C}G$ is not $*$–orderable. Then a finite non–trivial sum of permuted products of elements of the above form is 0. As each element of A is a $\mathbb{C}$–linear combination of elements from G, in such a non–trivial sum only a finite number of elements from G can appear. The subgroup of G generated by these elements is clearly finitely generated and is not $*$–orderable by the Artin–Schreier theory mentioned above. Hence a finitely generated subgroup of G is not $*$–orderable, as required. ∎

2.2. Corollary. *Every locally nilpotent torsion–free group is $*$–orderable.*

Proof. By Theorem 5.1 in [CKM], residually torsion–free nilpotent groups are $*$–orderable. Now a group is locally nilpotent iff every finitely generated subgroup is nilpotent. As nilpotent torsion–free groups are obviously residually torsion–free nilpotent, Theorem 2.1 finishes the proof. ∎

To prove the claim given in the title of this section, we give an example of a locally nilpotent torsion–free group which is not residually torsion–free nilpotent. We use a method introduced by McLain [McL] to construct groups which are essentially generalizations of groups of unitriangular matrices. More precisely, let Λ be a linearly ordered set and let F be a field of characteristic zero. Let V be an F–vector space with basis $\{v_\lambda \mid \lambda \in \Lambda\}$. For each pair $(\lambda, \mu) \in \Lambda \times \Lambda$ with $\lambda < \mu$ we define a linear transformation $e_{\lambda\mu} : V \to V$ by the following rules:

$$e_{\lambda\mu} v_\mu := v_\lambda, \quad e_{\lambda\mu} v_\nu = 0 \text{ if } \nu \neq \mu.$$

We claim that for $a \in F$ the linear transormation $1 + a e_{\lambda\mu}$ is invertible. To see this, observe first that $e_{\lambda\mu} e_{\mu\nu} = e_{\lambda\nu}$ and $e_{\lambda\mu} e_{\xi\nu} = 0$ if $\mu \neq \xi$. Then we have $(1 + a e_{\lambda\mu})(1 - a e_{\lambda\mu}) = 1 - a^2 e_{\lambda\mu}^2 = 1$, hence

$$(1 + a e_{\lambda\mu})^{-1} = 1 - a e_{\lambda\mu}.$$

Now we define the McLain group $M(\Lambda, F)$ determined by Λ and F to be the group generated by all $1 + a e_{\lambda\mu}$, where $\lambda, \mu \in \Lambda$, $\lambda < \mu$ and $a \in F$. Clearly, the elements $e_{\lambda\mu}$ are F–linearly independent, hence every element of $M(\Lambda, F)$ can be uniquely presented as

$$1 + \sum_{\lambda < \mu} a_{\lambda\mu} e_{\lambda\mu}, \tag{$\star$}$$

where only finitely many of the $a_{\lambda\mu} \in F$ are nonzero. Conversely, it is straightforward to see that every element of the form $(\star)$ belongs to $M(\Lambda, F)$.

It is not difficult to observe that if Λ is a finite set, then the McLain group $M(\Lambda, F)$ is isomorphic to the group of all $|\Lambda| \times |\Lambda|$ upper triangular matrices over F, hence it is nilpotent. In order to obtain a group with desired properties, we will assume that Λ is dense (for instance, we may take $\Lambda = \mathbb{Q}$). Let H be any finitely generated subgroup of $M(\Lambda, F)$. Then there exist elements $\lambda_1, \lambda_2, \dots, \lambda_r$ of Λ with $\lambda_1 < \lambda_2 < \dots < \lambda_r$ such that $H \leqslant U$, where

$$U = \langle 1 + ae_{\lambda_i\lambda_{i+1}} \mid i = 1, 2, \dots, r-1,\ a \in F\rangle.$$

But the group U is isomorphic to the group of all upper unitriangular $r \times r$ matrices over F, hence it is torsion–free nilpotent (here we use the fact that F has characteristic zero). This implies that $M(\Lambda, F)$ is a locally nilpotent torsion–free group.

Next we show that $M(\Lambda, F)$ is a perfect group. To do so, it suffices to prove that every generator of $M(\Lambda, F)$ belongs to the derived subgroup of $M(\Lambda, F)$. Let $1 + ae_{\lambda\mu}$ be such a generator. Since Λ is dense, there exists $\nu \in \Lambda$ such that $\lambda < \nu < \mu$. In this case we obtain

$$1 + ae_{\lambda\mu} = [1 + ae_{\lambda\nu}, 1 + e_{\nu\mu}],$$

which proves the assertion. The following lemma shows that under these assumptions the group $M(\Lambda, F)$ is not residually torsion–free nilpotent:

2.3. Lemma. *Let G be a group and $\gamma_i(G)$ the i–th term of its lower central series. For $H \leqslant G$ write $\sqrt{H} := \{g \in G \mid \exists n \in \mathbb{N} :\ g^n \in H\}$. Then:*

1. *G is residually nilpotent iff $\cap_{i=1}^{\infty}\gamma_i(G) = \{1\}$.*
2. *$\sqrt{\gamma_i(G)}$ is a normal subgroup of G.*
3. *G is residually torsion–free nilpotent iff $\cap_{i=1}^{\infty}\sqrt{\gamma_i(G)} = \{1\}$.*

Proof. (1) Let $\mathcal{N} := \{N \lhd G \mid G/N \text{ is nilpotent}\}$. Clearly we have $\gamma_i(G) \in \mathcal{N}$ for every $i \in \mathbb{N}$. Conversely, if $N \in \mathcal{N}$, then G/N is nilpotent, hence there exists $i \in \mathbb{N}$ such that $\gamma_i(G) \leqslant N$. This proves that $\cap\mathcal{N} = \cap_{i=1}^{\infty}\gamma_i(G)$, hence (1) is proved.

(2) For $x \in G$ and $y \in \sqrt{\gamma_i(G)}$, say $y^n \in \gamma_i(G)$, we have $x^{-1}y^nx = (x^{-1}yx)^n \in \gamma_i(G)$, hence $x^{-1}yx \in \sqrt{\gamma_i(G)}$. Obviously, $\sqrt{\gamma_i(G)}$ is closed under taking inverses. To see that $\sqrt{\gamma_i(G)}$ is multiplicative, observe that $G/\gamma_i(G)$ is nilpotent and hence the set of all its elements of finite order is a subgroup.

(3) Let $\mathcal{N} := \{N \lhd G \mid G/N \text{ is torsion–free nilpotent}\}$. As $\gamma_i(G) \subseteq \sqrt{\gamma_i(G)}$, $G/\sqrt{\gamma_i(G)}$ is nilpotent. It is torsion–free by definition. Hence $\sqrt{\gamma_i(G)} \in \mathcal{N}$. Conversely, if $N \in \mathcal{N}$, then G/N is torsion–free and nilpotent. Hence there exists $i \in \mathbb{N}$ such that $\gamma_i(G) \leqslant N$. This implies $\sqrt{\gamma_i(G)} \subseteq \sqrt{N} = N$ and thus $\cap\mathcal{N} = \cap_{i=1}^{\infty}\sqrt{\gamma_i(G)}$, proving (3). ∎

3. $*$–ORDERABLE GROUPS ARE ELEMENTARY

Some examples of $*$–orderable groups are known. But in general not much is known about the class of $*$–orderable groups. By Theorem 5.5 in [CKM], every $*$–orderable group is orderable. There is even an operation that associates an ordering of G to every $*$–ordering of $\mathbb{C}G$. In this section we prove that the class of $*$–orderable groups is elementary in the language of groups (i.e., described by a set of 1st–order axioms).

3.1. Lemma. *The class of $*$–orderable rings is elementary in the language of rings with involution.*

Proof. By the definition of a $*$–ordering, a ring A is $*$–orderable iff

$$\begin{aligned} A \models \exists \geqslant\colon & \forall x\, \forall y\, (x \geqslant 0 \wedge y \geqslant 0 \;\longrightarrow\; x+y \geqslant 0) \;\wedge\; \forall r\, \forall x\, (x \geqslant 0 \;\longrightarrow\; rxr^* \geqslant 0) \;\wedge \\ & \forall x\, (x^* = x \;\longrightarrow\; x \geqslant 0 \vee -x \geqslant 0) \;\wedge\; \forall x\, (x \geqslant 0 \wedge -x \geqslant 0 \;\longrightarrow\; x = 0) \;\wedge \\ & \forall x\, \forall y\, (x \geqslant 0 \wedge y \geqslant 0 \;\longrightarrow\; xy + yx \geqslant 0) \;\wedge \\ & \forall x\, \forall y\; (x^* = x \wedge y^* = y \wedge xyx = 0 \;\longrightarrow\; x = 0 \vee y = 0). \end{aligned}$$

Now e.g. by Exercise 6.6.12 (page 306) in Hodges [Hod], this sentence (although 2nd–order) is equivalent to a set of universal 1st–order sentences.

Alternatively, one can use the Artin–Schreier theory of Marshall [Ma1] to express (in infinitely many 1st–order sentences) that the sum of nonzero elements from the smallest $*$–preordering of A is nonzero. ■

3.2. Theorem. *The class of $*$–orderable groups is elementary in the language of groups.*

Proof. We make use of a known result from model theory. A class of structures $\mathcal{K}$ is elementary iff it is closed under ultraproducts, isomorphic copies and if an ultrapower of a structure A is in $\mathcal{K}$, then $A \in \mathcal{K}$ (see e.g. Corollary 9.5.10 [Hod]).

It is clear that the class of $*$–orderable groups is closed under isomorphic copies. Moreover, if for some group A an ultrapower $A^I/\mathcal{U}$ is $*$–orderable, then so is A (as it is canonically a subgroup of $A^I/\mathcal{U}$). So it suffices to show that the class of $*$–orderable groups is closed under ultraproducts.

Let G_i $(i \in I)$ be a class of $*$–orderable groups and denote $G := \prod_{i\in I} G_i/\mathcal{U}$ for some ultrafilter $\mathcal{U}$ on I. This ultraproduct is taken in the language of groups. We claim that G is $*$–orderable. Write $A_i := \mathbb{C}G_i$. As usual, we endow A_i with the canonical involution sending $g \mapsto g^{-1}$ for $g \in G$ and mapping $\lambda \mapsto \overline{\lambda}$ for $\lambda \in \mathbb{C}$. Define $A := \prod_{i\in I} A_i/\mathcal{U}$ (in the language of rings with involution). Clearly, $G \hookrightarrow A$. We have to prove more: $\mathbb{C}G$ embeds into A so as to extend the canonical embedding $G \hookrightarrow A$. As the possibility is only one, we denote this mapping $\mathbb{C}G \longrightarrow A$ by Φ.

Assume $\Phi(\lambda_1 g_1 + \cdots \lambda_r g_r) = 0$ for some $0 \neq \lambda_j \in \mathbb{C}$ and $g_j \in G$. We may assume that $g_j \neq g_k$ for $j \neq k$. Write $g_k = \big[(g_i^{(k)})_{i\in I}\big]$ for $k = 1, \ldots, r$. Then the given equality implies $\big[(\lambda_1 g_i^{(1)})_{i\in I} + \cdots + (\lambda_r g_i^{(r)})_{i\in I}\big] = 0$. Hence $\{i \in I \mid \sum_{k=1}^{r} \lambda_k g_i^{(k)} = 0\} =: J \in \mathcal{U}$. As all λ_k are nonzero, for all $i \in J$ there must be an index $\ell \in \{1, \ldots, r\}$ such that $g_1^{(i)} = g_\ell^{(i)}$. Hence

$$J \subseteq \{i \in I \mid g_1^{(i)} = g_2^{(i)}\} \cup \cdots \cup \{i \in I \mid g_1^{(i)} = g_r^{(i)}\}.$$

As $\mathcal{U}$ is an ultrafilter, this implies $\{i \in I \mid g_1^{(i)} = g_2^{(i)}\} \cup \cdots \cup \{i \in I \mid g_1^{(i)} = g_r^{(i)}\} \in \mathcal{U}$. Thus one of these sets must be in $\mathcal{U}$, say $\{i \in I \mid g_1^{(i)} = g_2^{(i)}\} \in \mathcal{U}$. But this implies $g_1 = g_2$, a contradiction.

This shows that $\mathbb{C}G$ embeds as a $*$–ring into A. By Lemma 3.1, A is $*$–orderable. Hence so is $\mathbb{C}G$. In other words, G is $*$–orderable. ∎

The downside of this theorem is that is does not give an axiomatization of $*$–orderable groups. It just proves there exists one. It would also be interesting to know whether the class of $*$–orderable groups is a quasi–variety. In view of the results deduced above this class is a quasi–variety iff it is closed under finite direct products.

3.3. Theorem. *Being a $*$–orderable group is a residual property, i.e., if for every $1 \neq x \in G$ there exists a normal subgroup N of G avoiding x such that G/N is $*$–orderable, then so is G.*

Proof. Let G be residually $*$–orderable. Thus $\mathcal{O} := \{N \lhd G \mid G/N \text{ is } *\text{–orderable}\}$ satisfies $\bigcap \mathcal{O} = \{1\}$. If G is not $*$–orderable, then by Theorem 2.1 a finitely generated subgroup H of G is not $*$–orderable. For $N \in \mathcal{O}$, $H/(H \cap N)$ is a subgroup of G/N and so H is residually $*$–orderable. In other words, we may assume that G is finitely generated.

Since G is residually $*$–orderable, we may choose a normal series

$$G = N_1 \supseteq N_2 \supseteq \cdots$$

such that G/N_i is $*$–orderable for every i and $\cap_{i\in\mathbb{N}} N_i = \{1\}$. Without loss of generality, $N_i \neq \{1\}$ for every i. We have a canonical homomorphism

$$\begin{aligned} \varphi : G &\longrightarrow \prod_{i\in\mathbb{N}} G/N_i \\ g &\longmapsto (gN_i)_{i\in\mathbb{N}}. \end{aligned}$$

It is easy to see that φ is an embedding.

Let $\mathcal{U}$ be any non–principal ultrafilter on $\mathbb{N}$ and set $G^\star := \prod_{i\in\mathbb{N}} \big(G/N_i\big)/\mathcal{U}$. Let Ψ denote the canonical homomorphism $\prod_{i\in\mathbb{N}} G/N_i \to G^\star$ and $\Phi := \Psi \circ \varphi$. We claim that Φ is an embedding. Obviously, $\Phi(x) = [xN_i]_{i\in\mathbb{N}} = 1$ iff $J := \{i \in \mathbb{N} \mid x \in N_i\} \in \mathcal{U}$. Since $\mathcal{U}$ is non–principal, J is infinite. So $\cap_{i\in J} N_i = \{1\}$ and thus $x = 1$. Now by Theorem 3.2, $G^\star$ is $*$–orderable and hence so is G. ■

4. On normal series associated to $*$–orderable groups

In this section we use valuations to construct certain normal series for $*$–orderable groups. These series are used to construct orderings of $*$–orderable groups.

Let G be a $*$–orderable group and $P \subseteq \mathbb{C}G$ a $*$–ordering. Let $v : \mathbb{C}G \to \Gamma \cup \{\infty\}$ denote the natural valuation. Since $v(g) = 0$ for all $g \in G$, $\mathcal{O}_v = \mathbb{C}G$. Hence the weak $*$–place $\wp$ maps $\mathbb{C}G \to \mathbb{C}$. As in [CKM], by replacing G with $G/\wp(G)$ we may assume $\wp(G) = \{1\}$. In particular, $v(g-1) > 0$ for all $g \in G$. We use this to define certain subsets of G. For $\gamma \in \Gamma$ set $G_\gamma := \{g \in G \mid v(g-1) \geqslant \gamma\}$, $G_\gamma^+ := \{g \in G \mid v(g-1) > \gamma\} = \cup_{\mu>\lambda} G_\mu$ and $\overline{G_\gamma} := G_\gamma/G_\gamma^+$. Clearly, $\lambda < \mu$ implies $G_\mu^+ \subseteq G_\mu \subseteq G_\lambda^+ \subseteq G_\lambda$.

4.1. Lemma. *G_γ and G_γ^+ are normal subgroups of G. Furthermore, $[G_\lambda, G_\mu] \subseteq G_{\lambda+\mu}^+$. The quotient groups G/G_γ, G/G_γ^+ and $\overline{G_\gamma}$ are torsion–free and $\overline{G_\gamma}$ is abelian.*

Proof. If $x \in G_\gamma$, then $v(x-1) \geqslant \lambda$. As $v(x^{-1}) = 0$, this implies $v(1-x^{-1}) = v(x-1) \geqslant \lambda$. Assume $x' \in G_\gamma$. Then $xx'-1 = (x-1)x' + (x'-1)$ implies $v(xx'-1) \geqslant \min\{v(x-1), v(x'-1)\} \geqslant \gamma$. This shows that G_γ is a subgroup of G. Let $y \in G$ be arbitrary. Clearly, $v(yxy^{-1}-1) = v(y)+v(x-1)+v(y^{-1}) = v(x-1) \geqslant \gamma$. This proves that G_γ is a normal subgroup. Similarly, G_γ^+ is a normal subgroup.

To prove that $[G_\lambda, G_\mu] \subseteq G_{\lambda+\mu}^+$, we use the quasi–commutativity of v. Let $x \in G_\lambda$ and $y \in G_\mu$. Obviously, $v(xyx^{-1}y^{-1}-1) = v(xy-yx)$. Observe that

$$xy - yx = (x-1)(y-1) - (y-1)(x-1).$$

Hence $v(xy-yx) > v(x-1) + v(y-1) \geqslant \gamma + \mu$ by the quasi–commutativity of v. This also proves that $\overline{G_\gamma}$ is abelian.

Let us show that G/G_γ is torsion–free. Assume $v(x^n-1) \geqslant \gamma$ for some $x \in G$. As $v(x^n-1) = v(x-1)+v(1+x+\cdots+x^{n-1})$ and $\wp(1+x+\cdots+x^{n-1}) = n \neq 0$, we have $v(1+x+\cdots+x^{n-1}) = 0$ and thus $v(x-1) = v(x^n-1) \geqslant \gamma$. This proves that G/G_γ is torsion–free. Similarly one can show that G/G_γ^+ (and hence also $\overline{G_\gamma}$) is torsion–free. ■

Let us summarize:

4.2. Theorem. *To every $*$–orderable group G a positively ordered abelian semigroup Γ without 0 and a series of normal subgroups $\{G_\lambda \mid \lambda \in \Gamma\}$ is associated such that:*

(N1) *for all $\lambda, \mu \in \Gamma$, $\lambda < \mu$ implies $G_\mu \subseteq G_\lambda$,*
(N2) *$\bigcap\{G_\lambda \mid \lambda \in \Gamma\} = \{1\}$ and $\bigcup\{G_\lambda \mid \lambda \in \Gamma\} = G$,*
(N3) *all groups G/G_λ are torsion free,*
(N4) *$[G_\lambda, G_\mu] \subseteq G^+_{\lambda+\mu}$, where $\lambda, \mu \in \Gamma$ and $G^+_\gamma := \cup_{\mu>\gamma} G_\mu$.*

Moreover, there exists an ordering of G such that all G_λ are convex.

Proof. Take any $*$–ordering of $\mathbb{C}G$, let $v : \mathbb{C}G \to \Gamma_0 \cup \{\infty\}$ denote the natural valuation and $\Gamma := \Gamma_0 \setminus \{0\}$. As explained above, we may assume $v(g-1) > 0$ for all $g \in G$. Let $G_\lambda := \{g \in G \mid v(g-1) \geqslant \lambda\}$. Assertions (N1) – (N4) follow from what has been deduced above and we prove the last statement.

Since $\overline{G_\lambda}$ is a torsion–free abelian group, it is orderable. Fix an ordering $\overline{P_\lambda}$ of $\overline{G_\lambda}$. Take an arbitrary $1 \neq x \in G$ and set $\gamma := v(x-1)$. We define $x \in P \;:\Leftrightarrow\; xG^+_\gamma \in \overline{P_\gamma}$. Furthermore, let $1 \in P$. We claim that P is an ordering of G.

Clearly, $P \cap P^{-1} = \{1\}$. Let $x \in P \cap G_\lambda$, $x \notin G^+_\lambda$ and $g \in G$. We claim that $g^{-1}xgG^+_\lambda = xG^+_\lambda$. Indeed, $v(x^{-1}g^{-1}xg-1) = v(xg-gx) = v\big((x-1)g-g(x-1)\big) > v(x-1) = \lambda$. This proves $g^{-1}xgG^+_\lambda = xG^+_\lambda$. Thus $g^{-1}xgG^+_\lambda \in \overline{P_\lambda}$ since $xG^+_\lambda \in \overline{P_\lambda}$.

Let us prove that P is closed under multiplication. For this, let $x \in P \cap (G_\lambda \setminus G^+_\lambda)$ and $y \in P \cap (G_\mu \setminus G^+_\mu)$. If $\mu = \lambda$, then $xyG^+_\lambda = (xG^+_\lambda)(yG^+_\lambda) \in \overline{P_\lambda}$ and thus $xy \in P$. If $\lambda \neq \mu$, say $\lambda < \mu$, then $xyG^+_\lambda = xG^+_\lambda \in \overline{P_\lambda}$. Hence $PP \subseteq P$. All this proves that P is an ordering of G.

To finish the proof, we must show that G_λ is convex in G w.r.t. P for all $\lambda \in \Gamma$. Assume $1 \neq x \in P \cap G_\lambda$ and $1 < y < x$. We claim that $y \in G_\lambda$. Assume otherwise. Then $y \in G_\mu \setminus G^+_\mu$ for some $\mu < \lambda$. Hence $xy^{-1} \in G_\mu$ and $xy^{-1}G^+_\mu = y^{-1}G^+_\mu \notin \overline{P_\mu}$. By definition, $xy^{-1} \notin P$. But this contradicts $1 < y < x$. ∎

4.3. Corollary. *A nontrivial $*$–orderable group G cannot be simple.*

Note that this is in sharp contrast with the situation in the ordered setting. Namely, by a result due to Neumann (cf. [MR]), every ordered group can be order–preserving embedded into a simple ordered group.

Proof. Keep the notation and assumptions from above. Let $1 \neq x \in G$ be arbitrary and $\gamma := v(x-1)$. By the previous lemma, G_γ is a normal subgroup of G. If $G_\gamma \neq G$, we are done. Otherwise $G_\gamma = G$. As $x \in G_\gamma \setminus G^+_\gamma$, we have $G^+_\gamma \neq G$. If $G^+_\gamma \neq \{1\}$, we are done. Otherwise $G^+_\gamma = \{1\}$ and hence $G = G_\gamma/G^+_\gamma$ is an abelian torsion–free group by Lemma 4.1. But such a group is never simple. ∎

Now assume G is a group and let Γ, $\{G_\gamma \mid \gamma \in \Gamma\}$ be as in Theorem 4.2. In this case a graded Lie ring $\mathcal{L}(G)$ can be formed as follows. As an abelian group,

$\mathcal{L}(G) := \oplus_{\gamma\in\Gamma}\overline{G_\gamma}$. The commutator operation on G induces an operation $[\ ,\ ]$ on $\mathcal{L}(G)$. By property (N4) of the normal series G_λ, this operation is trivial. Hence $\mathcal{L}(G)$ is an abelian Lie ring. Write $\mathcal{L}_{\mathbb{C}}(G) := \mathbb{C}\otimes_{\mathbb{Z}}\mathcal{L}(G)$.

On the other hand, the normal series G_γ gives us a filtration of $\mathbb{C}G$. First of all, we have a function $w : G\setminus\{1\}\to\Gamma$ defined by $w(g)=\gamma$ iff $g\in G_\gamma\setminus G_\gamma^+$. Now let A_γ be the $\mathbb{C}$–linear span of $\big\{(g_1-1)\cdots(g_r-1)\mid r\in\mathbb{N}_0,\, g_i\in G,\, \sum w(g_i)\geqslant\gamma\big\}$. Clearly, $\mu>\lambda$ implies $A_\mu\subseteq A_\lambda$.

4.4. Lemma. *A_γ and $A_\gamma^+ := \cup_{\mu>\gamma}A_\mu$ are ideals of $\mathbb{C}G$.*

Proof. To prove that A_γ is an ideal of $\mathbb{C}G$, it suffices to prove $GA_\gamma\subseteq A_\gamma$ and $A_\gamma G\subseteq A_\gamma$. Let $h\in G$ and $x:=(g_1-1)\cdots(g_r-1)\in A_\gamma$. Then $hx=(h-1)x+x\in A_\gamma+A_\gamma\subseteq A_\gamma$ and similarly, $xh\in A_\gamma$. This shows that A_γ and A_γ^+ are ideals of $\mathbb{C}G$. ∎

Now we can form the graded ring $\mathrm{gr}\big(\mathbb{C}G,\{G_\lambda\mid\lambda\in\Gamma\}\big)$ as follows. Write $\overline{A_\alpha} := A_\alpha/A_\alpha^+$ for $\alpha\in\Gamma$. Then $\mathrm{gr}\big(\mathbb{C}G,\{G_\lambda\mid\lambda\in\Gamma\}\big) := \oplus_{\alpha\in\Gamma}\overline{A_\alpha}$ is given the componentwise addition and the multiplication induced by $(\overline{a},\overline{b})\mapsto\overline{ab}$ for $a\in A_\alpha$ and $b\in A_\beta$.

We have a canonical Lie homomorphism $\mathcal{L}(G)\to\mathrm{gr}\big(\mathbb{C}G,\{G_\lambda\mid\lambda\in\Gamma\}\big)$ induced by $g_\lambda G_\lambda^+\mapsto(g_\lambda-1)+A_\lambda^+$ for $g_\lambda\in G_\lambda$. This mapping extends to the Lie homomorphism $\mathcal{L}_{\mathbb{C}}(G)\to\mathrm{gr}\big(\mathbb{C}G,\{G_\lambda\mid\lambda\in\Gamma\}\big)$ and by the Poincaré–Birkhoff–Witt theorem to an algebra homomorphism $\Phi:\mathcal{U}(\mathcal{L}_{\mathbb{C}}(G))\to\mathrm{gr}\big(\mathbb{C}G,\{G_\lambda\mid\lambda\in\Gamma\}\big)$. It is easy to see that Φ is onto. In particular, $\mathrm{gr}\big(\mathbb{C}G,\{G_\lambda\mid\lambda\in\Gamma\}\big)$ is commutative.

If $\Gamma=\mathbb{N}$, then $\cap_{n\in\mathbb{N}}A_n=\{0\}$ and Φ is known to be an isomorphism by a theorem due to Quillen (cf. [Pa]). It is not clear whether Φ is still an isomorphism or if $\cap_{\gamma\in\Gamma}A_\gamma=\{0\}$ in this, more general setting. If this holds, then the filtration $\{A_\gamma\mid\gamma\in\Gamma\}$ induces a valuation $v:\mathbb{C}G\to\Gamma\cup\{\infty\}$ by $0\neq x\mapsto\gamma$ iff $x\in A_\gamma\setminus A_\gamma^+$. The corresponding graded ring $\mathrm{gr}(\mathbb{C}G,v)$ is then isomorphic to $\mathcal{U}(\mathcal{L}_{\mathbb{C}}(G))$. Hence by Proposition 2.5 in [Ma2], certain $*$–orderings of $\mathrm{gr}(\mathbb{C}G,v)$ can be pulled back to $*$–orderings of $\mathbb{C}G$. In particular, G is $*$–orderable. Thus if Φ is an isomorphism and $\cap_{\gamma\in\Gamma}A_\gamma=\{0\}$, then Theorem 4.2 completely characterizes the class of $*$–orderable groups and hence the class of $*$–orderable groups is closed under finite direct products (i.e., is a quasi–variety).

5. $*$–orderability of metabelian groups

So far, to prove that a group is $*$–orderable, one usually used some kind of nilpotency condition. Hence it seems natural to study metabelian groups and ask when these groups are $*$–orderable. In [CKM] the authors gave an example of a 2–generated metabelian orderable group G such that the complex group ring $\mathbb{C}G$ is not $*$–orderable. This group is part of a countable family of 2–generated metabelian groups. For $n \in \mathbb{Z} \setminus \{0\}$ write $G_n := \langle a, b \mid ba = a^n b\rangle$. The example in [CKM] was G_2. Clearly, G_1 is abelian and isomorphic to the direct sum of two copies of $\mathbb{Z}$. We are mainly interested in the case $n \neq 1$.

5.1. Proposition.

(1) *All groups G_n are metabelian and torsion–free.*
(2) *The group G_n is orderable iff $n > 0$. More precisely, for $n > 1$ there are exactly 4 orderings of G_n, induced by $e < a < b$, $a < e < b$, $b < e < a$ and $b < a < e$.*

Proof. Note that every element of G_n has a unique canonical form $a^i b^j$ for some $i \in \bigcup_{m\in\mathbb{N}} \mathbb{Z}/n^m =: A$ and $j \in \mathbb{Z}$. Define $N := \{a^\alpha \mid \alpha \in A\}$. Clearly, $N \lhd G$ and $G/N = \langle bN\rangle \cong \mathbb{Z}$. As N and G/N are abelian, G is metabelian. A short calculation using the canonical form shows that G_n is torsion–free.

To prove (2), first observe that $bab^{-1} = a^n$. Assume G_n is orderable and fix an ordering $\leqslant$. If $n < 0$, a and a^n have different signs w.r.t. $\leqslant$. But they are conjugate, a contradiction. For the converse, note that

$$\left(a^\alpha b^\ell\right)\left(a^\beta b^q\right)\left(a^\alpha b^\ell\right)^{-1} = a^{\alpha + n^\ell\cdot\beta - n^q\cdot\alpha} b^q,$$

so two elements of the form $a^\alpha b$ are always conjugate. In particular, for every total ordering $\leqslant$ of G_n we have $\forall \alpha \in A :\ a^\alpha > b$ or $\forall \alpha \in A :\ a^\alpha < b$. This shows that N is convex in every ordering of G_ℓ. Hence there are at most 4 different orderings of G_n, induced by $e < a < b$, $a < e < b$, $b < e < a$ and $b < a < e$. To finish the proof, note that

$$\begin{aligned}
a^{\alpha_1} b^{n_1} \succ_1 a^{\alpha_2} b^{n_2} \ &:\Longleftrightarrow\ n_1 > n_2 \vee (n_1 = n_2 \wedge \alpha_1 > \alpha_2),\\
a^{\alpha_1} b^{n_1} \succ_2 a^{\alpha_2} b^{n_2} \ &:\Longleftrightarrow\ n_1 > n_2 \vee (n_1 = n_2 \wedge \alpha_1 < \alpha_2),\\
a^{\alpha_1} b^{n_1} \succ_3 a^{\alpha_2} b^{n_2} \ &:\Longleftrightarrow\ n_1 < n_2 \vee (n_1 = n_2 \wedge \alpha_1 > \alpha_2),\\
a^{\alpha_1} b^{n_1} \succ_4 a^{\alpha_2} b^{n_2} \ &:\Longleftrightarrow\ n_1 < n_2 \vee (n_1 = n_2 \wedge \alpha_1 < \alpha_2)
\end{aligned}$$

are explicit descriptions of these orderings as a short calculation shows. ■

5.2. Theorem. *G_n is ∗–orderable only for $n = 1$.*

Proof. Clearly, $\mathbb{C}G_n$ is ∗–orderable for $n = 1$. For the rest of the proof let us assume $n \neq 1$. As $\mathbb{C}G_n$ ∗–orderable implies G_n orderable, we may assume $n > 1$ by Proposition 5.1(2).

Assume G_n is ∗–orderable and fix a ∗–ordering of $\mathbb{C}G_n$. We write v for the natural valuation of $\mathbb{C}G_n$ compatible with this ∗–ordering. As v is a ∗–valuation, we have $v(g) = v(g^*) = v(g^{-1}) = -v(g)$, so $v(g) = 0$ for all $g \in G_n$. On the other hand, $v(ab - ba) > v(ab) = 0$ since the corresponding graded ring is commutative. In particular, $v\big(a^{n-1} - 1\big) = v\big(bab^{-1}a^{-1} - 1\big) > 0$.

Note that $ba^{n-1} = a^{(n-1)n}b$. Let us write $t := a^{n-1} - 1$. By the above, $v(t) > 0$. Moreover, $b(1+t) = (1+t)^n b = \big(1 + nt + \frac{n(n-1)}{2}t^2 + \cdots + t^n\big)a$. Hence $bt = \big(nt + \frac{n(n-1)}{2}t^2 + \cdots + t^n\big)b$, so $bt - tb = (n-1)tb + \frac{n(n-1)}{2}t^2 b + \cdots + t^n b$. Now $v(bt - tb) > v(tb) = v(t)$, but $v\big((n-1)tb + \frac{n(n-1)}{2}t^2 b + \cdots + t^n b\big) = v(t)$ since $v(t) < v(t^2) < \cdots < v(t^n)$ and $v(b) = 0$. This is a contradiction, finishing the proof. ∎

References

[Cim] Cimprič, J., *Free skew fields have many ∗–orderings*, J. Algebra (to appear).

[CKM] Cimprič, J., Kochetov, M., Marshall, M., *Orderings and ∗–orderings on cocommutative Hopf algebras*, preprint (2004), `http://vega.fmf.uni-lj.si/srag`.

[Cr] Craven, T., *Places on ∗–fields and the real holomorphy ring*, Comm. Algebra **18** (1990), 2791–2820.

[CS] Craven, T., Smith, T., *Ordered ∗–rings*, J. Algebra **238** (2001), 314–327.

[Hod] Hodges, W., *Model theory*, Encyclopedia of mathematics and its applications, 1993.

[Hol] Holland, S. S., *Strong orderings on ∗–fields*, J. Algebra **101** (1986), 16–46.

[Kl] Klep, I., *On ∗–places and group rings associated to ∗–ordered domains*, preprint (2004), `http://vega.fmf.uni-lj.si/srag`.

[Li] Lichtman, A. I., *Valuation methods in division rings*, J. Algebra **177** (1995), 870–898.

[Ma1] Marshall, M., *∗–orderings on a ring with involution*, Comm. Algebra **28** (2000), 1157–1173.

[Ma2] Marshall, M., *∗–orderings and ∗–valuations on algebras of finite Gelfand–Kirillov dimension*, J. Pure Appl. Algebra **179** (2003), 252–271.

[McL] McLain, D. H., *A characteristically–simple group*, Proc. Cambridge Philos. Soc. **50** (1954), 641–642.

[MR] Mura, B. R., Rhemtulla, A., *Orderable groups*, Marcel Dekker, Inc., 1977.

[Pa] Passi, Inder Bir S., *Group rings and their augmentation ideals*, LNM 715, Springer, 1979.

University of Ljubljana, Institute for Mathematics, Physics and Mechanics, Department of Mathematics, Jadranska 19, SI–1111 Ljubljana, Slovenia

E-mail address: `igor.klep@fmf.uni-lj.si`, `primoz.moravec@fmf.uni-lj.si`