IDEMPOTENT-FIXING AUTOMORPHISMS OF COMPLETELY REGULAR SEMIGROUPS

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ABSTRACT. The purpose of this note is to prove that if a completely regular semigroup with finite $\mathscr{H}$-classes posesses an idempotent-fixing automorphism of prime order, then it is an over-nilpotent semigroup.

1. INTRODUCTION

One of the prominent results in the theory of finite groups is the following theorem proved by Thompson:

Theorem 1.1 ([7]). *Let G be a finite group and α a fixed-point-free automorphism of G of prime order p. Then G is nilpotent, and its nilpotency class can be bounded in terms of p.*

This result was generalized to finite inverse semigroups by Araújo and Kinyon:

Theorem 1.2 ([1]). *Let S be a finite inverse semigroup that has an automorphism of prime order whose set of fixed points is precisely $E(S)$. Then S is a nilpotent Clifford semigroup.*

A problem posed in [1] asks about possible extensions of the above result to other classes of regular semigroups. For a start, the notion of nilpotency would need to be suitably adapted. One of the possible ways originates from a paper by Neumann and Taylor [5] and goes as folows. Let F be a free semigroup of countable rank. For $x, y, z_0, z_1, \ldots$ in F define a sequence of words $q_n(x, y, z_0, \ldots, z_{n-1})$ by $q_0(x, y) = x$ and

$$q_{n+1}(x, y, z_0, \ldots, z_n) = q_n(x, y, z_0, \ldots, z_{n-1}) z_n q_n(y, x, z_0, \ldots, z_{n-1})$$

for $n \geq 0$. A semigroup S is said to be *nilpotent of class c* if it satisfies the identity $q_c(x, y, z_0, \ldots, z_{c-1}) = q_c(y, x, z_0, \ldots, z_{c-1})$ for all $x, y \in S$, $z_i \in S^1$, and c is the least positive integer with this property. Neumann and Taylor [5] showed that a group is nilpotent of class $\leq c$ in the classical sense if and only if it satisfies the above identity.

We show the following:

Theorem 1.3. *Let S be a completely regular semigroup with finite $\mathscr{H}$-classes. If S has an automorphism of prime order p whose set of fixed points is precisely $E(S)$, then there exists $c = c(p)$ such that the $\mathscr{H}$-classes of S are nilpotent of class $\leq c$, and thus S satisfies the identity*

$$q_c(exe, eye, ez_0e, \ldots, ez_{c-1}e) = q_c(eye, exe, ez_0e, \ldots, ez_{c-1}e),$$

where $e = (xyz_0 \ldots z_{c-1})^0$.

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If S is as in Theorem 1.3, then its $\mathscr{H}$-classes are nilpotent of class $\leq c$; we say that S is an *over-(nilpotent of class $\leq c$) semigroup*. However, S itself need not be nilpotent (in the sense of Neumann and Taylor). We will exhibit an example of a finite completely simple semigroup with an automorphism of prime order that fixes all idempotents, but is not nilpotent.

Araújo and Kinyon [1] also show that if S is a uniquely 2-divisible inverse semigroup with an automorphism α of order 2 whose set of fixed points is precisely $E(S)$, then $x\alpha = x^{-1}$ for all $x \in S$. They pose a conjecture [1, Conjecture 1] that a similar result holds true for completely regular semigroups. We confirm this by showing the following result:

Theorem 1.4. *Let S be a uniquely 2-divisible completely regular semigroup with an automorphism α of order 2 that fixes precisely $E(S)$. Then $x\alpha = x^{-1}$ for all $x \in S$.*

As noted by Araújo and Kinyon [1], the above conclusion does not imply that S is commutative, as opposed to the case of inverse semigroups. The proof of Theorem 1.4 implies that, under the above assumptions, S is over-abelian and thus satisfies the identity $(xyx)^0yx = xy(xyx)^0$, see [6, Proposition II.7.2].

2. Proofs

We briefly collect some facts on completely regular semigroups. A reference for this is for instance a book of Petrich and Reilly [6]. A semigroup S is said to be *completely regular* if every $\mathscr{H}$-class of S is a subgroup. Given $x \in S$, we denote by x^0 the identity element of the $\mathscr{H}$-class H_x, and x^{-1} stands for the inverse of x in H_x.

Proof of Theorem 1.3. Let α be an automorphism of S of prime order whose set of fixed points is precisely $E(S)$, and let $x \in S$. Then [6, Lemma II.2.4], together with the fact that α fixes the idempotents of S, implies $x^0 = (x^0)\alpha = (x\alpha)^0$. By [6, p.63], this implies that $x\alpha \in H_x$. Thus the restriction of α to H_x is a fixed-point-free automorphism of prime order p of the group H_x. By Thompson's theorem, H_x is a nilpotent group whose nilpotency class can be bounded uniformly in terms of p only. This, together with [6, Proposition II.7.1], concludes the proof. □

A special class of completely regular semigroups are completely simple semigroups. These can be described as Rees matrix semigroups $S = \mathcal{M}(I, G, \Lambda; P)$ over a group G with $\Lambda \times I$ sandwich matrix $P = (p_{\lambda i})$. Elements of this semigroup are triples (i, g, λ), where $i \in I$, $g \in G$, $\lambda \in \Lambda$, and the multiplication is given by $(i, g, \lambda)(j, h, \mu) = (i, gp_{\lambda j}h, \mu)$. Moreover, P may be taken to be normalized, i.e., there exist $1 \in I$ and $1 \in \Lambda$ such that $p_{1i} = p_{\lambda 1} = e$ for all $i \in I$, $\lambda \in \Lambda$; here e is the identity element of G. If $x = (i, g, \lambda)$, then $H_x = \{(i, h, \lambda) : h \in G\}$ is isomorphic to G. By [3, Proposition 4.1] therefore have:

Corollary 2.1. *Let S be a completely simple semigroup with a finite structure group. If S has an automorphism of prime order p that fixes $E(S)$, then there exists $c = c(p)$ such that S satisfies the identity*

$$q_c(a^0, a, x_0, \ldots, x_{c-1}) = q_c(a, a^0, x_0, \ldots, x_{c-1})$$

for all $a, x_0, \ldots, x_{c-1} \in S$.

Proposition 2.2. *Let $S = \mathcal{M}(I, G, \Lambda; P)$ with P normalized. An automorphism α of S fixes $E(S)$ if and only if α is of the form $(i, g, \lambda)\alpha = (i, g\omega, \lambda)$, where $\omega \in \operatorname{Aut} G$ and the set of fixed points of ω contains $\langle P \rangle$, the subgroup of G generated by all entries of P.*

Proof. Let $(\varphi, \omega, \psi) \in \mathcal{T}(I) \times \mathrm{End}(G) \times \mathcal{T}(\Lambda)$ be a triple satisfying

$$p_{\lambda i}\omega = p_{1\psi,1\varphi} p^{-1}_{\lambda\psi,1\varphi} p_{\lambda\psi,i\varphi} p^{-1}_{1\psi,i\varphi}.$$

Define a map $\theta = \theta(\varphi, \omega, \psi) : S \to S$ by the rule

$$(i, g, \lambda)\theta = (i\varphi, p^{-1}_{1\psi,i\varphi}(g\omega)p_{1\psi,1\varphi}p^{-1}_{\lambda\psi,1\varphi}, \lambda\psi).$$

Then θ is an endomorphism of S. Moreover, every endomorphism of S can be obtained uniquely in this way [6, Lemma III.3.11].

Let $\alpha = \alpha(\varphi, \omega, \psi) \in \mathrm{Aut}\, S$ fix $E(S)$. This means that

$$(i, p^{-1}_{\lambda i}, \lambda) = (i, p^{-1}_{\lambda i}, \lambda)\alpha = (i\varphi, p^{-1}_{1\psi,i\varphi}((p^{-1}_{\lambda i})\omega)p_{1\psi,1\varphi}p^{-1}_{\lambda\psi,1\varphi}, \lambda\psi)$$

for all $\lambda \in \Lambda$, $i \in I$. It follows that φ and ψ are identity mappings, $\omega \in \mathrm{Aut}\, G$, and $p_{\lambda i}\omega = p_{\lambda i}$, therefore $\langle P \rangle$ is fixed by ω. The converse is obvious. □

Example 2.3. Let G be the Heisenberg group of 3×3 unitriangular matrices over $\mathrm{GF}(p)$, where p is an odd prime. It is generated by the matrices

$$a = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

The group G is finite of order p^3, non-abelian and nilpotent. Denote $c = [a, b]$. There is an automorphism ω of G defined by

$$a \mapsto ab^{-1}, \qquad b \mapsto bc^{-1}.$$

It is easily verified that ω has order p and the set of fixed points of ω is $\langle c \rangle = \{1, c, c^{-1}\}$. Now let $\Lambda = I = \{1, 2, 3\}$ and

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & c & c \\ 1 & c & c^{-1} \end{pmatrix}.$$

Put $S = \mathcal{M}(I, G, \Lambda; P)$ and define $\alpha : S \to S$ by the rule $(i, g, \lambda)\alpha = (i, g\omega, \lambda)$. By Proposition 2.2, α is an automorphism of order p of S whose set of fixed points is precisely $I \times \langle c \rangle \times \Lambda$. The semigroup S is not nilpotent, as can be easily seen by plugging in, for example, $x = (1, a, 2)$ and $y = (2, b, 3)$ into $q_n(x, y, \underline{z})$ and $q_n(y, x, \underline{z})$ and comparing the first and last components.

We note here that nilpotent completely (0)-simple semigroups were considered by Jespers and Okninski [2]. It is for example easy to see that a rectangular band $S = I \times \Lambda = \mathcal{M}(I, \{1\}, \Lambda; 1)$ is not nilpotent if $|I| > 1$ or $|\Lambda| > 1$. Since non-trivial left and right zero semigroups are not nilpotent, and the property of being nilpotent is preserved under taking sub- semigroups and quotients, it follows that a completely regular semigroup is nilpotent (of class $\leq c$) if and only if it is a semilattice of nilpotent groups (each of which is of class $\leq c$).

Proof of Theorem 1.4. Given $x \in S$, denote by $\sqrt{x}$ the unique element $y \in S$ such that $y^2 = x$. By [1] we have that $\sqrt{x\alpha} = (\sqrt{x})\alpha$ and $\sqrt{x^{-1}} = (\sqrt{x})^{-1}$ for all $x \in S$. Given $x \in S$, we can show as above that $(H_x)\alpha \subseteq H_x$. Furthermore, we claim that the equality holds. namely, if $y \in H_x$ and $z = y\alpha^{-1}$, then $x^0 = y^0 = (z\alpha)^0 = (z^0)\alpha = z^0$, hence $z \in H_x$. Thus the restriction of α to H_x is a fixed-point-free group automorphism of order 2 of H_x. We now claim that H_x is uniquely 2-divisible. Choose $y \in H_x$, that is, $y^0 = x^0$. We need to show that $\sqrt{y} \in H_x$. Note that $(\sqrt{y})^0 = \sqrt{y} \cdot (\sqrt{y})^{-1} = \sqrt{y}\sqrt{y^{-1}}$. It is easy to see that if $\sqrt{a}$ and $\sqrt{b}$ commute, then $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$. Thus $(\sqrt{y})^0 = \sqrt{yy^{-1}} = \sqrt{y^0}$. As y^0 is an idempotent, it follows that $\sqrt{y^0} = y^0 = x^0$, hence $\sqrt{y} \in H_x$. Now we can apply a result of Neumann

[4] to conclude that $x\alpha = x^{-1}$. Since α induces an automorphism $a \mapsto a^{-1}$ of any $\mathscr{H}$-class of S, it follows that S is over-abelian. $\square$

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