# ON $n$-CENTRAL SEMIGROUPS

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ABSTRACT. A semigroup $S$ is said to be $n$-central if $x^n$ belongs to the center of $S$ for every $x \in S$. We prove that every $n$-central semigroup is a semilattice of archimedean $n$-central semigroups. We obtain characterizations of simple (0-simple) $n$-central semigroups and describe subdirectly irreducible $n$-central semigroups. We also deal with the connection of $n$-central semigroups and E-$k$ semigroups.

## 1. INTRODUCTION

Let $n$ be a positive integer. A semigroup $S$ is said to be $n$-central if $x^n y = yx^n$ for any $x, y \in S$. $n$-central groups are extensively studied in [5], [7] and [8]. It is easy to see that a group $G$ is $n$-central if and only if the factor group $G/Z(G)$ is of exponent $n$ and this is further equivalent to the fact that $G$ satisfies the law $(xy)^n = (yx)^n$. In the semigroup case, we define a semigroup $S$ to be a $PC_n$-semigroup if $(xy)^n = (yx)^n$ for any $x, y \in S$. It is evident that every $PC_n$-semigroup is power commutative (PC) in the sense of [4]. As opposed to groups, the class of $PC_n$-semigroups does not coincide with the class of $n$-central semigroups; consider for instance the semigroup $S$ on three elements $a$, $b$ and $c$, where $a$ is a zero element, $b^2 = cb = a$, $bc = b$ and $c$ is an idempotent. It can be easily verified that $S$ is a $PC_n$-semigroup for every $n \geq 2$, but $c^n b \neq bc^n$ for any $n \geq 1$.

In the first part of this paper we deal with the semilattice decompositions of $n$-central ($PC_n$) semigroups. Since most of the results for $PC_n$-semigroups follow from the fact that every $PC_n$-semigroup is weakly commutative, we focus on $n$-central semigroups. We prove that every $n$-central semigroup is a semilattice of archimedean $n$-central semigroups. This enables us to prove that an $n$-central semigroup is simple (0-simple) if and only if it is isomorphic to an $n$-central group (an $n$-central group with zero). As a consequence we give a characterization of $n$-central archimedean semigroups containing at least one idempotent and describe regular $n$-central semigroups. We also present some results concerning subdirectly irreducible $n$-central semigroups.

Let $k > 1$ be a positive integer. A semigroup $S$ is said to be an E-$k$ semigroup if $(xy)^k = x^k y^k$ for every $x, y \in S$ (when $S$ is a group, it is also called $k$-abelian; see [5], [7] or [8]). In [5] it is proved that if $n$ is such that the free Burnside group $B(2, n)$ of rank two and exponent $n$ is finite, then every $n$-central group is also an E-$k$ group, where $k$ depends only on $n$. The question whether the free Burnside group $B(r, n)$ of rank $r$ and exponent $n$ is finite

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is known as the Burnside problem for groups; see [11] for a comprehensive survey about this problem. In particular, it is well–known that $B(r,n)$ is finite for $n = 2,3,4,6$ and for any $r$, but it is also known that $|B(r,n)| = \infty$ if $r > 1$ and $n$ is odd and $n \geq 665$. Motivated by the result of [5], one can ask the following question: Does there exist for a given positive integer $n$, an integer $k > 1$ depending only on $n$ such that every $n$-central semigroup is an E-$k$ semigroup? This question is answered negatively in general, even for groups, in [1]. On the other hand, it is proved in [5] that every 2-central group is an E-4 group, every 3-central group is an E-9 group, every 4-central group is an E-32 group and every 6-central group is an E-$(2^{29}3^{26})$ group. These results are improved in [8], where it is shown that every 4-central group is an E-16 group and every 6-central group is an E-36 group (see [8] for some other results about $n$-central groups). These two bounds are best possible, since there exist finite 4-central groups which are not E-$k$ groups for any $1 < k < 16$, and there are also examples of finite 6-central groups which are not E-$k$ groups for any $1 < k < 36$. The techniques used to prove these results for groups involve delicate commutator calculations and therefore they cannot be applied to the semigroup case. However, we are able to prove the following result.

**Theorem.** *Let $S$ be a semigroup.*

(a) *If $S$ is a 2-central $PC_2$-semigroup, then it is also an E-4 semigroup.*
(b) *If $S$ is a 3-central $PC_3$-semigroup, then it is also an E-54 semigroup.*
(c) *If $S$ is a 4-central $PC_4$-semigroup, then it is also an E-27840 semigroup.*

Comparing this result to the group case, the bounds 54 and 27840 appear to be exotic, but this is somewhat expected because of the restrictions in the structure. On the other hand, we are able to reduce these bounds at least in the case when $S$ is (left, right) cancellative by proving that every (left, right) cancellative 3-central semigroup is an E-9 semigroup and every (left, right) cancellative 4-central semigroup is an E-32 semigroup. It is unlikely that the bound 32 in the 4-central cancellative case can be reduced to 16, because the proof of the fact that every 4-central group is an E-16 group [8] relies on the solution of the Burnside problem for groups of exponent four [11]. The case of 6-central $PC_6$-semigroups seems to be very difficult to handle and it is still an open problem whether there is a positive integer $n \notin \{1,2,3,4,6\}$ such that every $n$-central group is an E-$k$ group for some integer $k > 1$.

## 2. Semilattice decomposition of $n$-central semigroups

A semigroup $S$ is said to be weakly commutative if for every $a, b \in S$ there exist $x, y \in S$ and a positive integer $n$ such that $(ab)^n = xa = by$. Since every $PC_n$-semigroup is weakly commutative, we immediately obtain the following result.

**Theorem 2.1.** *Let $S$ be a $PC_n$-semigroup.*

(a) *$S$ is a semilattice of archimedean $PC_n$-semigroups.*
(b) *$S$ is 0-simple if and only if $S$ is isomorphic to an $n$-central group with a zero adjoined.*

(c) *$S$ is archimedean and contains an idempotent if and only if it is an ideal extension of an $n$-central group by a nil $PC_n$-semigroup.*
(d) *$S$ is regular if and only if it is a Clifford $PC_n$-semigroup.*

*Proof.* See [2]. □

**Corollary 2.2.** *Let $S$ be an ideal extension of a regular semigroup $K$ by a nil semigroup $Q$. Then $S$ is a $PC_n$ semigroup if and only if $K$ is a Clifford $PC_n$-semigroup and $Q$ is a $PC_n$-semigroup.*

*Proof.* Let a $PC_n$-semigroup $S$ be an ideal extension of a regular semigroup $K$ by a nil semigroup $Q$. Then $K$ and $Q$ are $PC_n$-semigroups and $K$ is a Clifford semigroup by Theorem 2.1. Conversely, let $K$ be a Clifford $PC_n$-semigroup and let $Q$ be a nil $PC_n$-semigroup. If $S$ is an ideal extension of $K$ by $Q$, then this is a retract extension; one can check that $\phi : S \to K$ defined by $\phi(a) = ae$, where $e$ is an arbitrary idempotent of $K$, is a retract homomorphism. Therefore $S$ is also $PC_n$-semigroup, since varieties of semigroups are closed with respect to retract extensions. □

Our purpose is to obtain similar results for $n$-central semigroups. The following lemma is crucial:

**Lemma 2.3.** *Every $n$-central semigroup is a semilattice of archimedean $n$-central semigroups.*

*Proof.* Let $S$ be an $n$-central semigroup, $a, b \in S$ and suppose that $b \in S^1aS^1$. There exist $x, y \in S^1$ such that $b = xay$. Then we have $b^{n+1} = (xay)^{n+1} = x(ayx)^n ay = xa(ayx)^n y \in S^1a^2S^1$. By [9], $S$ is a semilattice of archimedean semigroups, which are clearly $n$-central. □

**Theorem 2.4.** *A semigroup is 0-simple and $n$-central if and only if it is isomorphic to $G^0$, where $G$ is an $n$-central group.*

*Proof.* Let $S$ be a 0-simple $n$-central semigroup. By Lemma 2.3, $S$ is a semilattice of archimedean $n$-central subsemigroups, therefore all non-zero elements of $S$ belong to the same semilattice component $C$. Beside that, $0 \notin C$, since the opposite would imply that $S$ is a nil semigroup, which would yield $S = \{0\}$. Hence we have $S = C^0$, where $C$ is a simple $n$-central semigroup. Pick $a \in C$. There exist $x, y \in C$ such that $a^n = xa^{2n}y = a^n xya^n$, hence $a^n$ is a regular element of $C$. This yields that $C$ contains an idempotent $f$. Suppose now that $C$ is not completely simple. Then $C$ contains bicyclic semigroup $C(p, q)$, where $pq = f$ is the identity element of $C(p, q)$. But then we have $q^np = pq^n = q^{n-1}$, which is a contradiction, hence $C$ is completely simple. By Theorem 3.3.1 of [6], $C$ is isomorphic to a Rees matrix semigroup $\mathcal{M}[G; I, J; P]$ over a group $G$ with the $J \times I$ sandwich matrix $P$. Moreover, we may assume that $P$ is normalized, i.e., $p_{j_0,i} = p_{j,i_0} = e$ for some $i_0 \in I$, $j_0 \in J$ and all $i \in I$, $j \in J$, where $e$ denotes the identity of the group $G$. Let $x = (i_0, g, j_0)$ and $y = (i, h, j)$ be the elements of $\mathcal{M}[G; I, J; P]$. Since $x^ny = yx^n$, we get $(i_0, g^nh, j) = (i, hg^n, j_0)$, hence $|I| = |J| = 1$ and $C$ is isomorphic to the group $G$, which is $n$-central. This proves the theorem. □

**Theorem 2.5.** *$S$ is an archimedean $n$-central semigroup and contains an idempotent if and only if it is an ideal extension of an $n$-central group by a nil $n$-central semigroup.*

*Proof.* Let $S$ be an archimedean $n$-central semigroup. By [3], $S$ is an ideal extension of a simple semigroup $K$ by a nil semigroup $Q$. Obviously $K$ and $Q$ are $n$-central and $K$ is a group by Theorem 2.4.

Conversely, suppose that $S$ is an ideal extension of an $n$-central group by a nil $n$-central semigroup. Then this is also a retract extension. Since varieties of semigroups are closed with respect to retract extension, $S$ is $n$-central. Using [3], we conclude that $S$ is archimedean and contains at least one idempotent. □

**Proposition 2.6.** *$S$ is a regular $n$-central semigroup if and only if it is an $n$-central Clifford semigroup.*

*Proof.* This follows directly from the fact that the idempotents of an $n$-central semigroup $S$ belong to the center of $S$. □

**Corollary 2.7.** *Let $S$ be an ideal extension of a regular semigroup $K$ by a nil semigroup $Q$. Then $S$ is an $n$-central semigroup if and only if $K$ is an $n$-central Clifford semigroup and $Q$ is an $n$-central semigroup.*

*Proof.* See the proof of Corollary 2.2. □

## 3. Subdirectly irreducible $n$-central semigroups

A semigroup $S$ is said to be subdirectly irreducible if whenever $S$ is a subdirect product of a family of semigroups $(S_\lambda)_{\lambda\in\Lambda}$, then there exists $\lambda_0 \in \Lambda$ such that the projection homomorphism $\pi_{\lambda_0} : \prod_{\lambda\in\Lambda} S_\lambda \to S_{\lambda_0}$ maps $S$ isomorphically onto $S_{\lambda_0}$. Subdirectly irreducible semigroups are building blocks of semigroups, since every semigroup is a subdirect product of subdirectly irreducible semigroups.

We mention here some results of [10]. For instance, it is known that $S$ is subdirectly irreducible if and only if $S^0$ is subdirectly irreducible. Every subdirectly irreducible semigroup contains a core, the least non-trivial ideal of $S$. If $K$ is a core of $S$, then we have either $K^2 = K$ or $K^2 = \{0\}$. In the first case we say that $K$ is globally idempotent, and in the second case $K$ is called nilpotent. A semigroup is said to be homogroup if it contains the least non-empty ideal (a kernel) which is a group. By [10], every subdirectly irreducible homogroup without zero is a group.

Our first result is a characterization of subdirectly irreducible $n$-central semigroups ($PC_n$-semigroups) with a globally idempotent core:

**Theorem 3.1.** *$S$ is a subdirectly irreducible $n$-central ($PC_n$) semigroup with a globally idempotent core $K$ if and only if it satisfies one of the following conditions.*

(i) *$S$ is isomorphic either to $G$ or $G^0$, where $G$ is a subdirectly irreducible $n$-central group.*
(ii) *$S$ is a two-element semilattice.*

*Proof.* If $0 \notin S$, then $K$ is a simple $n$-central ($PC_n$)-semigroup. By Theorem 2.4 (Theorem 2.1), $K$ is an $n$-central group. Therefore $S$ is a homogroup implying that $S$ is an $n$-central group. Hence we may assume that $S$ contains a zero element. Let $S^* = S\backslash\{0\}$. We prove that $S^*$ is a subsemigroup of $S$. Suppose that there exist $a, b \in S$, $a, b \neq 0$ such that $ab = 0$. By Lemma 2.3 (Theorem 2.1), $S$ is a semilattice of archimedean $n$-central $(PC_n)$ semigroups. Let $S_0$ be a semilattice component containing zero of $S$. This is an ideal of $S$. By Theorem 2.4 (Theorem 2.1), $K$ is a group with zero adjoined, hence 0 is the only element of $S$ contained in $S_0$. This implies that $I = \{x \in S : ax = 0\}$ is an ideal of $S$. As $I$ contains at least two elements, we have $K \subseteq I$, hence $aK = \{0\}$. Put $J = \{y \in S : yK = \{0\}\}$. This is a non-trivial ideal of $S$, hence $K \subseteq J$, from which it follows $K^2 = \{0\}$. This contradicts the assumption that $K$ is globally idempotent, hence $S^*$ is a subsemigroup of $S$.

Now, if $|S^*| = 1$, then $S$ is a two-element semilattice. If $|S^*| > 1$, then $S^*$ contains no zero element. As $S^*$ is subdirectly irreducible $n$-central $(PC_n)$ semigroup with a globally idempotent core, it is isomorphic to a subdirectly irreducible $n$-central group $G$, hence $S \cong G^0$.

The converse statement is clear. □

We introduce some more definitions from [10]. If $S$ is a semigroup with zero, let $A_S$ be the annihilator of $S$, that is, $A_S = \{a \in S : as = sa = 0 \text{ for every } s \in S\}$. This is an ideal of $S$. We also say that $s \in S$ is a disjunctive element of $S$ if the congruence $C_{\{s\}} = \{(a, b) \in S \times S : (\forall x, y \in S^1)\ xay = s \iff xby = s\}$ is the equality relation on $S$. The results of [10] give the following characterization of subdirectly irreducible $n$-central $(PC_n)$ semigroup with zero and a non-trivial annihilator:

**Proposition 3.2** ([10]). *An $n$-central $(PC_n)$ semigroup with a zero element and a non-trivial annihilator is subdirectly irreducible if and only if it has a non-zero disjunctive element.*

When the annihilator of $S$ is trivial, we have:

**Theorem 3.3.** *Let $S$ be a subdirectly irreducible $n$-central semigroup with zero and with a trivial annihilator. If $S$ has a nilpotent core, then $S$ is a monoid.*

*Proof.* Let $K$ be the nilpotent core of $S$ and put $L = \{x \in S : xK = \{0\}\}$ and $R = \{x \in S : Kx = \{0\}\}$. Suppose that there is $x \in R\backslash L$. Then $xK$ is a non-trivial ideal of $S$, hence $K \subseteq xK$ therefore $K = xK$. Since $S$ is $n$-central, we have $K = x^nK = Kx^n = \{0\}$, which is a contradiction. This proves that $R \subseteq L$. By the symmetry we also have $L \subseteq R$, hence $R = L$.

Now, let $N = S\backslash R$. This is a non-empty set, since the annihilator of $S$ is trivial. Suppose that there exist $x, y \in N$ such that $xy \in R$. Then $Ky \cup yK = K$ and $xK = xKy$, because $xy \in R$. This yields $xK = xKy^n = xy^nK = \{0\}$, which contradicts $x \in N$. Therefore we have proved that $N$ is a subsemigroup of $S$.

At this stage we fix $k \in K\backslash\{0\}$. Since $K$ is the core of $S$, we have $K = Nk \cup kN \cup NkN \cup \{0\}$. If $k = ke_1$ for some $e_1 \in N$, then $k = ke_1^n = e_1^nk$. If $k = e_2ke_3$ for some $e_2, e_3 \in N$, then $k = e_2^nke_3^n = e_2^ne_3^nk$. Therefore we may assume that $k = ek$ for some $e \in N$. We show that $e$ is the identity element of $S$.

Let $I = \{x \in S : e^m x = x$ for some positive integer $m\}$. $I$ is a non-empty set since $k \in I$. Let $x \in I$ and $s \in S$. Then there is a positive integer $m$ such that $e^m x = x$. This yields $xs = e^m(xs)$ and $sx = se^{nm}x = e^{nm}(sx)$, hence $I$ is an ideal of $S$. We have $K \subseteq I$, since $K$ is the core of $S$. Put

$$\rho = \{(x, y) \in S : e^m x = e^m y \text{ for some positive integer } m\}.$$

This is clearly a right congruence on $S$, but it is also a left congruence. To see this, let $(x, y) \in \rho$ and let $z \in S$. There is a positive integer $m$ such that $e^m x = e^m y$. This yields $e^{mn}zx = ze^{mn}x = ze^{mn}y = e^{mn}zy$, hence $(zx, zy) \in \rho$. We show that $\rho$ is the equality relation on $S$. Since $S$ is subdirectly irreducible, it suffices to show that $\rho$ is the equality relation on $K$. Let $(k_1, k_2) \in \rho$ for $k_1, k_2 \in K$. There is a positive integer $m$ such that $e^m k_1 = e^m k_2$. Since $K \subseteq I$, there exist positive integers $m_1$ and $m_2$ such that $e^{m_1}k_1 = k_1$ and $e^{m_2}k_2 = k_2$. But then we have $k_1 = e^{m_1 m_2 m}k_1 = e^{m_1 m_2 m}k_2 = k_2$, hence $\rho$ is indeed the equality relation.

To complete the proof, we observe that $s\rho se$ for any $s \in S$, hence $e$ is a right identity of $S$. But we also have $s = se = se^n = e^n s = es$, hence $S$ is a monoid, as required. □

## 4. Connection of $n$-central semigroups and E-$k$ semigroups

A semigroup $S$ is said to be an E-$k$ semigroup if $(xy)^k = x^k y^k$ for every $x, y \in S$. The results of [5] and [8] can be stated as follows.

**Theorem 4.1** ([5],[8])**.** *Let $n \in \{2, 3, 4, 6\}$. If $G$ is an $n$-central group, then it is an E-$n^2$ group.*

The question whether every $n$-central group is an E-$k$ group for some $k > 1$ is closely related to the Burnside problem [11] for groups of exponent $n$. The following result is an analogue of Theorem 4.1 for semigroups.

**Theorem 4.2.** *Let $S$ be a semigroup.*

(a) *If $S$ is a 2-central $PC_2$-semigroup, then it is also an E-4 semigroup.*
(b) *If $S$ is a 3-central $PC_3$-semigroup, then it is also an E-54 semigroup.*
(c) *If $S$ is a 4-central $PC_4$-semigroup, then it is also an E-27840 semigroup.*

*Proof.*

(a) Let $S$ be a 2-central $PC_2$-semigroup and let $x, y \in S$. By $PC_2$, $x$ commutes with $yxy$, hence $(xy)^4 = x(yxy)(xy)^2 = yxyx^2yxy = x^2(yxy)^2$, since $x^2 \in Z(S)$. Using $PC_2$ and 2-centrality once again, we conclude that $(yxy)^2 = (xy^2)^2 = x^2y^4$, therefore $(xy)^4 = x^4y^4$.

(b) Let $S$ be a 3-central $PC_3$-semigroup and let $x, y \in S$. We have that $x$ commutes with $yxyxy$, hence

$$(xy)^9 = (x \cdot yxyxy)^3 = x^3(yxyxy)^3 = x^3(y^2xyx)^3. \tag{1}$$

On the other hand, we have $(xy^2)^9 = x^3(y^2xy^2xy^2)^3 = x^3(y^4xy^2x)^3 = x^3y^9(yxy^2x)^3 = x^3y^9(y^2xyx)^3$. Using (1), we obtain

$$(xy^2)^9 = y^9(xy)^9. \tag{2}$$

Replacing $x$ by $xy$ in (2), using (2) and 3-centrality of $S$, we get

$$y^{18}(xy)^9 = x^9y^{27}. \tag{3}$$

Similarly, if we replace $y$ by $xy$ in (2) and observe (2) and $PC_3$, we conclude that $(x^2yxy)^9 = (xy)^{18}x^9$. Interchanging $x$ and $y$ and using (1) yield

$$(xy)^{27} = (xy)^{18}x^9y^9. \tag{4}$$

This equation, together with (3), yields $y^{27}(xy)^{27} = x^9y^{36}(xy)^{18} = x^{27}y^{54}$. Using (2), we arrive at

$$(xy^2)^{27} = x^{27}y^{54}. \tag{5}$$

Combining (2) and (5) gives $x^{27}y^{54} = y^{27}(xy)^{27}$. Using (4), this gives $x^{54}y^{54} = (xy)^{27}x^{27}y^{27} = (xy)^{18}x^9y^9 \cdot (xy)^9x^{18}y^{18} = (xy)^{36}x^{18}y^{18} = (xy)^{54}$, hence the result.

(c) We are left with the case when $S$ is a 4-central $PC_4$-semigroup. Let $x, y \in S$. Since $x$ commutes with $yxyxyxy$, we obtain

$$(xy)^{16} = x^4(y^2xyxyx)^4. \tag{6}$$

Replacing $y$ by $y^2$ in (6), we obtain

$$(xy^2)^{16} = x^4y^{16}(x^2y^2xy^2)^4, \tag{7}$$

whereas the replacement of $x$ by $xy$ in (6) gives

$$(xy^2)^{16} = (xy)^4(y^3xy^2xy^2x)^4. \tag{8}$$

Replacing $x$ by $x^2$ in (7), using $PC_4$ and 4-centrality of $S$, we get

$$(x^2y^2)^{16} = x^{32}y^{32}. \tag{9}$$

Further replacements of $x$ by $xy$ in (7), (8) and (9) lead to the following equalities:

$$(xy^3)^{16} = (xy)^4y^{16}(xyxy^3xy^3)^4, \tag{10}$$

$$(xy^3)^{16} = (xy^2)^4y^{16}(x^2y^3xy^3)^4, \tag{11}$$

$$(xyxy^3)^{16} = (xy)^{32}y^{32}. \tag{12}$$

Replacing $x$ by $xy$ in (12), we obtain

$$x^{32}y^{96} = (xy^2)^{32}y^{32} \tag{13}$$

and if we replace $x$ by $xy$ in (13), we get

$$(xy)^{32}y^{96} = (xy^3)^{32}y^{32}. \tag{14}$$

The identity (10) yields $(xy^3)^{128} = (xy)^{32}y^{128}(xyxy^3xy^3)^{32}$. Multiplying this equation by $(xy^3)^{32}$, we get $(xy^3)^{160} = (xy)^{32}y^{128}(xy \cdot (xy^3)^2)^{32}(xy^3)^{32}$. Applying (13) with $(xy, xy^3)$ instead of $(x, y)$, this yields $(xy^3)^{160} = (xy)^{64}y^{32}((xy^3)^{32}y^{32})^3$. Using (14), we obtain

$$(xy^3)^{160} = (xy)^{160}y^{320}. \tag{15}$$

From (8) we derive $(xy^2)^{640} = (xy)^{160}(y(y^2x)^3)^{160}$. Using (15), this gives $(xy^2)^{640} = (xy)^{160}(xy^3)^{160}(xy^2)^{320} = (xy)^{320}y^{320}(xy^2)^{320}$. Observing (13), we get

$$(xy^2)^{640} = (xy)^{320}x^{320}y^{960}. \tag{16}$$

Next we use (6) to obtain $(xy)^{2560} = x^{640}(xy^2 \cdot (xy)^2)^{640}$. Applying (16), we get $(xy)^{2560} = (xy)^{960}x^{640}(xy^2xy)^{320}(xy^2)^{320}$, hence $(xy)^{5120} = (xy)^{1920}x^{1280}(y\cdot(xy)^2)^{640}(xy^2)^{640}$. Using (16) once again, this yields $(xy)^{5120} = (xy)^{3200}x^{1600}y^{1280}(xy^2)^{320}$, which further gives, after some straightforward calculation,

$$(xy)^{10240} = (xy)^{6720}x^{3520}y^{3520}. \tag{17}$$

Replacing $x$ by $x^2$ in (11), we obtain $(x^2y^3)^{16} = (x^2y^2)^8x^{16}y^{32}$, which together with (9) implies

$$(x^2y^3)^{32} = x^{64}y^{96}. \tag{18}$$

If we replace $x$ by $x^2$ in (14) and use (18), we get $(x^2y)^{32}y^{96} = x^{64}y^{128}$, which is equivalent to

$$(xy^2)^{32}x^{96} = x^{128}y^{64}. \tag{19}$$

Now, if we replace $x$ by $xy$ in (18), we get $y^{128}(yx^2)^{32} = (xy)^{64}y^{96}$. Interchanging $x$ and $y$ and using (19), this yields

$$x^{160}y^{64} = (xy)^{64}x^{96}, \tag{20}$$

which is also equivalent to

$$x^{64}y^{160} = (xy)^{64}y^{96}. \tag{21}$$

The identity (16) now implies

$$(xy^2)^{640} = (xy)^{320}x^{320}y^{960} = ((xy)^{64}x^{96})^3((xy)^{64}y^{96})^2x^{32}y^{768},$$

which together with (20) and (21) yields

$$(xy^2)^{640} = x^{640}y^{1280}. \tag{22}$$

Combining (16) and (22) gives $x^{1280}y^{1280} = (xy)^{320}x^{960}y^{960}$. Using (17), we get

$$\begin{aligned}
x^{26880}y^{26880} &= (xy)^{6720}x^{20160}y^{20160}\\
&= (xy)^{10240}x^{16640}y^{16640}\\
&= (xy)^{6720}x^{3520}y^{3520}\cdot(xy)^{3520}x^{13120}y^{13120}\\
&= (xy)^{13760}x^{13120}y^{13120}\\
&= ((xy)^{6720}x^{3520}y^{3520})^2(xy)^{320}x^{6080}y^{6080}\\
&= (xy)^{20800}x^{6080}y^{6080}\\
&= (xy)^{24320}x^{2560}y^{2560}.
\end{aligned}$$

Multiplying this equation by $x^{960}y^{960}$, we obtain $x^{27840}y^{27840} = (xy)^{24320}x^{3520}y^{3520}$. Observing (17), we finally obtain

$$(xy)^{27840} = x^{27840}y^{27840}.$$

This proves the theorem. □

The following result is already proved in [5] for the case when $S$ is a group:

**Corollary 4.3.** *Let $S$ be a (left, right) cancellative semigroup.*

(a) *If $S$ is 2-central, then it is an $E$-4 semigroup.*
(b) *If $S$ is 3-central, then it is an $E$-9 semigroup.*
(c) *If $S$ is 4-central, then it is an $E$-32 semigroup.*

*Proof.* First we observe that if $S$ is a right cancellative $n$-central semigroup, then $(xy)^n y = y(xy)^n = (yx)^n y$ implies that $S$ is also a $PC_n$-semigroup. Similarly, if $S$ is left cancellative and $n$-central, then $y(yx)^n = (yx)^n y = y(xy)^n$ also implies that $S$ is a $PC_n$-semigroup. Now (a) follows from Theorem 4.2 (a) and (b) follows from the equation (3) of Theorem 4.2.

Let $S$ be a (left, right) cancellative 4-central semigroup. Replacing $(x, y)$ by $(x^2, y^2)$ in (8), we obtain, after the cancellation, $(x^2y^2)^8 = x^{16}y^{16}$. If we further replace $x$ by $xy$ in this equation, we get $(xyxy^3)^8 = (xy)^{16}y^{16}$. Replacing $x$ by $xy$ once again, we conclude that $(xy^2)^{16} = x^{16}y^{32}$. Now the equation (6) implies

$$(xy)^{64} = x^{16}(xyxyx \cdot y^2)^{16} = x^{16}(x(yx)^2)^{16}y^{32} = x^{32}y^{32}(xy)^{32},$$

which yields $(xy)^{32} = x^{32}y^{32}$, hence (c) is proved. □

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