GROUPS OF PRIME POWER ORDER AND THEIR NONABELIAN TENSOR SQUARES

PRIMOŽ MORAVEC

Abstract. We prove that the nonabelian tensor square of a powerful p-group is again a powerful p-group. Furthermore, If G is powerful, then the exponent of $G \otimes G$ divides the exponent of G. New bounds for the exponent, rank, and order of various homological functors of a given finite p-group are obtained. In particular, we improve the bound for the order of the Schur multiplier of a given finite p-group obtained by Lubotzky and Mann.

1. Introduction

intro

The concept of the nonabelian tensor product of groups was introduced by Brown and Loday in [1], following the ideas of Dennis [3]. This construction generalizes the notion of the usual 'abelian' tensor product of groups, and has its origins in the algebraic K-theory and homotopy theory. Group theoretical aspects of nonabelian tensor products have been studied extensively by several authors, starting with Brown, Johnson and Robertson [2]. We also mention here a survey paper by Kappe [9] for a rather thorough overview of known results and literature. Two important special cases of the above notion are the nonabelian tensor square $G \otimes G$ and exterior square $G \wedge G$ of a group G. Their interest lies in the fact that the Schur multiplier $H_2(G)$ of G is isomorphic to the kernel of the commutator map $G \wedge G \to G'$, whereas the kernel of the commutator map $G \otimes G \to G'$ is isomorphic to $\pi_3(SK(G,1))$, the third homotopy group of the suspension of an Eilenberg MacLane space $K(G,1)$ [1].

When G is a finite p-group, both $G \otimes G$ and $G \wedge G$ are finite p-groups as well [2]. The nonabelian tensor squares of finite p-groups have been studied, for instance, in [2, 4, 7, 12]. In this paper we consider the nonabelian tensor squares of powerful p-groups. Recall that a finite p-group G is said to be *powerful* if $p > 2$ and $G' \leq G^p$, or $p = 2$ and $G' \leq G^4$. The theory of these groups, which have proven to have fruitful applications in the theory of finite p-groups, was developed by Lubotzky and Mann [10]. One of our main results is that if G is a powerful p-group, then its nonabelian tensor square $G \otimes G$ is powerfully embedded in a group $\nu(G)$ introduced by Ellis and Leonard [6] and Rocco [12]. This result then enables us to prove that if G is a powerful p-group, the exponent of $G \otimes G$ divides the exponent of G. This generalizes a result of Lubotzky and Mann [10]. We then use techniques developed in [10] to estimate the order, exponent, and rank of $G \otimes G$ and $G \wedge G$ in terms of the exponent and rank of an arbitrary finite p-group G. Our bounds improve previously obtained estimates obtained by Jones [8], Rocco [12], and Ellis [4]. Furthermore, we obtain new bounds for $|H_2(G)|$ and $|\pi_3(SK(G,1))|$. We mention here that our

Date: January 31, 2008.

2000 *Mathematics Subject Classification.* 20J99, 20D15.

Key words and phrases. finite p-groups, the nonabelian tensor square.

The author was partially supported by the Ministry of Higher Education, Science and Technology of Slovenia.

bound for the order of $H_2(G)$ is asymptotically better than the one obtained by Lubotzky and Mann [10].

The paper is organized as follows. In Section 2 we briefly recall the definition and basic properties of nonabelian tensor squares of groups. Then we deal with tensor squares of powerful p-groups. The bounds for the order of the nonabelian tensor square of a given p-group are obtained in Section 3.

2. Powerful p-groups

powerful

At first we recall some basic properties of the nonabelian tensor squares of groups. We refer to [1, 2] for a more thorough account on the theory. Let G be a group and N a normal subgroup of G. We form the group $N \otimes G$, generated by the symbols $n \otimes g$, where $g \in G$ and $n \in N$, subject to the following relations:

$$nn' \otimes g = ({}^n n' \otimes {}^n g)(n \otimes g),$$
$$n \otimes gg' = (n \otimes g)({}^g n \otimes {}^g g'),$$

for all $n, n_1 \in N$ and $g, g_1 \in G$. Here we use the notation ${}^x y = xyx^{-1}$ for conjugation from the left (left hand actions are commonly used in this setting). We also define $N \wedge G = (N \otimes G)/\nabla(N, G)$, where $\nabla(N, G) = \langle n \otimes n : n \in N \rangle$. Note that we can similarly define $G \otimes N$ and $G \wedge N$, and that $G \otimes N \cong N \otimes G$ and $G \wedge N \cong N \wedge G$; this follows from the first principles of the more general construction of the nonabelian tensor product [1]. When $N = G$, the groups $G \otimes G$ and $G \wedge G$ are said to be the *nonabelian tensor square* and the *nonabelian exterior square* of G. We also denote $\nabla(G) = \nabla(G, G)$. By definition, the commutator maps $\kappa : G \otimes G \to G'$ and $\bar{\kappa} : G \wedge G \to G'$, given by $g \otimes h \mapsto [g, h]$ and $g \wedge h \mapsto [g, h]$, respectively, are well defined homomorphisms of groups. Here the commutator $[g, h]$ of elements g and h is defined by $[g, h] = ghg^{-1}h^{-1}$. Clearly $\ker \kappa$ is a central subgroup of $G \otimes G$, and $\ker \bar{\kappa}$ is central in $G \wedge G$. Furthermore, it turns out that $\ker \kappa \cong \pi_3(SK(G, 1))$ and $\ker \bar{\kappa} \cong H_2(G)$ [1]. We usually denote $J_2(G) = \ker \kappa$, and we clearly have that $J_2(G)$ is an extension of $\nabla(G)$ by $H_2(G)$.

There is another approach to the nonabelian tensor squares, introduced by Ellis and Leonard [6], and independently by Rocco [12]. Let G be a group and let G^φ be an isomorphic copy of G via the mapping $\varphi : g \to g^\varphi$ for all $g \in G$. We define the group $\nu(G)$ to be

$$\nu(G) = \langle G, G^\varphi \mid {}^x[g, h^\varphi] = [{}^x g, ({}^x h)^\varphi] = {}^{x^\varphi}[g, h^\varphi], \forall x, g, h \in G \rangle.$$

The motivation for considering $\nu(G)$ relative to the nonabelian tensor square is the fact that the map $\phi : G \otimes G \to [G, G^\varphi]$ defined by $\phi(g \otimes h) = [g, h^\varphi]$ for all g and h in G is an isomorphism of groups [6, 12]. Note that if G is a finite group (p-group), then $\nu(G)$ is also a finite group (p-group); this follows from the exact sequence [6]

$$1 \longrightarrow [G, G^\varphi] \longrightarrow \nu(G) \longrightarrow G \times G \longrightarrow 1.$$

Similarly we can define

$$\tau(G) = \langle G, G^\varphi \mid {}^x[g, h^\varphi] = [{}^x g, ({}^x h)^\varphi] = {}^{x^\varphi}[g, h^\varphi], [g, g^\varphi] = 1, \forall x, g, h \in G \rangle,$$

and we have that the subgroup $[G, G^\varphi]$ of $\tau(G)$ is naturally isomorphic to $G \wedge G$.

Let G be a finite p-group and let N be a normal subgroup of G. Recall that N is *powerfully embedded* in G if p is odd and $[N, G] \leq N^p$, or $p = 2$ and $[N, G] \leq N^4$. It is well known [2] that $G \wedge G$ is isomorphic to the derived subgroup of a covering group H of G. From [10, Theorem 2.1] it therefore follows that if G is a powerful p-group, an isomorphic copy of $G \wedge G$ can be powerfully embedded in H. In particular, we conclude that $G \wedge G$ is a powerful p-group. The main result of this section goes as follows.

main **Theorem 2.1.** *Let G be a powerful p-group. Then the groups $[\nu(G), \nu(G)]$ and $[G, G^{\varphi}]$ are powerfully embedded in $\nu(G)$.*

Proof. We only prove the assertion for $[\nu(G), \nu(G)]$, the proof for $[G, G^{\varphi}]$ follows along the same lines. Assume that p is odd and that $[\nu(G), \nu(G)]^p = 1$. We want to prove that $\gamma_3(\nu(G)) = 1$. We may assume without loss of generality that $\nu(G)$ is nilpotent of class ≤ 3. By the assumption we have that $[G, G]^p = [G^{\varphi}, G^{\varphi}]^p = [G, G^{\varphi}]^p = 1$. Since G is powerful, it follows from here that $\gamma_3(G) \leq [G, G]^p = 1$, hence G is nilpotent of class ≤ 2. As $\nu(G) = ([G, G^{\varphi}] \rtimes G) \rtimes G^{\varphi}$ [6] and $[G, G^{\varphi}]$ is normal in $\nu(G)$, we have

$$\gamma_3(\nu(G)) = \gamma_3(G)\gamma_3(G^{\varphi})[G, G, G^{\varphi}][G^{\varphi}, G^{\varphi}, G] = [G, G, G^{\varphi}][G^{\varphi}, G^{\varphi}, G].$$

Furthermore, for any $g, h \in G$ we have that $[g, h^{\varphi}][h, g^{\varphi}]$ is central in $\nu(G)$ [12]. As G is nilpotent of class ≤ 2 and $\nu(G)$ is nilpotent of class ≤ 3, we thus get $1 = [[g, h^{\varphi}][h, g^{\varphi}], k] = [g, h^{\varphi}, k][h, g^{\varphi}, k]$ for all $k \in G$. Therefore $[g, h^{\varphi}, k] = [g^{\varphi}, h, k] = [g, h, k^{\varphi}]$ (the last equality follows from [12]). We obtain that $[g^{\varphi}, h^{\varphi}, k] = [g, h^{\varphi}, k] = [g, h, k^{\varphi}]$ for all $g, h, k \in G$. This implies that $\gamma_3(\nu(G)) = [G, G, G^{\varphi}]$. As G is nilpotent of class ≤ 2, induction argument yields

A1 (2.1.1) $$[g^m, (h^{\varphi})^n] = [g, h^{\varphi}]^{mn}[h, [g, h]^{\varphi}]^{m\binom{n}{2}}[g, [g, h]^{\varphi}]^{n\binom{m}{2}}$$

for all integers m and n and $g, h \in G$. Let now $g, h, k \in G$ and consider the element $[g, h, k^{\varphi}]$. Since G is powerful, we have that $[g, h] \in G^p$, hence $[g, h] = x^p$ for some $x \in G$ [10, Proposition 1.7]. Since $G = G^2$, we can write $k = \prod_i y_i^2$ for some $y_i \in G$. By (2.1.1) we get

$$[x^p, (y_i^{\varphi})^2] = [x, y_i^{\varphi}]^{2p}[y_i, [x, y_i]^{\varphi}]^p[x, [x, y_i]^{\varphi}]^{2\binom{p}{2}} = 1,$$

hence $[x^p, k^{\varphi}] = 1$ by straightforward expansion. It follows from here that $[G, G, G^{\varphi}] = 1$, therefore $\nu(G)$ is nilpotent of class ≤ 2, as required.

The proof in the case $p = 2$ requires slight modifications. We may assume that $[\nu(G, \nu(G)]^4 = \gamma_4(\nu(G)) = 1$. Again we have that G is nilpotent of class ≤ 2 and $\gamma_3(\nu(G)) = [G, G, G^{\varphi}]$. Consider an element $[g, h, k^{\varphi}]$ of $[G, G, G^{\varphi}]$. Since G is powerful, there exists $x \in G$ such that $[g, h] = x^4$. As $G = G^3$, it suffices to consider the case when $k = y^3$ for some $y \in G$. Using (2.1.1), we obtain

$$[g, h, k^{\varphi}] = [x^4(, y^{\varphi})^3] = [x, y]^{12}[y, [x, y]^{\varphi}]^{12}[x, [x, y]^{\varphi}]^{18} = [x, [x, y]^{\varphi}]^2.$$

Again $[x, y] = z^4$ for some $z \in G$, and we only need to consider the case when $x = w^3$ for some $w \in G$. Now we get

$$[w^3, (z^{\varphi})^4] = [w, z^{\varphi}]^{12}[z, [w, z]^{\varphi}]^{18}[w, [w, z]^{\varphi}]^{12} = [z, [w, z]^{\varphi}]^2,$$

hence expansion gives that $[g, h, k^{\varphi}] = 1$. This proves the theorem. □

When G is a powerful p-group, $\nu(G)$ need not be powerful. For example, if G is a cyclic group of order p, then $\nu(G)$ is the nonabelian group of order p^3 and exponent p. Clearly $\nu(G)$ is not powerful. We also mention here that a similar argument as above shows that if G is powerful, the group $[\tau(G), \tau(G)]$ is powerfully embedded in $\tau(G)$.

If G is a group and N a normal subgroup of G, then we have canonical homomorphisms $\iota_1 : N \otimes G \to G \otimes G$ and $\iota_2 : G \otimes N \to G \otimes G$. Their images are normal subgroups of $G \otimes G$. A similar argument as above can be used to prove the following result:

pe **Proposition 2.2.** *Let G be a finite p-group and let N be powerfully embedded in G. Then $\operatorname{im}\iota_1$ and $\operatorname{im}\iota_2$ are powerfully embedded in $G\otimes G$.*

For a finite p-group G denote $|G:\Phi(G)| = p^{\mathrm{d}(G)}$. Lubotzky and Mann [10] proved that if G is a powerful p-group with $\mathrm{d}(G)=d$, then $G\wedge G$ and $H_2(G)$ can be generated by $\binom{d}{2}$ generators. By Theorem 2.1 we get the following extension of this result:

powergen **Corollary 2.3.** *Let G be a powerful p-group with $\mathrm{d}(G)=d$. Then $\mathrm{d}([\nu(G),\nu(G)])\le d(2d-1)$, $\mathrm{d}(G\otimes G)\le d^2$, and $\mathrm{d}(J_2(G))\le d^2$.*

Proof. Let G be generated by $\{x_1,\dots,x_d\}$. Then $[\nu(G),\nu(G)]$ is the normal closure in $\nu(G)$ of the set $\mathcal{S}=\{[x_i,x_j],[x_i^\varphi,y_j^\varphi],[x_i,x_k^\varphi] : 1\le j<i\le d, 1\le k\le d\}$. Similarly, the group $[G,G^\varphi]$ is the normal closure in $\nu(G)$ of the set $\mathcal{T}=\{[x_i,x_j^\varphi] : 1\le i,j\le d\}$. Since both $[\nu(G),\nu(G)]$ and $[G,G^\varphi]$ are powerfully embedded in $\nu(G)$ by Theorem 2.1, it follows that $[\nu(G),\nu(G)]$ is actually generated by $\mathcal{S}$, and that $[G,G^\varphi]$ is generated by $\mathcal{T}$ [10, Proposition 1.10]. This gives bounds for the number of generators of $[\nu(G),\nu(G)]$ and $G\otimes G$. As $J_2(G)$ is a subgroup of a powerful p-group $G\otimes G$, it follows that $\mathrm{d}(J_2(G))\le \mathrm{d}(G\otimes G)$ [10, Theorem 1.12]. This concludes the proof. □

Note that the bounds obtained in Corollary 2.3 are sharp; the equalities are attained for instance when G is a finite abelian p-group.

From [10, Theorem 2.1] it can be inferred that if G is a powerful p-group, then $\exp(G\wedge G)$ divides $\exp G$. Our next theorem generalizes this result.

powerexp **Theorem 2.4.** *Let G be a powerful p-group. Then the exponent of $[\nu(G),\nu(G)]$ divides $\exp G$.*

Proof. Let $H=\nu(G)$ and $\exp G = p^e$. By Theorem 2.1, the group H' is powerfully embedded in H. Thus it follows from [10] that $\gamma_k(H)^{p^i}=[\gamma_2(H)^{p^i},{}_{k-2}H]$ for all nonnegative integers i and $k\ge 2$. Assume first that p is odd. Then $[\gamma_2(H)^{p^i},{}_{k-2}H]\le (\gamma_2(H)^{p^i})^{p^{k-2}}=\gamma_2(H)^{p^{i+k-2}}$. This therefore gives

A2 $$\gamma_k(H)^{p^i}\le \gamma_2(H)^{p^{i+k-2}} \quad \text{for } k\ge 2. \tag{2.4.1}$$

This equation implies that if $\exp H'=p^n$, then $\exp\gamma_k(H)$ divides p^{n-k+2} for all $k\ge 2$. Let $a,b\in H$. Expanding $[a^{p^{n-1}},b]$ using Hall's Collection Process, we get

A3 $$[a^{p^{n-1}},b]\equiv [a,b]^{p^{n-1}}\gamma_2(\langle a,[a,b]\rangle)^{p^{n-1}}\prod_{i=1}^{n-1}\gamma_{p^i}(\langle a,[a,b]\rangle)^{p^{n-i-1}}. \tag{2.4.2}$$

We have that $\gamma_2(\langle a,[a,b]\rangle)^{p^{n-1}}\le \gamma_3(\langle a,b\rangle)^{p^{n-1}}=1$. Similarly we also have that $\gamma_{p^i}(\langle a,[a,b]\rangle)^{p^{n-i-1}}\le\gamma_{p^i+1}(\langle a,b\rangle)^{p^{n-i-1}}=1$, since $p\ge 3$ and $p^i\ge i+2$. It follows that $[a^{p^{n-1}},b]=[a,b]^{p^{n-1}}$ for all $a,b\in H$. In order to prove that the exponent of H', we may assume without loss of generality that $\gamma_2(H)^{p^{e+1}}=1$. The group H' is generated by the elements of the form $[g,h]$, $[g^\varphi,h^\varphi]$, and $[g,h^\varphi]$, where $g,h\in G$. As $\exp G=p^e$, we have that $[g,h]^{p^e}=[g^\varphi,h^\varphi]^{p^e}=1$. Besides, the above argument shows that $[g,h^\varphi]^{p^e}=[g^{p^e},h^\varphi]=1$. It follows that H' is generated by elements of order p^e. But H' is powerful, hence $\exp H'$ divides p^e by [10, Corollary 1.9].

For 2-groups, the equation (2.4.1) needs to be replaced with

A4 $$\gamma_k(H)^{2^i}\le\gamma_2(H)^{2^{i+2(k-2)}} \quad \text{for } k\ge 2. \tag{2.4.3}$$

Thus, if $\exp H'=2^n$, it follows that $\exp\gamma_k(H)$ divides $2^{n-2(k-2)}$. Using (2.4.2) with $p=2$ and observing that $2^{i+1}\ge i+3$, we conclude as above that $[a^{2^{n-1}},b]=[a,b]^{2^{n-1}}$ for all $a,b\in H$. This again implies that $\exp H'$ divides 2^e. □

Theorem 2.4 and Corollary 2.3, together with [10, Proposition 2.5], yield the following result:

powerord **Corollary 2.5.** *Let G be a powerful p-group, with* $\mathrm{d}(G) = d$ *and* $\exp G = p^e$. *Then* $|[\nu(G), \nu(G)]| \leq p^{ed(2d-1)}$, $|G \otimes G| \leq p^{ed^2}$, *and* $|G \wedge G| \leq p^{ed(d-1)/2}$.

Again, these bounds are best possible, since the equalities hold when G is an abelian p-group.

3. Bounds

bounds

Throughout this section (unless otherwise stated) let G be a finite p-group of exponent p^e, and let $r = \mathrm{sr}(G)$ be its special rank, i.e., $\mathrm{sr}(G) = \max\{\mathrm{d}(H) : H \leq G\}$. Note that $\mathrm{d}(G) \leq \mathrm{sr}(G) \leq \log_2 |G|$. Let

$$m = \begin{cases} \lceil \log_2 r \rceil & : \quad p > 2 \\ \lceil \log_2 r \rceil + 1 & : \quad p = 2 \end{cases}.$$

Lubotzky and Mann [10] found estimates for the order of G and $H_2(G)$ in terms of r and e. Their arguments were based on the following fact.

char **Lemma 3.1** ([10]). *Let G be a p-group of exponent p^e, with* $\mathrm{sr}(G) = r$. *Then G contains a characteristic powerful subgroup H such that* $|G : H| \leq p^{rm}$.

nabla **Lemma 3.2.** *Let G be a group generated by the set* $\{x_1, \ldots, x_r\}$. *Then $\nabla(G)$ is generated by* $\{x_i \otimes x_i, (x_i \otimes x_j)(x_j \otimes x_i) : i, j = 1, \ldots, r, j < i\}$.

Proof. Define the map $\phi : G \times G \to G \otimes G$ by the rule $\phi(g, h) = (g \otimes h)(h \otimes g)$ for all $g, h \in G$. It is straightforward to verify that ϕ is bimultiplicative, i.e., $\phi(gg', h) = \phi(g, h)\phi(g', h)$ and $\phi(g, hh') = \phi(g, h)\phi(g, h')$ for all $g, g', h, h' \in G$. Using this fact and the identity $gh \otimes gh = (g \otimes g)(h \otimes h)\phi(g, h)$, which holds true for all $g, h \in G$, the result follows. □

genrankodd **Proposition 3.3.** *Let G be as above. Then we have:*

(a) $\mathrm{d}(J_2(G)) \leq r^2(1 + m)$.
(b) $\mathrm{sr}(G \wedge G) \leq \binom{r+1}{2} + r^2 m$.
(c) $\mathrm{sr}(G \otimes G) \leq r + r^2(1 + m)$.

Proof. We have an exact sequence

B1 (3.3.1) $$0 \longrightarrow \nabla(G) \longrightarrow J_2(G) \longrightarrow H_2(G) \longrightarrow 0,$$

hence $\mathrm{d}(J_2(G)) \leq \mathrm{d}(\nabla(G)) + \mathrm{d}(H_2(G))$. From [10, Theorem 2.3 and Theorem 4.2.3] we get that $\mathrm{d}(H_2(G)) \leq \binom{r}{2} + r^2 m$. Besides, Lemma 3.2 implies that $\mathrm{d}(\nabla(G)) \leq \binom{r+1}{2}$. This gives (a). To prove (b) and (c), note that we have central extensions

B2 (3.3.2) $$0 \longrightarrow H_2(G) \longrightarrow G \wedge G \longrightarrow G' \longrightarrow 1$$

and

B3 (3.3.3) $$0 \longrightarrow J_2(G) \longrightarrow G \otimes G \longrightarrow G' \longrightarrow 1,$$

from which it is not difficult to get the assertions. □

Next we estimate the exponents of $J_2(G)$, $G \wedge G$, and $G \otimes G$. We note here that Ellis [4] proved that if G is any finite group, then $\exp J_2(G)$ divides the order of G, and that the exponent of $G \otimes G$ divides $|G| \exp G'$. On the other hand, it follows from [11] that if G is a locally finite group of finite exponent, then the exponent of $G \otimes G$ can be bounded in terms of $\exp G$ only. Precise bounds are however known only in some particular cases, cf. [11] for further details. Our aim here is to prove the following result.

genexpodd **Proposition 3.4.** *With the above notations, let*

$$k = \begin{cases} \lceil \log_2 r \rceil & : \quad p > 2 \\ \lceil \log_2 r \rceil^2 + 1 & : \quad p = 2 \end{cases}.$$

Then we have:

(a) $\exp(J_2(G)) \le p^{2e+rk}$.
(b) $\exp(G \wedge G) \le p^{2e+rk}$.
(c) $\exp(G \otimes G) \le p^{3e+rk}$.

Proof. At first note that $(g \otimes g)^{p^e} = g^{p^e} \otimes g = 1$ for all $g \in G$, therefore $\exp \nabla(G)$ divides p^e. By [10, Proposition 2.6 and Proposition 4.2.6], the exponent of $H_2(G)$ divides p^{e+rk}. From the exact sequence (3.3.1) it follows that $\exp J_2(G)$ divides $\exp H_2(G) \exp \nabla(G)$, which immediately gives (a). Similarly, (b) and (c) can now be proved by referring to the exact sequences (3.3.2) and (3.3.3). □

Now we focus on the order of $G \otimes G$ and $G \wedge G$. In principle we could use the above arguments to obtain the bounds. However, it turns out that a direct application of Lemma 3.1 provides substantially better estimates. Before formulating the result, we mention that Jones [8] proved that if G is a d-generator finite p-group of order p^n, then $p^{d(d-1)/2} \le |G \wedge G| \le p^{n(n-1)/2}$. Rocco [12] (see also Ellis [4]) proved that if G is a d-generator finite p-group of order p^n, with $|G'| = p^m$, then $p^{d^2} \le |G \otimes G| \le p^{n(n-m)}$. Similar bounds have been obtained by Ellis and McDermott [7] and Ellis [5] in a more general setting. We will apply here the following result:

EM **Lemma 3.5** ([5]). *Let G be a finite p-group and N a normal subgroup of G. Suppose that $|N| = p^n$, $d = \mathrm{d}(G)$, and $|N/N \cap \Phi(G)| = p^t$. Then $|G \wedge N| \le p^{dn - t(t+1)/2}$ and $|G \otimes N| \le p^{dn}$.*

genorderodd **Theorem 3.6.** *With the above notations we have:*

(a) $|G \wedge G| \le p^{r^2(e+m)}$.
(b) $|G \otimes G| \le p^{r^2(2e+m)}$.

Proof. Let H be as in Lemma 3.1. By the identity $g \wedge h = (h \wedge g)^{-1}$ which holds in $G \wedge G$ for all $g, h \in G$ we get an exact sequence

$$G \wedge H \longrightarrow G \wedge G \longrightarrow G/H \wedge G/H \longrightarrow 1.$$

By Lemma 3.5 we have that $|G \wedge H| \le |H|^r$. As H is powerful, we have that $|H| \le p^{er}$ [10, Proposition 2.5]. Thus $|G \wedge H| \le p^{er^2}$. As $|G : H| \le p^{rm}$, Lemma 3.5 implies that $|G/H \wedge G/H| \le p^{r^2 m}$, hence $|G \wedge G| \le p^{r^2(e+m)}$, as required. To prove the corresponding statement for $G \otimes G$, we use the exact sequence [5]

$$(G \otimes H)(H \otimes G) \longrightarrow G \otimes G \longrightarrow G/H \otimes G/H \longrightarrow 1,$$

which gives $|G \otimes G| \le |G \otimes H|^2 |G/H \otimes G/H|$. Using similar estimates as above, we get the result. □

It is proved in [10] that $|H_2(G)| \le p^{\binom{r}{2}e + er^2 m}$. On the other hand, Theorem 3.6 implies slightly better bounds:

H2J2p **Corollary 3.7.** *With the above notations we have that $|H_2(G)| \le p^{r^2(e+m)}$ and $|J_2(G)| \le p^{r^2(2e+m)}$.*

References

Bro1 [1] R. Brown, and J.-L. Loday, *Van Kampen theorems for diagrams of spaces*, Topology **26** (1987), no. 3, 311–335.

Bro2 [2] R. Brown, D. L. Johnson, and E. F. Robertson, *Some computations of nonabelian tensor products of groups*, J. Algebra **111** (1987), no. 1, 177–202.

Den1 [3] R. K. Dennis, *In search of new "homology" functors having a close relationship to K-theory*, Preprint, Cornell University, Ithaca, NY, 1976.

Ell0 [4] G. Ellis, *On the tensor square of a prime power group*, Arch. Math **66** (1996), 467–469.

Ell2 [5] G. Ellis, *On the relation between upper central quotients and lower central series of a group*, Trans. Amer. Math. Soc. **353** (2001), no. 10, 4219–4234.

Ell3 [6] G. Ellis, and F. Leonard, *Computing Schur multipliers and tensor products of finite groups*, Proc. Royal Irish Academy **(95A)** 2 (1995), 137-147.

Ell4 [7] G. Ellis, and A. McDermott, *Tensor products of prime-power groups*, J. Pure Appl. Algebra **132** (1998), 119–128.

Jon1 [8] M. R. Jones, *Some inequalities for the multiplicator of a finite group*, Proc. Amer. Math. Soc. **39** (1973), 450–456.

Kap1 [9] L.-C. Kappe, *Nonabelian tensor products of groups: the commutator connection*, Proc. Groups St. Andrews 1997 at Bath, London Math. Soc. Lecture Notes **261** (1999), 447–454.

Lub1 [10] A. Lubotzky, and A. Mann, *Powerful p-groups. I. Finite groups*, J. Algebra **105** (1987), 484–505.

Mor1 [11] P. Moravec, *The exponents of nonabelian tensor products of groups*, J. Pure Appl. Algebra, to appear. Available at `http://www.sciencedirect.com/`.

Roc1 [12] N. R. Rocco, *On a construction related to the nonabelian tensor square of a group*, Bol. Soc. Brasil. Mat. (N.S.) **22** (1991), 63–79.

Department of Mathematics, University of Ljubljana, Jadranska 21, 1000 Ljubljana, Slovenia

E-mail address: `primoz.moravec@fmf.uni-lj.si`