ON TWO GROUP FUNCTORS EXTENDING SCHUR MULTIPLIERS

HEIKO DIETRICH AND PRIMOŽ MORAVEC

Abstract. Liedtke (2008) has introduced group functors K and $\tilde{K}$, which are used in the context of describing certain invariants for complex algebraic surfaces. He proved that these functors are connected to the theory of central extensions and Schur multipliers. In this work we relate K and $\tilde{K}$ to a group functor τ arising in the construction of the non-abelian exterior square of a group. In contrast to $\tilde{K}$, there exist efficient algorithms for constructing τ, especially for polycyclic groups. Supported by computations with the computer algebra system GAP, we investigate when $K(G,3)$ is a quotient of $\tau(G)$, and when $\tau(G)$ and $\tilde{K}(G,3)$ are isomorphic.

1. Introduction

In the study of complex algebraic surfaces it is of interest to find strong invariants which are not too complicated to be useful. Towards this aim, Liedtke [Liedtke 2008] introduced group theoretical functors K and $\tilde{K}$ that are related to the fundamental groups of the associated Galois closures. More precisely, let X be a smooth projective surface, fix a generic projection $f\colon X \to \mathbb{P}^2$ of degree n, and let $f_{\mathrm{gal}}\colon X_{\mathrm{gal}} \to \mathbb{P}^2$ be its Galois closure. Let $\mathbb{A}^2$ be the complement of a fixed generic line in $\mathbb{P}^2$, and set $X^{\mathrm{aff}} = f^{-1}(\mathbb{A}^2)$ and $X^{\mathrm{aff}}_{\mathrm{gal}} = f^{-1}_{\mathrm{gal}}(\mathbb{A}^2)$. It is proved in [Liedtke 2008, Theorems 5.1 & 5.2] that $\pi_1(X^{\mathrm{aff}}_{\mathrm{gal}})$ has images isomorphic to $\tilde{K}(\pi_1(X^{\mathrm{aff}}), n)$ and to $K(\pi_1(X^{\mathrm{aff}}), n)$. It is the constructions of $K(-,n)$ and $\tilde{K}(-,n)$ that are central to Liedtke's investigation in [Liedtke 2008, Liedtke 2010]. As pointed out in these papers, it is important to have a better understanding of $\tilde{K}$ in order to describe the above mentioned fundamental groups.

The aim of this work is to extend the group theoretical analysis of the functors $\tilde{K}$ and K, and to relate these to a functor τ associated with Brown and Loday's construction of the non-abelian tensor square of a group [BL 1987]. The latter has applications in topology and K-theory, and can efficiently be computed for several classes of groups, such as polycyclic groups.

In Section 2, we set the notations and give the definitions of $K(G,n)$, $\tilde{K}(G,n)$, and $\tau(G)$. In Section 3, we elaborate on these and provide explicit descriptions that enable efficient computations for polycyclic groups. In Section 4, we introduce the concept of an AI-automorphism and show that the existence of such an automorphism for a group G yields a central extension

$$1 \longrightarrow H_2(G,\mathbb{Z}) \longrightarrow \tau(G) \longrightarrow K(G,3) \longrightarrow 1,$$

similar to the one proved in [Liedtke 2008, Theorem 2.2]:

$$1 \longrightarrow H_2(G,\mathbb{Z}) \longrightarrow \tilde{K}(G,3) \longrightarrow K(G,3) \longrightarrow 1.$$

School of Mathematics, Monash University, Clayton VIC 3800, Australia
Faculty of Mathematics and Physics, University of Ljubljana, Slovenia
E-mail addresses: heiko.dietrich@monash.edu, primoz.moravec@fmf.uni-lj.si.
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It is therefore natural to ask when $\tau(G)$ and $\tilde{K}(G,3)$ are isomorphic. In Section 5, we explore this question for several classes of groups. For example, we show that if G is a finite group and a Schur cover $H/M = G$ admits an AI-automorphism which acts as inversion on M, then $\tau(G) \cong \tilde{K}(G,3)$.

In Section 6, we show that $K(G,3)$ and $\tilde{K}(G,3)$ are closely related to the unramified Brauer group of the field of G-fixed points in a complex function field. This group is also known as the Bogomolov multiplier $B_0(G)$, and has various applications in algebraic geometry, in particular, to Noether's Problem. In Section 7 we comment on our computational experiments with the system GAP [GAP].

2. Definitions and preliminary results

Unless stated otherwise, all groups are finite and written multiplicatively. For a group G and integer $n > 0$ we denote by G^n the direct product of n copies of G. We write C_n for the cyclic group of size n. The commutator subgroup G' is the subgroup of G generated by all commutators $[g,h] = g^{-1}h^{-1}gh = g^{-1}g^h$ with $g,h \in G$. A free presentation for G is a free group F with normal subgroup $N \trianglelefteq F$ such that $G \cong F/N$; since G is assumed to be finite, we assume that F is finitely generated. A polycyclic presentation $\mathrm{pc}\langle g_1,\ldots,g_n \mid r_1,\ldots,r_m\rangle$ for G is a group presentation with abstract generators $g_1,\ldots,g_n$ and relations $r_1,\ldots,r_m$ that are power or conjugate relations, with the convention that trivial conjugate relations are omitted; see [EN 2008, Section 2.1] for details. For example, $\mathrm{pc}\langle g_1,g_2 \mid g_1^2, g_1^2\rangle$ describes the Klein 4-group $\langle g_1,g_2 \mid g_1^2, g_2^2, g_2^{g_1} = g_2\rangle$. A group extension of A by B is written $G = B.A$, meaning that $A \trianglelefteq G$ with quotient $G/A = B$.

2.1. Liedtke's constructions.

For a group G and integer $n \geqslant 2$, the group $K(G,n)$ is the kernel of the map $G^n \to G/G'$ that sends an n-tuple $(g_1,\ldots,g_n)$ to the product of its components modulo the commutator subgroups, that is,

$$K(G,n) = \{(g_1,\ldots,g_n) \in G^n : g_1 \cdots g_n \in G'\}.$$

Note that every permutation of the n factors in G^n defines an automorphism of $K(G,n)$, that is, we have $\mathrm{Sym}_n \leqslant \mathrm{Aut}(K(G,n))$. To define the group $\tilde{K}(G,n)$, choose a free presentation $G = F/N$ for G, and set

$$\tilde{K}(G,n) = K(F,n)/K(N,n)^{F^n},$$

where $K(N,n)^{F^n}$ is the normal closure of $K(N,n)$ in F^n; if $n \geqslant 3$, then this is simply the normal closure of $K(N,n)$ in $K(F,n)$, see [Liedtke 2008, p. 248]. It is shown in [Liedtke 2008, Theorem 2.2] that the definition of $\tilde{K}(G,n)$ does not depend on the choice of presentation for G.

2.2. Non-abelian exterior square.

Let G and G^* be groups, with isomorphism $G \to G^*$, $g \mapsto g^*$; we continue to use "$*$" to denote elements and subsets of G^*. Let $G \star G^*$ be the free product of G and G^*, and, following [Rocco 1991], define $\nu(G)$ as a quotient group of $G \star G^*$ via

$$\nu(G) = (G \star G^*)/\langle\{[x,y^*]^z[x^z,(y^z)^*]^{-1}, [x,y^*]^{(z^*)}[x^z,(y^z)^*]^{-1} : x,y,z \in G\}\rangle^{G\star G^*}.$$

To simplify notation, we identify elements in $\nu(G)$ with elements in $G \star G^*$, keeping in mind that further relations hold in $\nu(G)$. If we want to emphasise the parent group, then we sometimes use subscripts at generated groups $\langle - \rangle_A$ or at commutators $[-,-]_A$ to indicate that the corresponding structures are to be considered in the group A. For example, if $g \in G$ and $g^* \in G^*$, then $[g,g^*]_{\nu(G)}$ denotes their commutator in $\nu(G)$, not in $G * G^*$. With this convention, consider $\nabla(G) = \langle [x,x^*]_{\nu(G)} : x \in G\rangle$ as a subgroup of $\nu(G)$, and define

$$\tau(G) = \nu(G)/\nabla(G).$$

Note that the homomorphism $G \star G^* \to G \times G$, $g_1 h_1^* g_2 h_2^* \dots g_k h_k^* \mapsto (g_1 \cdots g_k, h_1 \cdots h_k)$, maps commutators $[x, y^*]$ to 1, hence it induces short exact sequences

$$1 \longrightarrow G \otimes G \longrightarrow \nu(G) \xrightarrow{c_\nu} G \times G \longrightarrow 1$$

$$1 \longrightarrow G \wedge G \longrightarrow \tau(G) \xrightarrow{c_\tau} G \times G \longrightarrow 1$$

where the kernels $G \otimes G$ and $G \wedge G$ are called the *non-abelian tensor square* and the *non-abelian exterior square* of G, respectively. In will be shown in Lemma 3.1 below that this coincides with the definitions given in [BL 1987]. We conclude with a lemma that is used later.

Lemma 2.1. *Let $H \to G$ be a surjective group homomorphism with kernel M. Then there are induced epimorphisms $\beta\colon \nu(H) \to \nu(G)$ and $\gamma\colon \tau(H) \to \tau(G)$ whose kernels are*

$$\langle M, M^* \rangle_{\nu(H)} [M, H^*]_{\nu(H)} [H, M^*]_{\nu(H)} \quad \textit{and} \quad \langle M, M^* \rangle_{\tau(H)} [M, H^*]_{\tau(H)} [H, M^*]_{\tau(H)}.$$

Proof. For β this is [Rocco 1991, Proposition 2.5]. Since β maps $\nabla(H)$ to $\nabla(G)$, this induces γ. Note that $\ker \gamma = \{x \cdot \nabla(H) : x \in \beta^{-1}(\nabla(G))\}$, and $\beta^{-1}(\nabla(G)) = (\ker \beta) \cdot \nabla(H)$; the claim follows. □

2.3. **Schur multiplier.** We recall some facts about the Schur multiplier of a finite group and refer to [Karpilovsky 1987] for more details, in particular, Proposition 2.1.1 and Theorems 2.1.4, 2.4.6, 2.5.1, 2.6.7, and 2.7.3. A *Schur cover* of G is a group H such that $H/M \cong G$ for some $M \leqslant H' \cap Z(H)$ isomorphic to the *Schur multiplier*

$$M(G) = H^2(G, \mathbb{C}^\times).$$

Note that $G' \cong H'/M$ since $M \leqslant H'$. Schur (1904-07) has shown that $M(G)$ is finite and if $F/N = G$ is a free presentation of G with F a free group of finite rank r, then $M(G) \cong (F' \cap N)/[F, N]$; the latter is known as Hopf's formula. Every Schur cover H of G is isomorphic to F/S for some normal subgroup $S \trianglelefteq F$ that defines a complement $S/[F, N]$ to $(F' \cap N)/[F, N] \cong M(G)$ in $N/[F, N]$; in particular, $S/[F, N]$ is free abelian of rank r and $(F' \cap N)/[F, N]$ is the torsion subgroup of $N/[F, N]$. The isomorphism type of a Schur cover is in general not uniquely determined. However, Schur proved that the isomorphism type of H' depends only on G, and not on the chosen cover H. Miller (1952) has shown that

$$M(G) \cong H_2(G, \mathbb{Z}).$$

We will see in Remark 3.2 below that we can identify $[G, G^*]_{\tau(G)} = G \wedge G$ via $[g, h^*] \mapsto g \wedge h$. This identification allows us to define the surjective commutator map

$$\kappa\colon G \wedge G \to G', \quad g \wedge h \mapsto [g, h],$$

which, according to [BJR 1987, Corollary 2], can be lifted to an isomorphism

$$G \wedge G \to H', \quad g \wedge h \to [g', h'],$$

where $g', h' \in H$ are lifts of $g, h \in G$. Since $G' = H'/M(G)$, this yields an exact sequence

$$1 \longrightarrow M(G) \longrightarrow G \wedge G \xrightarrow{\kappa} G' \longrightarrow 1$$

with $\ker \kappa \cong M(G)$ central in $G \wedge G$. This shows that if G is abelian, then $G \wedge G \cong M(G) \cong H'$, and a Schur cover of G is abelian if and only if G is cyclic if and only if $M(G) = 1$.

3. Explicit description

As a first step towards investigating the relation between $\tau(G)$ and $\tilde{K}(G, 3)$ we provide a more concrete description of these groups.

3.1. **An explicit description of τ.** The next lemma summarises some facts about $\tau(G)$ and $\nu(G)$.

Lemma 3.1. *Every $w \in \nu(G)$ can be written uniquely as $w = gh^*w'$ for some $w' \in [G, G^*]_{\nu(G)}$ and $g, h \in G$; the analogous statement holds in $\tau(G)$. Moreover, we have*

$$\ker c_\nu = [G, G^*]_{\nu(G)} \cong G \otimes G \quad \textit{and} \quad \ker c_\tau = [G, G^*]_{\tau(G)} \cong G \wedge G.$$

Proof. Let $g = g_1h_1^* \cdots g_nh_n^* \in \nu(G)$. The identities

$$h^*g = gh^*[h^*, g], \quad [h^*, g]k = k[(h^k)^*, g^k], \quad \text{and} \quad [h^*, g]k^* = k^*[(h^k)^*, g^k] \tag{3.1}$$

can be used to rewrite $g = g_1h_1^* \cdots g_nh_n^* = (g_1 \cdots g_n)(h_1 \cdots h_n)^*w$ with $w \in [G, G^*]_{\nu(G)}$. Recall that c_ν maps $[G, G^*]_{\nu(G)}$ to 1, hence $c_\nu(g) = (g_1 \cdots g_n, h_1 \cdots h_n)$, which proves $\ker c_\nu = [G, G^*]_{\nu(G)}$. The uniqueness of the expression of g follows from the exact sequence associated with c_ν. The argument for $\tau(G)$ and c_τ is exactly the same. Recall that above we have defined $G \otimes G = \ker c_\nu$ and $G \wedge G = \ker c_\tau$; it is shown in [Rocco 1991, Proposition 2.6] that the 'non-abelian tensor square' of [BL 1987] is isomorphic to $[G, G^*]_{\nu(G)}$ via $[g, h^*] \mapsto g \otimes h$, and from this it follows that the 'non-abelian exterior square' of [BL 1987] is is naturally isomorphic to $[G, G^*]_{\tau(G)}$. □

Remark 3.2. Using Lemma 3.1, we can identify

$$G \otimes G = [G, G^*]_{\nu(G)} \quad \text{and} \quad [G, G^*]_{\tau(G)} = G \wedge G$$

via $g \otimes h \to [g, h^*]$ and $g \wedge h \to [g, h^*]$, respectively.

Proposition 3.3. *The group $\tau(G)$ is isomorphic to $G^2.(G \wedge G)$ with multiplication*

$$(a, b; c)(g, h; d) = (ag, bh; (b^h \wedge g^h)c^{gh}d),$$

and derived subgroup $\tau(G)' \cong (G' \times G').(G \wedge G)$.

Proof. By Lemma 3.1, the element $gh^*w \in \tau(G)$ corresponds to $(g, h; w) \in G^2.(G \wedge G)$, and this correspondence defines the multiplication in $G^2.(G \wedge G)$. Note that $c \in G \wedge G$ corresponds to an element of the form $\prod_i [x_i, y_i^*]$, and so c^g and $c^{(g^*)}$ both correspond to $\prod_i [x_i^g, (y_i^g)^*]$. The last claim is [Rocco 1991, Theorem 3.1]. □

Remark 3.4. If $G \wedge G$ is abelian, then Proposition 3.3 shows that $\tau(G)$ is an extension of $G \wedge G$ by G^2 defined by a 2-cocycle $\gamma \in Z^2(G^2, G \wedge G)$ with $\gamma((a, b), (g, h)) = b^h \wedge g^h$; the G^2-module structure on $G \wedge G$ is defined by $(u \wedge v)^{(g,h)} = (u^{gh} \wedge v^{gh})$, cf. [Robinson 1982, §11.4].

Remark 3.5. The extension in Remark 3.4 is split if and only if there is a function $f\colon G^2 \to G \wedge G$ such that the subset $\{(a, b; f(a, b)) : a, b \in G\}$ is a subgroup of $G^2.(G \wedge G)$ isomorphic to G^2 via $(a, b) \mapsto (a, b; f(a, b))$. In this case $A = \{(a, 1; f(a, 1)) : a \in G\}$ and $B = \{(1, b; f(1, b)) : b \in G\}$ are commuting and disjoint subgroups of $G^2.(G \wedge G)$ isomorphic to G. In particular, the maps $a \to f(a, 1)$ and $b \to f(1, b)$ are 1-cocycles $G \to G \wedge G$; recall that a 1-cocycle $r\colon G \to G \wedge G$ is a map satisfying $r(gh) = r(g)^h r(h)$ for all $g, h \in G$. Conversely, for every pair of 1-cocycles $l, r\colon G \to G \wedge G$ the sets $L = \{(a, 1; l(a)) : a \in G\}$ and $R = \{(1, b; r(b)) : b \in G\}$ are disjoint subgroups of $G^2.(G \wedge G)$ isomorphic G. Together they form a complement to $G \wedge G$ if and only if they commute, that is, if and only if $l(a)^b r(b) = (b \wedge a) r(b)^a l(a)$ for all $a, b \in G$. The existence of such 1-cocycles is a necessary and sufficient condition for the extension to be split.

Remark 3.6. It follows from [BL 1987, Proposition 2.5] that G acts trivially on the kernels of the maps $\kappa\colon G \wedge G \to G'$ and $\kappa'\colon G \otimes G \to G'$, both induced by the commutator map. This proves that $\ker \kappa \trianglelefteq \tau(G)$ and $\ker \kappa' \trianglelefteq \nu(G)$ are central. Since $(G \wedge G)/\ker \kappa \cong G'$, we have $\tau(G)/\ker \kappa \cong G^2.G'$ with multiplication $(a, b; c)(g, h; d) = (ag, bh; [b, g]^h c^{gh} d)$. An analysis similar to that in Remark 3.5 can be used to determine necessary and sufficient conditions for this extension to be split.

3.2. **An explicit description of $\tilde{K}$.** The following result is based on [Liedtke 2008, Theorem 3.2]. We denote the components of a tuple g by $g_1, g_2, \ldots$, that is, $g \in G^{n-1}$ is $g = (g_1, \ldots, g_{n-1})$.

Proposition 3.7. *Let G be a group with Schur cover H and $H/M = G$. The following hold for $n \geqslant 3$.*

a) *We have $K(G,n) \cong G^{n-1}.G'$ where the product of $u = (g;c)$ and $v = (h;d)$ in $G^{n-1}.G'$ is defined as*
$$uv = (gh; \mu(g,h)c^h d)$$
where $c^h = c^{(h_1 \cdots h_{n-1})^{-1}}$ and $\mu(g,h) = (g_1h_1)\cdots(g_{n-1}h_{n-1})(g_1\cdots g_{n-1})^{-1}(h_1\cdots h_{n-1})^{-1}$; we have $\mu(g,h)(c^g)^h = c^{(gh)}\mu(g,h)$ for all $g, h \in G^{n-1}$ and $c \in G'$.

b) *Let μ be the map defining $K(H,n) \cong H^{n-1}.H'$ as in a). Identifying $H' = G \wedge G$ via the isomorphism in Section 2.3, we have $\tilde{K}(G,n) \cong G^{n-1}.(G \wedge G)$ where the product of $u = (g;c)$ and $v = (h;d)$ in $G^{n-1}.(G \wedge G)$ is defined as*
$$uv = (gh; \mu(g',h')c^h d);$$
here $g', h' \in H^{n-1}$ are elements that map onto $g, h \in G^{n-1}$, and c^h is defined as in a).

c) *There is a central extension*
$$1 \longrightarrow H_2(G,\mathbb{Z}) \longrightarrow \tilde{K}(G,n) \longrightarrow K(G,n) \longrightarrow 1.$$

Proof. a) By definition, $K(G,n) = \{(g_1, \ldots, g_{n-1}, g_{n-1}^{-1}\cdots g_1^{-1}c) : g_1, \ldots, g_{n-1} \in G, c \in G'\}$. The isomorphism from $G^{n-1}.G'$ to $K(G,n)$ maps $(g;c) \in G^{n-1}.G'$ to $(g, g_{n-1}^{-1}\cdots g_1^{-1}c) \in K(G,n)$; the definition of μ and c^h guarantee that this is an isomorphism.

b) It is shown in [Liedtke 2008, Theorem 3.2] that $\tilde{K}(G,n) \cong K(H,n)/K(M,n)$, independent of the chosen cover. The proof of a) shows that there is an isomorphism $\varphi\colon H^{n-1}.(G \wedge G) \to K(H,n)$. Recall that $M \leqslant Z(H)$ is central, hence it follows from the definition of μ that $M^{n-1}.1$ is a central subgroup of $H^{n-1}.(G \wedge G)$. This subgroup is mapped under φ onto $K(M,n)$, which proves that $\tilde{K}(G,n) \cong K(H,n)/K(M,n) \cong (H^{n-1}.(G \wedge G))/(M^{n-1}.1) \cong G^{n-1}.(G \wedge G)$. Note that the multiplication is well-defined since $M \leqslant Z(H)$.

c) This is [Liedtke 2008, Theorem 2.2]: note that the epimorphism from $\tilde{K}(G,n) \cong G^{n-1}.(G \wedge G)$ to $K(G,n) \cong G^{n-1}.G' \cong K(G,n)$ can be induced by $\kappa\colon G \wedge G \to G'$; recall from Section 2.3 and Remark 3.6 that $\ker \kappa \cong H^2(G,\mathbb{Z})$ is central. □

Remark 3.8. If G' is abelian, then Proposition 3.7a) shows that $K(G,n)$ is an extension of G' by G^{n-1} defined by the 2-cocycle $\mu \in Z^2(G^{n-1}, G')$ as in the proposition and G^{n-1}-module structure on G' defined by $c^h = c^{(h_1\cdots h_{n-1})^{-1}}$; since G' is abelian, this is indeed a group action. A similar consideration as in Remark 3.5 can be used to obtain a (quite technical) criterion for splitness.

Corollary 3.9. *If H has nilpotency class at most 2, then $K(H,n) \cong H^{n-1}.H'$ with multiplication*
$$(g;c)(h,d) = (gh; cd \prod\nolimits_{i=1}^{n-1} \prod\nolimits_{j=i}^{n-1} [g_i, h_j]).$$

Proof. This follows from the formula given in Proposition 3.7a), together with $c, d \in H' \leqslant Z(H)$ and $[h, g^{-1}] = [h, g^{-1}]^{(g^h)} = [g,h]$ for all $g, h \in G$. □

3.3. **Abelian groups.** For a group G let $Z^\wedge(G) = \{g \in G : g \wedge x = 1 \text{ for all } x \in G\}$ be the *epicentre* of G. Note that $Z^\wedge(G)$ is equal to the projection of the center of a Schur cover of G on G, see [Ellis 1995, p. 254], therefore the next result agrees with [Liedtke 2008, Proposition 4.7]. It is shown in [Ellis 1995, Proposition 16(vii)] that there exists H with $H/Z(H) \cong G$ if and only if $Z^\wedge(G) = 1$.

Proposition 3.10. *If G is an abelian group, then $\tilde{K}(G,n)$ is isomorphic to the group $G^{n-1}.(G\wedge G)$ with multiplication*

$$(g;c)(h;d) = (gh; cd\prod\nolimits_{i=1}^{n-1} g_i\wedge (h_i\cdots h_{n-1})).$$

Under this identification,

$$\begin{aligned} Z(\tilde{K}(G,n)) &= \{(u,uy_2,\ldots,uy_{n-1};c)\in G^{n-1}.(G\wedge G) : y_2,\ldots,y_{n-1},u^n\in Z^\wedge(G)\} \\ &\cong Z^\wedge(G)^{n-2}\times (G\wedge G)\times\{u\in G : u^n\in Z^\wedge(G)\}. \end{aligned}$$

Proof. Let H be a Schur cover of G with $H/M = G$. It follows from Corollary 3.9 and Proposition 3.7b) that $\tilde{K}(G,n)\cong G^{n-1}.H'$ with multiplication

$$(g;c)(h,d) = (gh; cd\prod\nolimits_{i=1}^{n-1}\prod\nolimits_{j=i}^{n-1}[g_i',h_j']),$$

where each g_i' and h_j' is a lift of $g_i,h_j\in G$ to H; note that $H'=M\leqslant Z(H)$ and $H'=M\cong G\wedge G$ since G is abelian. Recall that $G\wedge G=\ker c_\tau$, that is, $G\wedge G=\langle g\wedge h : g,h\in G\rangle$ with the convention $g\wedge h=[g,h^*]_{\tau(G)}$. In particular, if $[g',h']_H\in H$ where $g',h'\in H$ are lifts of $g,h\in G$, then $H'\cong G\wedge G$ via $[g',h']\mapsto g\wedge h$. The first claim follows.

If $(a;c)\in Z(\tilde{K}(G,n))$, then the following holds for all $(g;d)\in\tilde{K}(G,n)$:

$$\prod\nolimits_{i=1}^{n-1} a_i\wedge g_i\cdots g_{n-1} = \prod\nolimits_{i=1}^{n-1} g_i\wedge a_i\cdots a_{n-1}.$$

Considering g with only one nontrivial entry $g_i=h$, this forces

$$a_1\ldots a_{i-1}a_i^2a_{i+1}\ldots a_{n-1}\wedge h = 1 \quad \text{for all } h\in G \text{ and } i\in\{1,\ldots,n-1\}.$$

Write $z_i = a_1\ldots a_{i-1}a_i^2a_{i+1}\ldots a_{n-1}$ and note that each $z_i\in Z^\wedge(G)$; now for $i=2,\ldots,n-1$ we have $z_{i-1}^{-1}z_i = a_{i-1}^{-1}a_i\in Z^\wedge(G)$, so $a_i=a_1y_i$ for some $y_i\in Z^\wedge(G)$. Now $z_1\in Z^\wedge(G)$ yields $a_1^n\in Z^\wedge(G)$. Conversely, it is easy to check that every such element yields a central $(a;c)$. □

Proposition 3.11. *If G is an abelian group, then $\tau(G)$ is isomorphic to the group $G^2.(G\wedge G)$, where the multiplication is given by $(g_1,g_2;c)(h_1,h_2;d)=(g_1h_1,g_2h_2;cd(g_2\wedge h_1))$. Under this identification, $Z(\tau(G))=\{(a,b;c) : a,b\in Z^\wedge(G), c\in G\wedge G\}\cong Z^\wedge(G)^2\times(G\wedge G)$.*

Proof. The first claim follows from Proposition 3.3. As above, $(a,b;c)\in Z(\tau(G))$ if and only if $b\wedge g = h\wedge a$ for all $g,h\in G$. If $g=1$, then $a\in Z^\wedge(G)$; if $h=1$, then $b\in Z^\wedge(G)$. Conversely, every such $(a,b;c)$ lies in the centre; the claim follows. □

4. Relating $\tau(G)$ with $\tilde{K}(G,3)$ and $K(G,3)$

The aim of this section is to relate $\tau(G)$ with $\tilde{K}(G,3)$. As a first step, we consider a construction of an epimorphism $\tau(G)\to K(G,3)$. Our construction requires an automorphism of G which acts as inversion on the abelianisation of G.

4.1. AI-automorphisms.

An automorphism $\alpha\in\mathrm{Aut}(G)$ of a group G is an *AI-automorphism* if it induces the inversion automorphism on the abelianisation G/G'; this is not to be confused with an *IA-automorphism* introduced by Bachmuth (1966), which is an automorphism that induces the identity on the abelianisation. Clearly, the composition of two AI-automorphisms is an IA-automorphism; for abelian groups the only AI-automorphism is inversion.

Example 4.1. Let F be a free group on X. The map $X \to X$ given by $x \mapsto x^{-1}$ for all $x \in X$ induces an AI-automorphism ι_F of F. If a group G is given by a free presentation $G = F/N$ such that $\iota_F(N) = N$, then ι_F induces an AI-automorphism of G. Note that if F/N is abelian, then $F' \leqslant N$, hence $\iota_F(N) = N$ and ι_F induces inversion on G. If $\iota_F(N) \neq N$, then define $M = \iota_F(N)N \trianglelefteq F$. By definition, $\iota_F(M) = M$, and F/M is the largest quotient of G on which ι_F induces an AI-automorphism. In particular, every group G has such a quotient since ι_F induces inversion on $F/F'N \cong G/G'$. We give two examples. First, the dihedral group of order $2n$ can be defined as $D_{2n} = F/N$ where F is free on $\{a, b\}$ and N is the normal closure of $\{a^n, b^2, a^b a\}$. Clearly, $\iota_F(a^n) = (a^{-1})^n$ and $\iota_F(b^2) = b^{-2}$ lie in N; moreover, $(\iota_F(a^b a)^{-1})^b = (aa^{b^{-1}})^b = a^b a \in N$, hence ι_F induces an AI-automorphism on F/N. Second, consider $G = F/N$ where F is free on $\{g, h\}$ and N is the normal closure of $\{g^4, h^5, h^g h^2\}$, that is, G is a semidirect product $C_4 \ltimes C_5$. A direct computation (by hand or with GAP [GAP]) shows that G does not admit an AI-automorphism, which implies that $\iota_F(N) \neq N$. If M is the normal closure of $\{g^4, h^5, (h^{-1})^{(g^{-1})}h^{-2}, h^g h^2\}$, then $\iota_F(M) = M$, and $G/M \cong C_4$ is the largest quotient of G on which ι_F induces an AI-automorphism.

Example 4.2. Let $\alpha \in \mathrm{Aut}(G)$ be an automorphism which inverts every element of a generating set X of G. Such an automorphism is called *GI-automorphism* in [Boston 2006], where GI can be interpreted as "generator inversion". (Originally, GI stands for "generator-involutions" because $\langle\alpha\rangle \ltimes G$ is generated by involutions $\{(\alpha, x) : x \in X\}$.) Clearly, every GI-automorphism is an AI-automorphism. The map ι_F in Example 4.1 is an example. To give another example, consider the alternating group Alt_n of rank $n \geqslant 3$: Conjugation by the transposition $(1\,2)$ defines an automorphism α of Alt_n that inverts every element of the generating set $\{(1\,2\,3), (1\,2\,4), \ldots, (1\,2\,n)\}$; thus α is a GI- and AI-automorphism.

4.2. **An epimorphism.** Suppose G has an AI-automorphism α; we use α to construct $K(G, 3)$ as a quotient of $\tau(G)$. Note that the homomorphism

$$G \star G^* \to G^3, \quad g_1 h_1^* \ldots g_k h_k^* \mapsto (g_1 \ldots g_k, h_1 \ldots h_k, \alpha(g_1 h_1 \ldots g_k h_k))$$

maps commutators $[x, x^*]$ to 1; since the above map forgets "$*$", it also maps the relations of $\tau(G)$ to 1. Thus there is an induced homomorphism

$$\Phi_\alpha \colon \tau(G) \to G^3.$$

Recall that the commutator map

$$\kappa \colon G \wedge G = [G, G^*]_{\tau(G)} \to G'$$

has central kernel $H_2(G, \mathbb{Z}) \cong M(G)$, see Section 2.3. We now show the following:

Theorem 4.3. *If $\alpha \in \mathrm{Aut}(G)$ is an AI-automorphism, then*

$$\operatorname{im} \Phi_\alpha = K(G, 3) \quad \textit{and} \quad \ker \Phi_\alpha = \ker \kappa \leqslant Z(\tau(G)).$$

Proof. The inclusion $\operatorname{im} \Phi_\alpha \leqslant K(G, 3)$ follows immediately from the definition and the fact that α is an AI-automorphism. If $(g, h, k) \in K(G, 3)$, then $k = h^{-1}g^{-1}c$ for some $c \in G'$. Note that Φ_α maps gh^* to $(g, h, \alpha(gh)) \in K(G, 3)$, and $\alpha(gh) = h^{-1}g^{-1}d$ for some $d \in G'$, thus

$$\Phi_\alpha(gh^*)^{-1} \cdot (g, h, k) = (1, 1, d^{-1}c);$$

now $d^{-1}c = \prod_i [x_i, y_i] \in G'$, and so $(1, 1, d^{-1}c) = \Phi_\alpha(\prod_i [\alpha^{-1}(x_i), (\alpha^{-1}(y_i))^*])$. This shows that $(g, h, k) \in \operatorname{im} \Phi_\alpha$, thus $K(G, 3) \leqslant \operatorname{im} \Phi_\alpha$. Now we consider the kernel. Note that

$$\ker \Phi_\alpha = \{g_1 h_1^* \ldots g_k h_k^* : g_1 \cdots g_k = h_1 \cdots h_k = (g_1 h_1) \cdots (g_k h_k) = 1\}.$$

If $w = g_1 h_1^* \ldots g_k h_k^* \in \ker \Phi_\alpha$, then use Lemma 3.1 to rewrite $w = g_1 \cdots g_k (h_1 \cdots h_k)^* w' = w'$ for some $w' = \prod_i [x_i, y_i^*] \in [G, G^*]_{\tau(G)}$; thus, $w = w' \in G \wedge G$ and applying κ yields $\kappa(w) =$

$\kappa(w') = \prod_i [x_i, y_i] \in G$. Note that rewriting w to w' involves a sequence of commutator rules as in (3.1), replacing elements such as a^*b by $ba^*[a^*, b]$, etc. Obviously, this rewriting process can be reversed which yields a sequence of commutator rules that bring w' back into the form w. Applying this reversed process not to w', but to $\prod_i [x_i, y_i]_G$, we obtain $g_1h_1 \ldots g_kh_k \in G$, which is the element w without the "$*$". Recall that $g_1h_1 \ldots g_kh_k = 1$ by assumption, which shows that

$$\kappa(w) = \kappa(w') = \prod\nolimits_i [x_i, y_i] = g_1h_1 \ldots g_kh_k = 1,$$

so $w \in \ker \kappa$. Conversely, let $w \in \ker \kappa$, that is, $w = \prod_i [g_i, h_i^*] \in [G, G^*]_{\tau(G)}$ with $\prod_i [g_i, h_i] = 1$. Writing w as $w = \prod_i g_i^{-1}(h_i^{-1})^* g_i h_i^*$ and applying Φ_α shows that

$$\Phi_\alpha(w) = (1, 1, \alpha([g_1, h_1] \ldots [g_k, h_k])) = (1, 1, 1),$$

hence $\ker \kappa \leqslant \ker \Phi_\alpha$. In conclusion, $\ker \Phi_\alpha = \ker \kappa$. By Remark 3.6, the group $\ker \Phi_\alpha$ is central. □

Corollary 4.4. *The existence of an AI-automorphism of G yields a central extension*

$$1 \longrightarrow H_2(G, \mathbb{Z}) \longrightarrow \tau(G) \longrightarrow K(G, 3) \longrightarrow 1.$$

Together with Proposition 3.7c), it seems natural to ask when $\tau(G) \cong \tilde{K}(G, 3)$. We will see in Proposition 5.4 that the lack of AI-automorphisms may prevent this.

4.3. **A subgroup $T(G)$.** Recall that $K(G, 3)$ is the kernel of $G^3 \to G/G'$, $(g, h, k) \mapsto ghkG'$. We now consider the kernel

$$T(G) = \{(g, h, cgh) : g, h \in G, c \in G'\}$$

of the homomorphism $\pi\colon G^3 \to G/G'$, $(g, h, k) \mapsto ghk^{-1}G'$. We now provide a short alternative proof that $K(G, 3) \cong \tau(G)/\ker \kappa$ if G has an AI-automorphism, cf. Theorem 4.3.

Lemma 4.5. *We have $T(G) \cong \tau(G)/\ker \kappa$. If G has AI-automorphisms, then $K(G, 3) \cong T(G)$.*

Proof. Recall that we can identify $\tau(G)/\ker \kappa = G^2.G'$ and $K(G, 3) = G^2.G'$ via Remark 3.6 and $(g, h, h^{-1}g^{-1}c) = (g, h; c)$, respectively, with the following multiplications

$$\begin{aligned} K(G,3): &\quad (a, b; c)(g, h; d) = (ag, bh; [a^{-1}, g^{-1}][b^{-1}a^{-1}, h^{-1}]^{g^{-1}} c^{(gh)^{-1}} d) \text{ and} \\ \tau(G)/\ker\kappa: &\quad (a, b; c)(g, h; d) = (ag, bh; [b, g]^h c^{gh} d). \end{aligned}$$

Every element in $T(G)$ can be written as (a, b, abc) for unique $a, b \in G$ and $c \in G'$. This allows us to identify $T(G) = G^2.G'$ via $(a, b, abc) = (a, b; c)$, and a short calculation confirms that the induced multiplication in $T(G) = G^2.G'$ is the same as for $\tau(G)/\ker \kappa = G^2.G'$, so $T(G) \cong \tau(G)/\ker \kappa$. If α is an AI-automorphism of G, then $\beta = 1 \times 1 \times \alpha$ is an automorphism of G^3 that interchanges $K(G, 3)$ and $T(G)$, hence $T(G) \cong K(G, 3)$. □

Identifying $T(G) = \tau(G)/\ker \kappa$, the isomorphism $\beta\colon T(G) \to K(G, 3)$ in the proof of Lemma 4.5 coincides with the isomorphism $\tau(G)/\ker G \cong K(G, 3)$ induced by Φ_α in Theorem 4.3.

5. Some isomorphisms

Our computations in Section 7 suggest that $\tau(G) \cong \tilde{K}(G, 3)$ only if G admits an AI-automorphism, cf. Corollary 4.4. As mentioned above, the lack of AI-automorphisms may prevent isomorphisms, but one may ask whether an AI-automorphism implies $\tau(G) \cong \tilde{K}(G, 3)$. In general, the answer is no, as illustrated by Proposition 5.12b) and Examples 5.11 and 7.1. However, there is strong evidence that $\tau(G)$ is closely related to $\tilde{K}(G, 3)$ when AI-automorphisms exists; the next theorem is a useful tool for establishing various isomorphisms.

Theorem 5.1. *Suppose G has an AI-automorphism that lifts to an AI-automorphism of a Schur cover inverting the Schur multiplier.*

a) *We have $\tilde{K}(G,3) \cong \tau(G)$.*
b) *We have $\tilde{K}(G,3) \cong T(H)/T(M)$.*

Proof. a) Let H be a Schur cover with $H/M = G$ and let $\alpha \in \mathrm{Aut}(H)$ be the induced AI-automorphism with $\alpha(m) = m^{-1}$ for all $m \in M$. Corollary 4.4 shows that $\Phi_\alpha\colon \tau(H) \to K(H,3)$ is an epimorphism with kernel $H_2(H,\mathbb{Z})$. It is shown in [Liedtke 2008, Theorem 3.2] that $\tilde{K}(G,n)$ is isomorphic to $K(H,n)/K(M,n)$, so we obtain an epimorphism $\tau(H) \to \tilde{K}(G,3)$. By Lemma 2.1, the projection $H \to G$ yields a surjection $\gamma\colon \tau(H) \to \tau(G)$ with kernel $(\langle M, M^*\rangle [M,H^*][H,M^*])_{\tau(H)}$. We can construct an induced epimorphism $\tau(G) \to \tilde{K}(G,3)$ if $\Phi_\alpha(\ker\gamma) \leqslant K(M,3)$. If $m \in M$, then $\Phi_\alpha(m) = (m,1,\alpha(m))$, which lies in $K(M,3)$ since $\alpha(m) = m^{-1}$; similarly for $m^* \in M^*$. If $[m,h^*]$ is a generator of $[M,H^*]$, then this is mapped under Φ_α to $(1,1,\alpha([m,h])) = (1,1,1)$ since $M \leqslant Z(H)$; similarly for elements in $[H,M^*]$. Thus, $\Phi_\alpha(\ker\gamma) \leqslant K(M,3)$ and the claim follows.

b) We have $T(M) = \{(a,b,ab) : a,b \in M\}$ and $K(M,3) = \{(a,b,a^{-1}b^{-1}) : a,b \in M\}$. The isomorphism $T(H) \cong K(H,3)$ of Lemma 4.5 maps $T(M)$ onto $K(M,3)$; recall that α inverts M. This implies that $\tilde{K}(G,3) \cong K(H,3)/K(M,3) \cong T(H)/T(M)$. □

Remark 5.2. a) If G has an abelian Schur cover, say $H/M = G$, then $M \leqslant H'$ implies that $M = 1$, so $G = H$ is cyclic, the assumptions of Theorem 5.1 are satisfied, and $\tau(G) \cong \tilde{K}(G,3) \cong K(G,3)$.

b) If a Schur cover H of G admits an AI-automorphism α that leaves M invariant, then α induces an AI-automorphism of $G \cong H/M$ since $H/H' \cong G/G'$.

c) Based on Theorem 5.1 and example computations, we conjecture that $\tau(G) \cong \tilde{K}(G,3)$ only if G admits an AI-automorphism. The results that follow and Example 7.1 below support this conjecture. A stronger conjecture would be that $\tau(G) \cong \tilde{K}(G,3)$ if and only if G admits an AI-automorphism that lifts to an AI-automorphism of a Schur cover inverting the Schur multiplier. However, this is not true as can be shown by a direct computation with GAP: the group $G = C_4 \times C_4$ has Schur multiplier $M \cong C_4$, has an AI-automorphism, and satisfies $\tau(G) \cong \tilde{K}(G,3)$; up to isomorphism G has three Schur covers H_1, H_2, and H_3, with GAP SmallGroup id [64,18], [64,19], and [64,28], respectively. Each H_i has a unique $M_i \leqslant H_i' \cap Z(H_i)$ with $M_i \cong M$ and $H_i/M_i \cong G$. Only H_1 and H_2 have AI-automorphisms, but all of those act trivially on M. A similar statement holds for the non-abelian $C_4 \times (C_4 \ltimes C_3)$ with GAP id [48,11]. This illustrates several things: First, whether or whether not an AI-automorphism of G lifts to an AI-automorphism of a Schur cover depends on the isomorphism type of the Schur cover. Second, we can have $\tau(G) \cong \tilde{K}(G,3)$ even though there is no lift of an AI-automorphism of G that inverts the Schur multiplier.

Corollary 5.3. *If G is a group with $\exp(G/G'), \exp(M(G)) \in \{1,2\}$, then $\tau(G) \cong \tilde{K}(G,3)$.*

Proof. Let H be a Schur cover of G with $H/M \cong G$. Since $H/H' \cong G/G'$ and $\exp(M) \mid 2$, the identity automorphism is an AI-automorphism inverting M. Now Theorem 5.1 proves the claim. □

The next result considers the finite groups all whose Sylow subgroups are cyclic, see [Robinson 1982, 10.1.10]. Note that every group of square-free order has this property.

Proposition 5.4. *Let G be a group all whose Sylow subgroups are cyclic, that is,*

$$G = \langle a,b \mid b^n, a^m, a^b = a^r\rangle \cong C_n \ltimes C_m$$

where $|G| = mn$ with m odd, and $0 \leqslant r < m$ with $r^n \equiv 1 \bmod m$ and $\gcd(m, n(r-1)) = 1$. Then G has trivial Schur multiplier, hence $\tilde{K}(G,3) = K(G,3)$, and the following hold.

a) *The group G has AI-automorphisms if and only if $r^2 \equiv 1 \bmod m$.*
b) *If G is square-free, then G has AI-automorphisms if and only if G has a cyclic $2'$-Hall subgroup.*
c) *The group G satisfies $\tilde{K}(G,3) \cong \tau(G)$ if and only if G has AI-automorphisms.*

Proof. It follows from Hölder's classification [Robinson 1982, 10.1.10] that the finite groups all whose Sylow subgroups are cyclic are exactly the groups having a presentation as in the proposition. It follows from [Karpilovsky 1987, Corollary 2.1.3] that G has trivial Schur multiplier, hence $\tilde{K}(G,3) = K(G,3)$ by definition. If G is abelian, then G is cyclic and Theorem 5.1 proves the claim where the AI-automorphism is inversion. The condition $\gcd(m, n(r-1)) = 1$ guarantees that $r \neq 1$, hence G is abelian if and only if $r = 0$ and $m = 1$; note that in this case $r^2 \equiv 1 \bmod m$ holds trivially. Thus, in the following we assume that G is non-abelian, that is, $r > 1$.

a) Note that $[a,b] = a^{r-1}$, so $G' = \langle a \rangle$. If G has an AI-automorphism, then there exist u, v with $\gcd(u,m) = 1$ such that $b^{-1}a^v$ and a^u satisfy the relations of b and a in G. The conjugacy relation forces $a^b = a^{b^{-1}}$, that is, $r^2 \equiv 1 \bmod m$. Conversely, if $r^2 \equiv 1 \bmod m$, then $(b,a) \mapsto (b^{-1}, a)$ describes an AI-automorphism of G.

b) If G is square-free with cyclic $2'$-Hall subgroup $V \cong C_{nm/2}$, then there is a subgroup $U \cong C_2$ with $G = U \ltimes V \cong C_2 \ltimes C_{mn/2}$, see [Robinson 1982, Ex. 1.3(13) and (9.1.2)]. In particular, V is the unique Hall $2'$-subgroup, which shows that $V = \langle b^2, a \rangle$ and we can choose $U = \langle b^{n/2} \rangle$. Thus, by renaming the generators, we can assume that $G = \langle a, b \mid a^m, b^2, a^b = a^r \rangle$ where $r^2 \equiv 1 \bmod m$, m is odd, $0 \leqslant r < m$, and $\gcd(m, r-1) = 1$. Now by part a), the identity defines an AI-automorphism of G. Conversely, if G is square-free with AI-automorphisms, then a) implies that $G \cong \langle b^{n/2} \rangle \ltimes \langle b^2, a \rangle \cong C_2 \ltimes C_{mn/2}$, so G has a cyclic Hall $2'$-subgroup.

c) If G has an AI-automorphism, then Theorem 5.1 proves that $\tau(G) \cong \tilde{K}(G,3) \cong K(G,3)$; recall that $M(G) = 1$. Conversely, suppose that $\tau(G) \cong \tilde{K}(G,3) = K(G,3)$; abbreviate $T = \tau(G)$ and $K = K(G,3)$. If we interpret T via Proposition 3.3, we get generators $y_1 = (b,1,1)$, $x_1 = (a,1,1)$, $y_2 = (1,b,1)$, $x_2 = (1,a,1)$, and $x_3 = (1,1,a)$, and it follows that $T' = \langle x_1, x_2, x_3 \rangle \cong C_m^3$ and $T/T' = \langle y_1T', y_2T' \rangle$. The elements y_iT' act on T' from the right via matrices

$$m_1 = \begin{pmatrix} r & 0 & 0 \\ 0 & 1 & r-1 \\ 0 & 0 & r \end{pmatrix} \quad \text{and} \quad m_2 = \begin{pmatrix} 1 & 0 & r-1 \\ 0 & r & 0 \\ 0 & 0 & r \end{pmatrix},$$

both given with respect to x_1, x_2, x_3. Similarly, K is generated by $\tilde{y}_1 = (b, 1, b^{-1})$, $\tilde{x}_1 = (a,1,1)$, $\tilde{y}_2 = (1, b, b^{-1})$, $\tilde{x}_2 = (1,a,1)$, and $\tilde{x}_3 = (1,1,a)$, and it follows that $K' = \langle \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \rangle \cong C_m^3$, and $K/K' = \langle \tilde{y}_1K', \tilde{y}_2K' \rangle$. Here the elements $\tilde{y}_iK'$ act on K' from the right via the matrices

$$n_1 = \begin{pmatrix} r & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s \end{pmatrix} \quad \text{and} \quad n_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & s \end{pmatrix},$$

where s is the multiplicative inverse of r modulo m. Now consider the subgroups $A = \langle m_1, m_2 \rangle$ and $B = \langle n_1, n_2 \rangle$ of $\mathrm{GL}_3(m)$. As T and K are isomorphic, it follows that A and B are conjugate in $\mathrm{GL}_3(m)$. Since B is contained in $\mathrm{SL}_3(m)$, the same holds for A. This forces $r^2 \equiv 1 \bmod m$, and now part a) shows that G admits an AI-automorphism. □

Proposition 5.5. *Let G be an extra-special p-group with p odd.*

a) *Let $\exp(G) = p$. If $|G| = p^3$, then $\tau(G) \cong \tilde{K}(G,3)$; if $|G| = p^{2n+1}$ with $n \geqslant 2$, then there exist Schur covers of G that admit AI-automorphisms, but none of these inverts the Schur multiplier.*
b) *If $\exp(G) = p^2$, then G does not have an AI-automorphism.*
c) *If $|G| = p^3$ and $\exp(G) = p^2$, then $\tau(G) \ncong \tilde{K}(G,3)$.*

Proof. a) Let G be an extra-special p-group of exponent p and order p^{2n+1}. It follows from [Huppert 1967, Satz III.13.7] that G is a central product of n extra-special groups of size p^3 and exponent p, that is, we can assume that $G = \mathrm{pc}\langle g_1, \ldots, g_{2n}, c \mid \forall i, j : [g_{2i}, g_{2i-1}] = c^{-1},\ g_j^p = c^p = 1\rangle$. First suppose that $n = 1$. By [Karpilovsky 1987, Theorem 3.3.6], the Schur multiplier is isomorphic to C_p^2, and it is straightforward to verify that the group

$$H = \mathrm{pc}\langle g_1, g_2, c, h_1, h_2 \mid g_1^p, g_2^p, c^p, h_1^p, h_2^p, [g_2, g_1] = c^{-1}, [c, g_1] = h_1, [c, g_2] = h_2\rangle,$$

is a Schur cover of G with $H/M = G$ for $M = \langle h_1, h_2\rangle \cong C_p^2$. The elements $g_1^{-1}c$, $g_2^{-1}c^{-1}$, c, h_1^{-1}, h_2^{-1} satisfy the relations of H, so von Dyck's Theorem [Robinson 1982, 2.2.1] shows that $(g_1, g_2, c, h_1, h_2) \mapsto (g_1^{-1}c, g_2^{-1}c^{-1}, c, h_1^{-1}, h_2^{-1})$ extends to an automorphism α of H. This is an AI-automorphism of H that inverts elements of M, so Theorem 5.1 proves the claim for $n = 1$.

Now let $n > 1$. Beyl and Tappe (1982) [Karpilovsky 1987, Theorem 3.3.6] proved that $M = M(G)$ is elementary abelian of rank $2n^2 - n - 1$ and that every Schur cover H of G with $H/M = G$ is unicentral, that is, $Z(H)$ is the full preimage of $Z(G)$ under the projection $H \to G$; in particular, we have $Z(G) = G' = H'/M$ and $H' = Z(H)$. Thus, if $g, h \in H$, then $\alpha([g, h]) = [\alpha(g), \alpha(h)] = [g^{-1}, h^{-1}] = [g, h]$, so α fixes H' (and so $M \leqslant H'$) elementwise; in particular, α does not invert M.

An explicit Schur cover H of G can be defined by abstract generators $g_1, \ldots, g_{2n}, c$ and $h_{i,j}$ for $1 \leqslant i < j \leqslant n$ except $(i, j) = (1, 2)$, all of order p, with each $h_{i,j}$ and c central, and the following nontrivial commutator relations: each commutator relation $[g_j, g_i] = w$ in G with $i < j$ (except for $[g_2, g_1]$) becomes a relation $[g_i, g_j] = wh_{i,j}$ in H. Let N be the subgroup generated by all $h_{i,j}$; it follows from the construction that $N \leqslant Z(H) \cap H'$, that $Z(H) = \langle c, N\rangle$, and that $H/N \cong G$. Standard *consistency checks* (see [HEO, Section 8.7.2]) can be used to show that this presentation is *consistent*: since every element has order p, the only tests that have to be carried out are for the equations $(g_i g_j)g_k = g_i(g_j g_k)$ with $k < j < i$, but all those lead to the conditions $h_{j,i}h_{k,i}h_{k,j} = h_{k,i}h_{j,i}h_{k,j}$ which are trivially satisfied. Consistency of the presentation implies $|H| = p^{2n+1+2n^2-n-1}$, so $H/N \cong G$ proves that $N \cong C_p^{2n^2-n-1}$ is isomorphic to the Schur multiplier. This shows that H is indeed a Schur cover of G. In particular, AI-automorphisms exist: take the isomorphism that is defined by mapping each generator g_i to g_i^{-1}.

b) Let G be extra-special of order p^{1+2n} with $Z(G) = \langle c\rangle = G'$. It follows from [Huppert 1967, Satz III.13.7] that G is a central product of n extra-special groups of size p^3, at least one of them of exponent p^2. Thus, there are $g, h \in G$ such that $\langle g, h, c\rangle$ is extraspecial of order p^3 and exponent p^2; we can assume that $g^p = c$, $h^p = c^p = 1$, and $[h, g] = c^{-1}$. If $\alpha \in \mathrm{Aut}(G)$ is an AI-automorphism, then $\alpha(c^{-1}) = [\alpha(h), \alpha(g)] = [h^{-1}, g^{-1}] = [h, g]^{(-1)^2} = c^{-1}$, so $\alpha(c) = c$. Now if $\alpha(g) = g^{-1}d$ with $d \in Z(G)$, then $c = \alpha(g)^p = g^{-p}d^p = g^{-p} = c^{-1}$ forces $|c| = 2$, a contradiction.

c) We consider $G = \langle g, h, c \mid g^p = c, h^p, c^p, [h, g] = c^{-1}, [c, g], [c, h]\rangle$ with $M(G) = 1$. Recall that $\tilde{K}(G, 3) = K(G, 3)$ and $\tau(G) \cong T(G)$, see Lemma 4.5. We show that $T = T(G)$ and $K = K(G, 3)$ are not isomorphic, which proves the claim. For $i \in \{1, 2, 3\}$ and $x \in G$ let x_i be the element x in the i-th copy of G^3. Note that K is generated by $\{x_1x_3^{-1}, x_2x_3^{-1}, c_1, c_2, c_3 : x \in G\}$, whereas T is generated by $\{x_1x_3, x_2x_3, c_1, c_2, c_3 : x \in G\}$. One can show that $Z(K) = \langle c_1, c_2, c_3\rangle = Z(T)$ and $\mho(K) = \langle c_1c_3^{-1}, c_2c_3^{-1}\rangle$ and $\mho(T) = \langle c_1c_3, c_2c_3\rangle$. Moreover, $\Omega(T) = \langle h_1h_3, h_2h_3, c_1, c_2, c_3\rangle$ and $\Omega(K) = \langle h_1h_3^{-1}, h_2h_3^{-1}, c_1, c_2, c_3\rangle$. Recall that $\mho$ and Ω denote the subgroups generated by p-th powers and elements of order p, respectively. Let $\mathcal{B} = \{c_1c_3^{-1}, c_2c_3^{-1}, c_3\}$ be a basis of the $\mathbb{Z}_p$-space $Z(K)$. The commutator map $K/\Omega(K) \times \Omega(K)/Z(K) \to Z(K)$ is induced by $[g_1g_3^{-1}, h_1h_3^{-1}] = c_1c_3$, $[g_1g_3^{-1}, h_2h_3^{-1}] = c_3$, $[g_2g_3^{-1}, h_1h_3^{-1}] = c_3$, and $[g_2g_3^{-1}, h_2h_3^{-1}] = c_2c_3$. Moreover, the power map $K/\Omega(K) \to \mho(K) \leqslant Z(K)$ satisfies $(g_1g_3^{-1})^p = c_1c_3^{-1}$ and $(g_2g_3^{-1})^p = c_2c_3^{-1}$. Note that with respect to $\mathcal{B}$, the element c_1c_3 is represented as $(1, 0, 2)$, etc. All together, these commutator and power maps are encoded by the $\mathbb{Z}_p$-matrix $\mathcal{M}(K)$ below; analogously, we obtain the matrix $\mathcal{M}(T)$

with respect to the basis $\mathcal{B}' = \{c_1c_3, c_2c_3, c_3\}$ of the $\mathbb{Z}_p$-space $Z(T)$:

$$\mathcal{M}(K) = \left(\begin{array}{ccc|ccc} 1&0&2&1&0&0\\ 0&0&1&1&0&0\\ 0&0&1&0&1&0\\ 0&1&2&0&1&0 \end{array}\right) \qquad \mathcal{M}(T) = \left(\begin{array}{ccc|ccc} 1&0&0&1&0&0\\ 0&0&1&1&0&0\\ 0&0&1&0&1&0\\ 0&1&0&0&1&0 \end{array}\right).$$

If $K \cong T$, then there exist $A, B \in \mathrm{GL}_2(p)$ and $C \in \mathrm{GL}_3(p)$ with $(A \otimes B) \cdot \mathcal{M}(T) = \mathcal{M}(K) \cdot (I_2 \otimes C)$; since $\mho(T)$ and $\mho(K)$ are characteristic subgroups, C has entry 0 in position $(1,3)$ and $(2,3)$. A straightforward but technical calculation shows that such A, B, C cannot exist, thus, $T \ncong K$. $\square$

Remark 5.6. In general, deciding (non)-isomorphism for $\tau(G)$ and $\tilde{K}(G,3)$ seems to be an intricate matter since already for extra-special G of order 3^5, both $\tau(G)$ and $\tilde{K}(G,3)$ are extensions of C_3^8 by C_3^8. As explained in Section 7, even advanced computational group theory methods fail for such isomorphism tests.

Next, for $n \geqslant 1$ we consider the generalised quaternion group Q_{4n} and dihedral group D_{2n} of order $4n$ and $2n$, respectively, which are defined as

$$Q_{4n} = \langle a, b \mid a^{2n}, b^2 = a^n, a^b = a^{-1} \rangle \quad \text{and} \quad D_{2n} = \langle a, b \mid a^n, b^2, a^b = a^{-1} \rangle.$$

Proposition 5.7. *We have $\tau(Q_{4n}) \cong \tilde{K}(Q_{4n}, 3)$ and $\tau(D_{2n}) \cong \tilde{K}(D_{2n}, 3)$.*

Proof. For $Q_4 = C_4$ and $D_2 = C_2$ the claim is obvious, so let $n \geqslant 2$. It follows from [Karpilovsky 1987, Example 2.4.8] that $M(Q_{4n}) = 1$. Note that $\{a^{-1}, b^{-1}\}$ also satisfies the relations of Q_{4n}, so $(a, b) \mapsto (a^{-1}, b^{-1})$ extends to a GI-automorphism of Q_{4n} by von Dyck's Theorem. Now $\tau(Q_{4n}) \cong \tilde{K}(Q_{4n}, 3)$ by Theorem 5.1. Let H be a Schur cover of D_{2n} with $H/M = D_{2n}$. By [Karpilovsky 1987, Proposition 2.11.4], we have $M = 1$ and $H = D_{2n}$ if n is odd, and $M = C_2$ and $H = Q_{4n}$ otherwise. As seen above and in Example 4.1, the group H admits an AI-automorphism which fixed M elementwise. Again, the claim follows with Theorem 5.1. $\square$

Proposition 5.8. *We have $\tau(\mathrm{Sym}_n) \cong \tilde{K}(\mathrm{Sym}_n, 3)$ and $\tau(\mathrm{Alt}_n) \cong \tilde{K}(\mathrm{Alt}_n, 3)$.*

Proof. For $n \leqslant 3$ the claim can be verified directly, so let $n \geqslant 4$ in the following. Schur (1911) proved that the Schur multiplier of Sym_n is cyclic of order 2 for $n \geqslant 4$, and trivial otherwise, see [Karpilovsky 1987, Theorem 2.12.3]. Now Corollary 5.3 proves the claim for Sym_n. Similarly, if $n \notin \{4, 6, 7\}$, then the claim for Alt_n follows from [Karpilovsky 1987, Theorem 2.12.5] and Corollary 5.3 for Alt_n. The case $n = 4$ can be checked directly, and if $n \in \{6, 7\}$, then the outer automorphism extending Alt_n to Sym_n inverts the Schur multiplier: this also follows directly from the presentations given in [Karpilovsky 1987, Theorem 2.12.5]. $\square$

The next result shows that Theorem 5.1 cannot be applied to abelian groups G in general. Recall that if M is a trivial G-module of exponent 2, then a 2-coboundary $\delta \in B^2(G, M)$ is a function $G \times G \to M$ defined by a map $\kappa \colon G \to M$ with $\kappa(1) = 1$ such that $\delta(g, h) = \kappa(gh)\kappa(g)\kappa(h)$ for all $g, h \in G$. In the following, for an abelian group G, we write $G = G_2 \times G_{2'}$ where G_2 is the Sylow 2-subgroup of G.

Proposition 5.9. *Let G be an abelian group with Schur cover H, say $H/M = G$. Then H admits an AI-automorphism whose restriction to M is inversion if and only if $G_{2'}$ is cyclic, M has exponent dividing 2, and the map $G_2 \times G_2 \to G_2 \wedge G_2$ defined by $(g, h) \mapsto g \wedge h$ is a 2-coboundary; in particular, any such AI-automorphism has order dividing 2.*

Proof. First suppose that H admits an AI-automorphism, say α, whose restriction to M is inversion. Since G is abelian, $H' \leqslant M$, and now $M \leqslant H' \cap Z(H)$ implies $M = H' \leqslant Z(H)$. We decompose $G = G_2 \times G_{2'}$ as above. It follows from [Karpilovsky 1987, Lemma 2.9.1] that the Schur cover H of G is the direct product of Schur covers of G_2 and $G_{2'}$, respectively. Thus, we first assume that

$G = G_{2'}$ and show that G is cyclic. By assumption, for every $h \in H$ we can write $\alpha(h) = h^{-1}c_h$ for some $c_h \in H'$. Now

$$h^{-1}g^{-1}c_{gh} = \alpha(gh) = \alpha(g)\alpha(h) = g^{-1}c_g h^{-1}c_h = h^{-1}g^{-1}[g^{-1}, h^{-1}]c_g c_h$$

implies that $c_{gh} = [g^{-1}, h^{-1}]c_g c_h$ for all $g, h \in H$. Note that $[g, h] = [g^{-1}, h^{-1}]^{gh} = [g^{-1}, h^{-1}]$ since H' is central, so $c_{gh} = c_g c_h[g, h]$. Moreover, $1 = c_1 = c_{gg^{-1}}$ yields $c_{g^{-1}} = (c_g)^{-1}$. This can be used to show that $\alpha^{2n+1}(g) = g^{-1}c_g^{2n+1}$ and $\alpha^{2n}(g) = gc_g^{-2n}$ for all $g \in H$ and $n \geqslant 1$. Since $G = G_{2'}$ has odd order, $m = |M| = |G \wedge G|$ is odd, and so $\alpha^m(g) = g^{-1}$ describes an isomorphism of H. This is only possible if H is abelian, that is, if G is cyclic, see Section 2.3. Back to the general case $G = G_2 \times G_{2'}$, the same argument shows that $G_{2'}$ must be cyclic, hence it remains to consider the case $G = G_2$ in the following. Since

$$[h, g] = \alpha([g, h]) = [\alpha(g), \alpha(h)] = [g^{-1}c_g, h^{-1}c_h] = [g^{-1}, h^{-1}] = [g, h]^{h^{-1}g^{-1}} = [g, h]$$

for all $g, h \in H$, we must have that $H' = M$ has exponent 2. Thus, α is the identity on M, and so $\alpha^2(h) = \alpha(h^{-1}c_h) = hc_{h^{-1}}c_h = h$ for all $h \in H$ proves that α has order 2. Note also that $[g, h] = c_{gh}c_g c_h$. The map $\gamma\colon H \times H \to H'$, $(g, h) \mapsto [g, h]$, is a 2-cocycle in $Z^2(H, H')$. Since H' is central, γ induces a 2-cocycle $\delta \in Z^2(G, H')$. Since G is abelian, an isomorphism $G \wedge G \to H'$ is given by $g \wedge h \to [g', h']$, where $g', h' \in H$ are lifts of $g, h \in G$. This shows that the induced 2-cocycle δ lies in $Z^2(G, G \wedge G)$ and $\delta(g, h) = g \wedge h$ for all $g, h \in G$. Recall that if $h \in H$ and $z \in H'$, then $\alpha(h) = h^{-1}c_h$ and $(hz)^{-1}c_{hz} = \alpha(hz) = \alpha(h)\alpha(z) = h^{-1}c_h z$, which shows that $c_{hz} = c_h$. Thus for $g \in G$ we can define $\kappa(g) = c_{g'}$ where $g' \in H$ is a lift of g. This shows that $\delta(g, h) = \kappa(gh)\kappa(g)\kappa(h)$ with $\kappa(1) = 1$, that is, δ is a 2-coboundary in $B^2(G, G \wedge G)$.

Conversely, let $G = G_2 \times G_{2'}$ be abelian with cyclic $G_{2'}$ and $G \wedge G$ of exponent 2 such that $\delta(g, h) = g \wedge h$ defines a 2-coboundary in $B^2(G_2, G_2 \wedge G_2)$; by what is said above, it is sufficient to consider $G = G_2$. Since δ defines a 2-coboundary, we have $g \wedge h = \delta(g, h) = \kappa(gh)\kappa(g)\kappa(h)$ for some map $\kappa\colon G \to G \wedge G$ with $\kappa(1) = 1$. Let H be a Schur cover of G with natural projection $\pi\colon H \to G$, such that $M = \ker \pi$ satisfies $M = H' \leqslant Z(H)$. Note that under the isomorphism $H' \to G \wedge G$, $[h, k] \mapsto \pi(h) \wedge \pi(k)$ we have $[h, k] = \delta(\pi(h), \pi(k)) = \kappa(\pi(hk))\kappa(\pi(h))\kappa(\pi(k))$. Now define $\alpha \in \mathrm{Aut}(H)$ by $\alpha(h) = h^{-1}c_h$ where $c_h = \kappa(\pi(h))$; note that

$$\alpha(hk) = k^{-1}h^{-1}c_{hk} = h^{-1}k^{-1}[k^{-1}, h^{-1}]c_{hk} = h^{-1}k^{-1}[k, h]c_{hk} = h^{-1}c_h k^{-1}c_k = \alpha(h)\alpha(k),$$

so α is indeed a homomorphism. Clearly, α acts as inversion (that is, as identity) on M, and as inversion on H/M. This proves the claim. □

Proposition 5.10. *If G is an abelian 2-group such that $\exp(G \wedge G)$ divides 2, then $G \cong C_2^n \times C_{2^m}$ for some n, m and $\tau(G) \cong \tilde{K}(G, 3)$.*

Proof. It is straightforward to see that an abelian G as in the statement must be isomorphic to $C_2^n \times C_{2^m}$ for some n and m. If $m \in \{0, 1\}$, then G is elementary abelian and Corollary 5.3 proves the claim. Now let $m \geqslant 2$ and let $\{x_1, \ldots, x_n, y\} \subseteq G$ be a generating set with y of order 2^m and each x_i of order 2. A Schur cover of G is $H = \langle x_1', \ldots, x_n', y', M\rangle$ where

$$M = \langle z_{i,j}, z_k : 1 \leqslant i < j \leqslant n, 1 \leqslant k \leqslant n\rangle \cong M(G) \cong G \wedge G$$

is 2-elementary abelian and central in H, subject to the relations $(x_i')^2 = (y')^{2^m} = 1$, $[x_i', x_j'] = z_{i,j}$ and $[x_k', y'] = z_k$. Now inversion of G lifts to an AI-automorphism of H that fixes M element-wise, and therefore $\tau(g) \cong \tilde{K}(G, 3)$ by Theorem 5.1. □

Example 5.11. For $A = C_4^3$, we determine $\tau(A) \not\cong \tilde{K}(A, 3)$ by computing with GAP that $\mathrm{Aut}(\tau(A))$ and $\mathrm{Aut}(\tilde{K}(G, 3))$ have orders 94575592174780416 and 283726776524341248, respectively. Since $A \wedge A$ has exponent 4, this is also an example showing that the assumptions in Proposition 5.10

cannot be relaxed. Similarly, it shows that Proposition 5.12 cannot be extended to higher rank. A comparison of the automorphism group orders also shows that $\tau(B) \not\cong \tilde{K}(B,3)$ for $B = C_5^3$.

Proposition 5.12. *Let G be an abelian group.*

a) *Suppose all Sylow p-subgroups of G have rank at most 2. Then $\tau(G) \cong \tilde{K}(G,3)$ if and only if the Sylow 3-subgroup of G is cyclic.*
b) *If the Sylow 3-subgroup of G has rank at least 2, then $\tau(G) \not\cong \tilde{K}(G,3)$.*

Proof. Let $G = \prod_p G_p$ be the decomposition of G into its Sylow subgroups. By [Liedtke 2008, Proposition 4.1] and [Rocco 1991, Corollary 3.7], we can also decompose $\tau(G) = \prod_p \tau(G_p)$ and $\tilde{K}(G,n) = \prod_p \tilde{K}(G_p,n)$, and every isomorphism $\tau(G) \to \tilde{K}(G,n)$ induces an isomorphism from $\tau(G_p)$ to $\tilde{K}(G_p,n)$ for every p. (We note that [Rocco 1991, Corollary 3.7] considers $\nu(G)$, but it implies the fact needed for $\tau(G)$.) Thus it is sufficient to assume that G is an abelian p-group.

a) We have $G \cong C_m \times C_n$ with $m = p^a$ and $n = p^b$ for $a \geqslant b$. Recall from Section 2.3 that $M(G) \cong G \wedge G \cong C_n$. Let g and h be generators of C_m and C_n, respectively. Considering the description of $\tau(G)$ as in Proposition 3.11, set $g_1 = (g,1;1)$, $h_1 = (h,1;1)$, $g_2 = (1,g;1)$, $h_2 = (1,h;1)$, and $k = (1,1;h\wedge g)$. These elements form a polycyclic generating sequence of $\tau(G)$, with corresponding polycyclic presentation

$$\tau(G) = \mathrm{pc}\langle g_1,h_1,g_2,h_2,k \mid g_1^m, g_2^m, h_1^n, h_2^n, k^n, g_2^{h_1} = g_2k^{-1}, h_2^{g_1} = h_2k\rangle.$$

Using the identification of $\tilde{K}(G,3) = G^2.(G\wedge G)$ as in Proposition 3.10, we obtain

$$\begin{aligned}\tilde{K}(G,3) \quad = \quad & \mathrm{pc}\langle g_1,h_1,g_2,h_2,k \mid g_1^m, g_2^m, h_1^n, h_2^n, k^n, h_1^{g_1} = h_1k^2, \\ & \qquad g_2^{h_1} = g_2k^{-1}, h_2^{g_1} = h_2k, h_2^{g_2} = h_2k^2\rangle.\end{aligned}$$

If $p \neq 3$, then, by von Dyck's Theorem, $(g_1,h_1,g_2,h_2,k) \mapsto (g_1g_2^2, h_1, g_2g_1^2, h_2, k)$ extends to an isomorphism $\tilde{K}(G,3) \to \tau(G)$. If $p = 3$ and G has rank 2, then $\tau(G) \not\cong \tilde{K}(G,3)$, see part b). If G is a cyclic 3-group, then $M(G) = 1$, hence $\tau(G) = \tilde{K}(G,3)$ by Theorem 5.1.

b) Let G be an abelian 3-group. As G is not cyclic, hence $Z^\wedge(G) \neq G$, it follows that there exists $u \in G \setminus Z^\wedge(G)$ with $u^3 \in Z^\wedge(G)$. Now Propositions 3.10 and 3.11 imply $\tau(G) \not\cong \tilde{K}(G,3)$. $\square$

Proposition 5.12a) together with Proposition 5.9 shows that there are infinitely many abelian groups G such that $\tau(G) \cong \tilde{K}(G,3)$, but no Schur cover of G has an AI-automorphism whose restriction to the Schur multiplier is inversion.

6. Bogomolov multiplier

Let G be a group with AI-automorphism α, and let $\Phi_\alpha\colon \tau(G) \to K(G,3)$ be the epimorphism in Section 4.2. Set

$$M^\flat(G) = \langle [x,y^*] : x,y \in G, [x,y] = 1\rangle_{\tau(G)}$$

and note that $M^\flat(G)$ is contained in the kernel of the commutator map $\kappa\colon [G,G^*]_{\tau(G)} \to G'$. Define

$$\tau^\flat(G) = \tau(G)/M^\flat(G).$$

If x and y commute in G, then $\Phi_\alpha([x,y^*]) = (x^{-1}x, y^{-1}y, \alpha([x,y])) = (1,1,1)$, therefore Φ_α induces an epimorphism $\Phi_\alpha^\flat\colon \tau^\flat(G) \to K(G,3)$. Theorem 4.3 implies that the kernel of this map is $(\ker\kappa)/M^\flat(G)$, which is isomorphic to the *Bogomolov multiplier* $B_0(G)$ of G, see [Moravec 2012].

Corollary 6.1. *The existence of an AI-automorphism of G yields a central extension*

$$1 \longrightarrow B_0(G) \longrightarrow \tau^\flat(G) \longrightarrow K(G,3) \longrightarrow 1.$$

Proposition 6.2. *Let H be a Schur cover of a group G with $H/M = G$. If α is an AI-automorphism of H, then $\tilde{K}(G,3) \cong \tau^\flat(H)/\operatorname{im}\iota$ for the monomorphism $\iota\colon M^2 \to \tau^\flat(H)$ given by*

$$(m_1, m_2) \mapsto m_1 m_2^* \prod\nolimits_{i=1}^{\ell} [\alpha^{-1}(k_i), (\alpha^{-1}(h_i))^*]_{\tau^\flat(H)},$$

where the elements $h_i, k_i \in G$ are defined by $\alpha(m_1 m_2) = m_2^{-1} m_1^{-1} [h_\ell, k_\ell] \ldots [h_1, k_1]$.

Proof. Since M is abelian, $M^2 \cong K(M,3)$ with isomorphism $(m_1, m_2) \mapsto (m_1, m_2, m_1^{-1} m_2^{-1})$. Note that $K(M,3)$ is naturally embedded in $K(H,3)$. From [MM 1999, Proposition 6.12] we conclude that $B_0(H)$ is trivial, therefore $\Phi_\alpha^\flat\colon \tau^\flat(H) \to K(H,3)$ is an isomorphism by Corollary 6.1. Note that ι is the map that makes the following diagram commutative; in particular, ι is an injective homomomorphism, and the results follows from taking quotients in diagram:

$$\begin{array}{ccc} M^2 & \xrightarrow{\cong} & K(M,3) \\ \Big\downarrow{\scriptstyle\iota} & & \Big\downarrow \\ \tau^\flat(H) & \xrightarrow{\Phi_\alpha^\flat} & K(H,3). \end{array}$$

□

7. Computations

If G is a finite polycyclic group, then also $\tilde{K}(G,3)$ is polycyclic, see [Liedtke 2008, Proposition 1.5]. In this situation, the algorithms described in [EN 2008] can be used to compute $\tau(G)$; these algorithms are implemented in the software package Polycyclic, distributed with the computer algebra system GAP [GAP]. Our explicit formulas in Section 3 can be used to compute a polycyclic presentation for $\tilde{K}(G,3)$. We have done this to test whether $\tau(G)$ and $\tilde{K}(G,3)$ are isomorphic for certain examples of groups (abelian, Frobenius, extra-special, ...). Even though there exist powerful algorithms for working with polycyclic groups, approaching this isomorphism problem with conventional methods poses a serious computational challenge. This is due to the fact that if G is an abelian group of order p^n, then $\tilde{K}(G,3)$ and $\tau(G)$ are both large central extensions of $G \wedge G$ by G^2; they have class 2, order $p^{2n}|G \wedge G|$, and often seem indistinguishable. The latter is not a surprise, given the folklore conjecture that *most* p-groups have class 2: for example, note that among the 49499125314 groups of order at most 1024 (up to isomorphism), 99.976% of these are 2-groups and 98.595% are 2-groups of class 2, see [CDO 2008, Section 4]. A computational isomorphism test for these groups reduces to orbit calculations of huge matrix groups on very large vector spaces; often these computations turn out to be infeasible. For example, the powerful implementations of the p-group algorithms for automorphism groups and isomorphisms (provided by the GAP package Anupq) struggle to compute automorphisms and isomorphisms for $\tau(G)$ and $\tilde{K}(G,3)$ already for moderately sized p-groups such as $G = C_7^3$. Most of our computer experiments have therefore focused on groups of cubefree order, that is, groups whose order is not divisible by any prime power p^3.

Example 7.1. In Table 1 we report on some example computations: there are 237 cubefree groups of order at most 100. Of these, 113 groups are abelian, 123 groups are non-abelian solvable, and 1 group is simple. Every abelian G admits AI-automorphisms and, being cubefree, $\tau(G) \cong \tilde{K}(G,3)$ if and only if G has a cyclic Sylow 3-subgroup, see Proposition 5.12. Our computations show that, with two exceptions, $\tau(G) \cong \tilde{K}(G,3)$ if and only if G has AI-automorphisms. The exceptions are

$A = C_3 \times \mathrm{Alt}_4$ and $B = C_3^2 \times D_{10}$; we have $Z(\tilde{K}(A,3)) = C_6 \times C_3$ and $Z(\tau(A)) = C_6$, and $Z(\tilde{K}(B,3)) = C_3^3$ and $Z(\tau(B)) = C_3$.

TABLE 1. Statistics for solvable non-abelian groups of cubefree order at most 100

$\tau \cong \tilde{K}$	has AI	# groups
yes	yes	96
yes	no	0
no	yes	2
no	no	25

Example 7.2. Running over GAP's group database, there are 6505 non-abelian solvable groups of order < 256; of these groups, 6127 have AI-automorphisms. Note that every simple and every abelian group admits AI-automorphisms. This computation suggests that for many groups we can apply Corollary 4.4 to describe $\tau(G)$ as a central extension of $H_2(G,\mathbb{Z})$ by $K(G,3)$. Table 1 suggests that the existence of AI-automorphisms for G is strongly connected to the property $\tau(G) \cong \tilde{K}(G,3)$; cf. also Proposition 5.5b,c).

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