

SCHUR MULTIPLIERS OF n -ENGEL GROUPS

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ABSTRACT. We find a bound for the exponent of the Schur multiplier of a finite p -group in terms of the exponent and Engel length of the given group. It is also proved that if G is a 3-Engel group of finite exponent, then the exponent of $H_2(G)$ divides $\exp G$. When G is a 4-Engel group of exponent e , the exponent of $H_2(G)$ divides $10e$.

1. INTRODUCTION

Let G be a group. The second integral homology group $H_2(G, \mathbb{Z})$ of G (or $H_2(G)$ in shorthand notation) is known as the *Schur multiplier* of G . When the group G is finite, $H_2(G)$ is isomorphic to the cohomology group $H^2(G, \mathbb{C}^\times)$. This group has several important applications in the theory of projective representations and the theory of central extensions. For a thorough account on the theory of Schur multipliers see e.g. Beyl and Tappe [3], Huppert [15], or Karpilovsky [16].

One of the fundamental problems regarding Schur multipliers is the determination of the exponent of $H_2(G)$. Schur proved that if G is a finite group and if $\exp H_2(G) = e$, then e^2 divides the order of G [15, Satz V.23.9]. This bound is best possible, since $H_2(\mathbb{Z}_e \oplus \mathbb{Z}_e) \cong \mathbb{Z}_e$. On the other hand, this simple example shows that the exponent of the Schur multiplier is in a sense close to the exponent of the given group. In fact, it had been conjectured for a long time that the exponent of $H_2(G)$ divides $\exp G$ for every finite group G , see [16, p. 152]. By standard homological arguments [3] it suffices to consider this question for finite p -groups. It is now known that the conjecture is false in general. Namely, Bayes et al. [2] constructed a finite group G of exponent 4 with $\exp H_2(G) = 8$ (see also [22, 24]). Surprisingly, examples of this kind are relatively rare. For instance, it is still not known whether or not the above conjecture holds true for all groups of odd order. On the other hand, the conjecture holds true for all finite groups that are nilpotent of class ≤ 3 (cf. Jones [18] and [24]), and for groups of odd order that are nilpotent of class ≤ 4 [24].

The above mentioned result of Schur, together with Zelmanov's solution of the restricted Burnside problem [29], implies that for all positive integers d and e there exists an integer $f = f(d, e)$ depending only on d and e such that for every finite d -generator group G of exponent e , the exponent of $H_2(G)$ divides f . It is proved in [22] that we can actually bound $\exp H_2(G)$ by a function depending only on e for every locally finite group G of exponent e . In some cases, precise bounds can be obtained. A sample result is that if G is any group of exponent 4, then the exponent of $H_2(G)$ divides 8 [22]. In general it is possible to prove [22] that if G is a locally finite group of exponent e , then $\exp H_2(G) \leq |R(2, e)|$, where $R(2, e)$ is the restricted 2-generator Burnside group of exponent e , yet this estimate is highly

Date: June 15, 2008.

2000 Mathematics Subject Classification. 20F45, 20F18.

Key words and phrases. Engel groups, Schur multiplier.

The author was partially supported by the Slovenian Research Agency.

unrealistic. Much better bounds can be obtained in terms of the nilpotency class of the group in question. Ellis [7] showed that if G is a finite p -group of class $c \geq 2$ and exponent p^e , then the exponent of $H_2(G)$ divides $p^{e\lceil c/2 \rceil}$. It is shown in [22] that this can be slightly improved; one can replace $\lceil c/2 \rceil$ with $2\lfloor \log_2 c \rfloor$.

A group G is said to be n -Engel if it satisfies the law $[x, {}_n y] = 1$. Groups satisfying the n -Engel condition are analogs of n -Engel Lie algebras. It is an important open problem whether n -Engel groups are locally nilpotent. This is now known to be true for $n \leq 4$. Hopkins [14] showed that 2-Engel groups are nilpotent of class ≤ 3 (this result is usually attributed to Levi). Even before that, Burnside [6] had proved that every 2-Engel group without elements of order 3 is nilpotent of class ≤ 2 . The local nilpotency of 3-Engel groups was proved by Heineken [13]. The corresponding result for 4-Engel groups was obtained recently by Havas and Vaughan-Lee [11], based on the work of Traustason [26, 27, 28]. On the other hand, finite n -Engel groups are always nilpotent by a result of Zorn [30].

The purpose of this paper is to study the relationship between the exponent of $H_2(G)$ and the Engel length of a finite p -group G . At first we note that, given a positive integer n , there exists an integer $m = m(n)$, such that, for every prime $p > m$, the exponent of $H_2(G)$ divides $\exp G$ for every finite n -Engel p -group G . Then we obtain a bound for $\exp H_2(G)$, where G is a finite n -Engel group of exponent p^e , which involves n , p , e , and certain invariants corresponding to the action of G on some of its power subgroups. Here we apply results from Abdollahi and Traustason [1], as well as the nonabelian exterior product of groups, a topological construction introduced by Brown and Loday [4]. Then we focus on n -Engel groups for $n \leq 4$. As every 2-Engel group is nilpotent of class ≤ 3 , it follows from [24] that $\exp H_2(G)$ divides the exponent of G for every finite 2-Engel group G . We generalize this result by proving that for every 3-Engel group G of finite exponent, the exponent of $H_2(G)$ divides the exponent of G . A similar result fails to be true for 4-Engel groups. An example of a finite group G of class 4 and exponent 4 with $\exp H_2(G) = 8$ was constructed in [24]. What we prove is that if G is a 4-Engel group of exponent e , then $\exp H_2(G)$ divides $10e$. When e is not divisible by 2 or 5, then $\exp H_2(G)$ divides e . It is not clear whether this result is best possible. In particular, we do not know whether $\exp H_2(G)$ divides $\exp G$ for every finite 4-Engel group G . If so, then we could replace the bound $10e$ by $2e$.

The basic idea behind the proofs of the above results is first to reduce the question to finite p -groups. Then, instead of estimating $\exp H_2(G)$ directly, we try to bound the exponent of the derived subgroup of a covering group H of G which always contains a copy of $H_2(G)$. If G is n -Engel, then H is center-by- n -Engel. Thus we describe the power commutator structure of such groups. This method is convenient as it allows us to exploit the full power of commutator calculus in n -Engel groups. Its downside lies in the fact that the exponent of H' can be larger than $\exp H_2(G)$, cf. [23]. We also mention here that computer calculations with GAP [9] are used in the arguments. Nevertheless, their use is reduced to the minimum.

2. A GENERAL BOUND

We first recall a notion that is closely related to the Schur multiplier of a given group. Let G be a group. Then H is a *covering group* of G if there exists $Z \leq H' \cap Z(H)$ such that $Z \cong H_2(G)$ and $H/Z \cong G$. It is known that every finite group has a covering group. Covering groups of a given group need not be unique, yet they have isomorphic derived subgroups. For further details on the subject cf. [3] or [15].

Let G be a finite group and let p be a prime. Then it is well known that the p -part of the abelian group $H_2(G)$ can be embedded into $H_2(P)$, where P is a Sylow p -subgroup of G [15, Satz V.25.1]. This result shows that, when estimating the exponent of the Schur multiplier, one can obtain bounds by considering only finite p -groups. In [24] we showed that if G is a finite p -group of nilpotency class c with $p > c + 1$, then the exponent of $H_2(G)$ divides $\exp G$. Here we slightly extend this by showing that a similar result holds true for groups of fixed Engel length.

Proposition 2.1. *For every positive integer n there exists an integer $m = m(n)$ such that if p is a prime larger than m , and G is a finite n -Engel p -group, then $\exp H_2(G)$ divides $\exp G$.*

Proof. By a result of Endimioni [8, Theorem 1], there exist integers $k = k(n)$ and $c = c(n)$ such that, for all primes $p > k$, every (locally) finite n -Engel p -group is nilpotent of class $\leq c$. Let $m = \max\{k, c + 1\}$, $p > m$, and let G be a finite n -Engel p -group. Then G is nilpotent of class $< p - 1$. By [24, Proposition 11], the exponent of $H_2(G)$ divides $\exp G$, as required. \square

In general it is hard to determine precise values of m in Proposition 2.1. When $n \leq 4$, some estimates are known [13, 26]. We will elaborate on these in sections 4 and 5.

Let G be a finite n -Engel p -group of exponent p^e . It is known [22] that $\exp H_2(G)$ can be bounded in terms of p and e only. Explicit bounds of this type are however not known in general, or they are rather crude as they depend on the solution of the restricted Burnside problem [29]. In the rest of the section we find a more realistic bound for $\exp H_2(G)$ which depends only on n, p, e , and certain invariants of G which will be introduced below. In order to do that, we recall the notion of the nonabelian exterior product of groups which comes from the notion of the nonabelian tensor product of groups introduced by Brown and Loday [4]. What we need is only a rather special case of this construction. Let G be a group and N a normal subgroup of G . Let $N \wedge G$ be the group generated by the symbols $x \wedge g$, where $x \in N$ and $g \in G$, subject to the following relations:

$$\begin{aligned} x_1 x \wedge g &= (x_1^x \wedge g^x)(x \wedge g), \\ x \wedge g_1 g &= (x \wedge g)(x^g \wedge g_1^g), \\ x \wedge x &= 1, \end{aligned}$$

for all $x, x_1 \in N$ and $g, g_1 \in G$. It is a common practice in the theory of nonabelian exterior (or tensor) products that groups act upon each other from the left. For our purposes it is convenient to use the so called ‘right hand action’ notation. From the defining relations it becomes clear that G acts on $N \wedge G$ via $(x \wedge g)^h = x^h \wedge g^h$, where $x \in N$ and $g, h \in G$. Moreover, the commutator map $\kappa_{N,G} : N \wedge G \rightarrow [N, G]$, defined by $x \wedge g \mapsto [x, g]$, is a surjective homomorphism of groups. What relates this construction to Schur multipliers of groups is a result of Miller [20] which states that $H_2(G)$ is isomorphic to $\ker \kappa_{G,G}$ for any group G . Furthermore, Brown, Johnson, and Robertson [5] proved that when G is a finite group, $G \wedge G$ is isomorphic to the derived group of any covering group of G . Thus it suffices to find an estimate for the exponent of the group $G \wedge G$ to obtain a bound for $\exp H_2(G)$.

Let G be a finite n -Engel p -group of exponent p^e . Let r be the integer satisfying $p^{r-1} < n \leq p^r$. Let $N = \mathcal{U}_r(G)$ if p is odd, or $N = \mathcal{U}_{r+1}(G)$ when $p = 2$. Here we use the notation $\mathcal{U}_k(G) = \langle g^{p^k} : g \in G \rangle$. By a result of Abdollahi and Traustason [1], we have that N is a powerful p -group, i.e., $N/\mathcal{U}_1(N)$ is abelian if p is odd or

$N/\mathcal{U}_2(N)$ is abelian when $p = 2$ (we refer to a paper of Lubotzky and Mann [17] for further details). By [7] we have an exact sequence

$$(2.1.1) \quad N \wedge G \longrightarrow G \wedge G \longrightarrow G/N \wedge G/N \longrightarrow 1.$$

Denote

$$f(p, e, n) = \sup\{\exp G \wedge G \mid G \text{ is a finite } n\text{-Engel } p\text{-group of exponent } p^e\}.$$

From the short exact sequence

$$0 \longrightarrow H_2(G) \longrightarrow G \wedge G \longrightarrow G' \longrightarrow 1$$

and [22] it follows that $f(p, e, n) < \infty$, moreover, $f(p, e, n)$ can be bounded in terms of p and e only (this can also be inferred from a result of Mann [19]). Note that the group G/N has exponent dividing p^r if p is odd, or 2^{r+1} when $p = 2$. The sequence (2.1.1) thus implies that $\exp(G \wedge G)$ divides $\exp(N \wedge G)f(p, r, n)$ if p is odd, or $\exp(N \wedge G)f(2, r+1, n)$ if $p = 2$. Note also that since N is powerful, we have that $\mathcal{U}_i(N) = \mathcal{U}_{r+i}(G)$ for all i in the case when p is odd, and $\mathcal{U}_i(N) = \mathcal{U}_{r+i+1}(G)$ when $p = 2$. Thus $\exp N$ divides p^{e-r} . A result of Ellis [7, Proposition 9] gives the following theorem.

Theorem 2.2. *Let G be a finite n -Engel p -group and let the function f be defined as above. Denote by r be the integer satisfying $p^{r-1} < n \leq p^r$, and let g be equal to $f(p, r, n)$ if p is odd, or to $f(2, r+1, n)$ if $p = 2$. Let $N = \mathcal{U}_r(G)$ if p is odd, or $N = \mathcal{U}_{r+1}(G)$ if $p = 2$. For each $1 \leq i \leq e - r$ let k_i be the smallest integer such that $[\mathcal{U}_{i-1}(N),_{k_i} G] \leq \mathcal{U}_i(N)$. Then*

$$\exp H_2(G) \text{ divides } p^{\lceil k_1/2 \rceil + \dots + \lceil k_{e-r}/2 \rceil} g.$$

Note that the number g in Theorem 2.2 is bounded in terms of p and n . On the other hand, the values of k_i depend on the action of G upon N . For instance, if N is powerfully embedded in G , then $k_i = 1$ for all i [17].

Precise values of f are also known in some cases. For instance, Mann [19] showed that $f(2, 1, n) = 2$ and $f(3, 1, n) = 3$. It is also known that $f(2, 2, n) = 4$ for $n \leq 3$ and $f(2, 2, n) = 8$ for $n > 3$ [22], and that $f(5, 1, n) = 25$ for n large enough [24].

3. TWO-GENERATOR CENTER-BY-4-ENGEL GROUPS

Havas and Vaughan-Lee [11] proved that 4-Engel groups are locally nilpotent. It follows from here that, in principle, one can construct relatively free r -generator center-by-4-Engel groups using Nickel's nilpotent quotient algorithm implemented in GAP [9]. In reality however, free center-by-4-Engel groups of rank ≥ 3 seem to be out of reach of GAP. On the other hand, the free 2-generator center-by-4-Engel group C can be constructed within minutes. Relations satisfied by this group represent two variable identities satisfied by any center-by-4-Engel group.

By [12], finite 4-Engel 5-groups are regular. A corresponding result for center-by-4-Engel groups is that every finite center-by-4-Engel 7-group is regular. This follows from the following lemma that can be proved by straightforward computation in the group C using GAP.

Lemma 3.1. *Let G be a center-by-4-Engel group and $x, y \in G$. Then*

$$\begin{aligned} (xy)^7 &= x^7 y^7 [y, x, x]^{21} [y, x, x]^{35} [y, x, y]^{91} [y, x, x, x]^{35} [y, x, y, x]^{175} [y, x, y, y]^{175} \\ &\quad \times [y, x, x, x, x]^{21} [y, x, x, x, y]^{-42} [y, x, y, x, x]^{231} [y, x, y, x, y]^{-546} [y, x, y, y, x]^{917} \\ &\quad \times [y, x, y, y, y]^{189} [y, x, y, x, x, x]^{-7} [y, x, y, y, x, x]^{3031} [y, x, y, y, x, y]^{28} \\ &\quad \times [y, x, y, x, x, y, x]^{-21} [y, x, y, y, x, x, x]^{42} [y, x, y, y, x, x, y]^{-3269}. \end{aligned}$$

4. 4-ENGEL GROUPS

As already mentioned in the introduction, our approach to estimating $\exp H_2(G)$ is to obtain a bound for the exponent of H' , where H is a covering group. Note that H is a central extension of G . In this section we deal with the case when G is a 4-Engel group. Thus we first determine the power structure of finite center-by-4-Engel p -groups. Our main result is the following.

Theorem 4.1. *Let H be a finite p -group and let Z be a central subgroup of H such that $G = H/Z$ is a 4-Engel group. If $p \neq 2, 5$, then $\mathcal{U}_e(H') \leq [\mathcal{U}_e(H), H]$. For $p = 2$ or $p = 5$ we have that $\mathcal{U}_{e+1}(H') \leq [\mathcal{U}_e(H), H]$.*

Before embarking on the proof, we mention the following consequence of this result:

Corollary 4.2. *Let G be a 4-Engel group of exponent e . If e is not divisible by 2 or 5, then $\exp H_2(G)$ divides e . Otherwise, $\exp H_2(G)$ divides $10e$.*

Proof. Since the functor H_2 commutes with direct limits [3, Proposition I.5.10], we may assume that G is finitely generated. As 4-Engel groups are locally nilpotent [11], it follows that G is finite. For each prime p , the p -part of the abelian group $H_2(G)$ embeds into $H_2(P)$, where P is any Sylow p -subgroup of G [15, Satz V.25.1]. Let e have the prime decomposition $e = p_1^{e_1} \cdots p_k^{e_k}$. Given $i \in \{1, \dots, k\}$, let P_i be a Sylow p_i -subgroup of G and let H_i be a covering group of P_i . Then H_i is center-by-4-Engel and $[\mathcal{U}_{e_i}(H_i), H_i] = 1$. From Theorem 4.1 it follows that $\mathcal{U}_{e_i}(H'_i) = 1$ when $p_i \neq 2, 5$, and $\mathcal{U}_{e_i+1}(H'_i) = 1$ when $p_i \in \{2, 5\}$. As $H_2(P_i)$ can be embedded into H'_i , it follows that $\exp H_2(P_i)$ divides $p_i^{e_i}$ for $p_i \neq 2, 5$, or $p_i^{e_i+1}$ when $p_i \in \{2, 5\}$. This proves the assertion. \square

Let H be a finite p -group and let Z be a central subgroup of H such that the group $G = H/Z$ is 4-Engel. If $p \geq 7$, then G is nilpotent of class ≤ 7 by a result of Traustason [26, Corollary 1]. In this case, the nilpotency class of H does not exceed 8. If $p > 7$, then H is regular by [15, Satz III.10.2]. For $p = 7$, H is also regular by Lemma 3.1. Using [15, Satz III.10.8], we conclude that $\mathcal{U}_e(H') = [\mathcal{U}_e(H), H]$ for $p \geq 7$. Thus it suffices to prove Theorem 4.1 for 2-groups, 3-groups and 5-groups. We may assume that $[\mathcal{U}_e(H), H] = 1$, and it then suffices to prove that $\mathcal{U}_e(H') = 1$ when $p = 3$, and $\mathcal{U}_{e+1}(H') = 1$ for $p \in \{2, 5\}$. Before going into particular cases, we mention some auxiliary results.

Lemma 4.3 ([25]). *Let E_r be a free 4-Engel group on r generators.*

- (a) E_2 is torsion-free and nilpotent of class 6.
- (b) $\gamma_8(E_3)^{30} = \gamma_9(E_3)^3 = \gamma_{10}(E_3) = 1$.

Lemma 4.4. *Let $a, b \in H$. Then*

$$[a^k, b] = [a, b]^k [a, b, a]^{\binom{k}{2}} [a, b, a, a]^{\binom{k}{3}} [a, b, a, [a, b]]^{\binom{k}{2} + 2\binom{k}{3}} [a, b, a, a, a]^{\binom{k}{4}} \\ \times [a, b, a, a, [a, b]]^{2\binom{k}{3} + 3\binom{k}{4}} [a, b, a, [a, b], [a, b]]^{2\binom{k}{3} + 3\binom{k}{4}}$$

for all nonnegative integers k .

Proof. The group $\langle a, [a, b] \rangle = \langle a, a^b \rangle$ is center-by-metabelian and nilpotent of class ≤ 5 ; this follows from a result of Traustason [27], or from the presentation of the free 2-generator center-by-4-Engel group (see Section 3). Expanding $(a^b)^k = (a[a, b])^k$ using the collection process [15, Satz III.9.3], we get the result. \square

4-Engel 5-groups. Let H be a 5-group satisfying the above assumptions. First of all we note that Lemma 4.3 implies that every 2-generator subgroup of H is nilpotent of class ≤ 7 , and every 3-generator subgroup is nilpotent of class ≤ 9 . Moreover, if $r > 3$, every r -generator subgroup of H is nilpotent of class $\leq 2r + 1$ by [12, Theorem 3.3].

Let $x, y \in H$. As $\langle x, y \rangle$ has class ≤ 7 , we have $1 = [[x_1, x_2, x_3, x_4]^{5^e}, x_5] = [x_1, x_2, x_3, x_4, x_5]^{5^e}$ for all $x_i \in \langle x, y \rangle$. Thus we conclude that $\mathcal{U}_e(\gamma_5(\langle x, y \rangle)) = 1$. Lemma 4.4 now yields

$$(4.4.1) \quad 1 = [x, y]^{5^e} [x, y, x]^{5^e} [x, y, x, x]^{5^e}.$$

Replacing x by $[y, x]$ in (4.4.1), we get $[y, x, y]^{5^e} = 1$. Thus we conclude that $[x, y]^{5^e} = 1$ for all $x, y \in H$.

Now let $x, y \in H$ and $\omega \in H'$. Then we have

$$([x, y]\omega)^{5^{e+1}} \equiv [x, y]^{5^{e+1}} \omega^{5^{e+1}} \pmod{\mathcal{U}_{e+1}(\gamma_2(\langle [x, y], \omega \rangle)) \prod_{i=1}^{e+1} \mathcal{U}_{e+1-i}(\gamma_{5^i}(\langle [x, y], \omega \rangle))}.$$

By the above argument and Lemma 4.3 we conclude that $\mathcal{U}_e(\gamma_3(\langle [x, y], \omega \rangle)) = \mathcal{U}_1(\gamma_9(\langle [x, y], \omega \rangle)) = 1$, hence the last equation gives $([x, y]\omega)^{5^{e+1}} = \omega^{5^{e+1}}$. This implies that $\mathcal{U}_{e+1}(H') = 1$, hence Theorem 4.1 is proved for $p = 5$.

It is not clear whether or not we actually have that $\mathcal{U}_e(H') \leq [\mathcal{U}_e(H), H]$ in the case when $p = 5$. A potential counterexample would have to be, in a sense, rather large, as the following argument shows. Note that for every $x, y, z, w \in H$ we have that $K = \langle x, y, z, w \rangle$ is nilpotent of class ≤ 9 [12], hence K' is nilpotent of class ≤ 4 . It follows that K' is regular [15, Satz III.10.2], therefore $\mathcal{U}_e(K') \leq [\mathcal{U}_e(K), K]$. Thus a counterexample, if it exists, would have to be at least a 5-generator center-by-4-Engel 5-group of class ≥ 10 . These groups seem to be out of reach of the computational functionality currently available.

4-Engel 3-groups. If H is a center-by-4-Engel 3-group, then every two-generator subgroup of H has class at most 7, whereas every three-generator subgroup of K of H has class ≤ 10 and $\mathcal{U}_1(\gamma_9(K)) = 1$ by Lemma 4.3. As groups of exponent 3 are nilpotent of class ≤ 3 , we may assume that $e > 1$. We will also need the following result.

Lemma 4.5. *Let G be a 4-Engel 3-group and $a, b, c \in G$. If $\exp G = 3^e$, then $\mathcal{U}_{e-1}(\gamma_4(\langle a, b, c \rangle)) = 1$.*

Proof. As above we can assume that $e > 1$. Besides, we can assume without loss of generality that $\mathcal{U}_{e-1}(\gamma_5(\langle a, b, c \rangle)) = 1$. By a result of Traustason [27], the group $\langle a, [a, b] \rangle = \langle a, a^b \rangle$ has class ≤ 4 and is thus metabelian. We can expand $b^{-1}a^{3^e}b = (a[a, b])^{3^e}$ as in the proof of Lemma 4.4. Since G is 4-Engel, we get

$$\begin{aligned} 1 &= [a^{3^e}, b] \\ &= [a, [a, b]]^{3^e} [a, [a, b], [a, b]]^{3^e} [a, [a, b], a]^{3^e} \\ &= [a, [a, b], a]^{3^e}, \end{aligned}$$

whence $[a, b, a, a]^{3^{e-1}} = 1$. Replacing a by ab , we get $[a, b, a, b]^{3^{e-1}} [a, b, b, a]^{3^{e-1}} = 1$. On the other hand, the Hall-Witt identity implies that $[a, b, a, b]^{3^{e-1}} = [a, b, b, a]^{3^{e-1}}$. This implies $[a, b, a, b]^{3^{e-1}} = 1$, from which it follows that $\mathcal{U}_{e-1}(\gamma_4(\langle a, b \rangle)) = 1$ for all $a, b \in G$.

As $[a, bc, bc, bc]^{3^{e-1}} = 1$, we get

$$(4.5.1) \quad \begin{aligned} 1 &= [a, b, b, c]^{3^{e-1}} [a, b, c, b]^{3^{e-1}} [a, b, c, c]^{3^{e-1}} \\ &\quad \times [a, c, b, c]^{3^{e-1}} [a, c, c, b]^{3^{e-1}} [a, c, b, b]^{3^{e-1}}. \end{aligned}$$

Similarly, from $[a, bc, a, bc]^{3^{e-1}} = 1$ it follows that

$$(4.5.2) \quad 1 = [a, b, a, c]^{3^{e-1}} [a, c, a, b]^{3^{e-1}}.$$

Replacing c by ac in (4.5.1) and using (4.5.1) and (4.5.2), we obtain

$$(4.5.3) \quad 1 = [a, b, c, a]^{3^{e-1}} [a, c, b, a]^{3^{e-1}}.$$

By the Hall-Witt identity, the equation (4.5.3) implies

$$(4.5.4) \quad [c, a, b, a]^{3^{e-1}} = [b, c, a, a]^{3^{e-1}}.$$

Replacing a by ca in (4.5.3), we conclude that

$$(4.5.5) \quad 1 = [a, b, c, c]^{3^{e-1}} [c, b, c, a]^{3^{e-1}} [a, c, b, c]^{3^{e-1}},$$

which together with (4.5.4) implies $[c, b, c, a]^{3^{e-1}} = 1$. From here one readily obtains $[c, b, b, a]^{3^{e-1}} = 1$. Replacing a by ca in (4.5.4), we now get, after relabeling, $[c, a, b, a]^{3^{e-1}} = [a, b, c, a]^{3^{e-1}}$. In other words, if $a_i \in \{a, b, c\}$, $1 \leq i \leq 4$, then the value of $[a_1, a_2, a_3, a_4]^{3^{e-1}}$ is invariant with respect to cyclic permutations of the first three arguments of the commutator. Observe that

$$[a, b, c, a]^{3^{e-1}} [a, b, c, b]^{3^{e-1}} [a, b, c, c]^{3^{e-1}} = [a, abc, c, abc]^{3^{e-1}}.$$

Replacing a by ab and c by bc , we get $[a, b, c, a]^{3^{e-1}} [a, b, c, c]^{3^{e-1}} = 1$. If we replace c by bc in this equation, we obtain $[a, b, c, b]^{3^{e-1}} = 1$. This shows that $\mathcal{U}_{e-1}(\gamma_4(\langle a, b, c \rangle)) = 1$. \square

Let $x, y, z \in H$ and denote $K = \langle x, y, z \rangle$. From Lemma 4.5 it follows that $\mathcal{U}_{e-1}(\gamma_4(K)) \leq Z(K)$. This gives

$$\begin{aligned} 1 &= [[x_1, x_2, x_3, x_4]^{3^{e-1}}, x_5] \\ &= [x_1, x_2, x_3, x_4, x_5]^{3^{e-1}} [x_1, x_2, x_3, x_4, x_5, [x_1, x_2, x_3, x_4]]^{\binom{3^{e-1}}{2}} \\ &= [x_1, x_2, x_3, x_4, x_5]^{3^{e-1}}. \end{aligned}$$

As $\gamma_5(K)$ is nilpotent of class ≤ 2 , we obtain $\mathcal{U}_{e-1}(\gamma_5(K)) = 1$. By Lemma 4.4 we have

$$(4.5.6) \quad 1 = [x, y]^{3^e} [x, y, x]^{\binom{3^e}{2}} [x, y, x, x]^{\binom{3^e}{3}}.$$

Replacing x by $[y, x]$, we get $[y, x, y]^{3^e} = 1$, and since $\gamma_3(\langle x, y \rangle)$ is nilpotent of class ≤ 2 , this implies $\mathcal{U}_e(\gamma_3(\langle x, y \rangle)) = 1$. Therefore (4.5.6) can be rewritten as $1 = [x, y]^{3^e} [x, y, x]^{\binom{3^e}{3}}$. Replacing x by yx in this equation, we obtain $1 = [x, y, x, y]^{\binom{3^e}{3}} [x, y, y, x]^{\binom{3^e}{3}} [x, y, y, y]^{\binom{3^e}{3}}$. By the Hall-Witt identity we have that $[x, y, x, y]^{\binom{3^e}{3}} = [x, y, y, x]^{\binom{3^e}{3}}$, whence $[x, y, y, y]^{\binom{3^e}{3}} = [x, y, x, y]^{\binom{3^e}{3}}$. Replacing x by yx in this equation, we conclude that $[x, y, y, y]^{3^{e-1}} = 1$, which implies $\mathcal{U}_{e-1}(\gamma_4(\langle x, y \rangle)) = 1$. As a consequence we also obtain that $[x, y]^{3^e} = 1$. Now we prove that $\mathcal{U}_{e-1}(\gamma_4(K)) = 1$. As $[x, yz, yz, yz]^{3^{e-1}} = 1$, we get

$$(4.5.7) \quad \begin{aligned} 1 &= [x, y, y, z]^{3^{e-1}} [x, y, z, y]^{3^{e-1}} [x, y, z, z]^{3^{e-1}} \\ &\quad \times [x, z, y, z]^{3^{e-1}} [x, z, z, y]^{3^{e-1}} [x, z, y, y]^{3^{e-1}}. \end{aligned}$$

The relation $[x, yz, x, yz]^{3^{e-1}} = 1$ implies

$$(4.5.8) \quad 1 = [x, y, x, z]^{3^{e-1}} [x, z, x, y]^{3^{e-1}}.$$

If we substitute z by xz in (4.5.7) and apply (4.5.7) and (4.5.8), we arrive at

$$(4.5.9) \quad 1 = [x, y, z, x]^{3^{e-1}} [x, z, y, x]^{3^{e-1}}.$$

Using the Hall-Witt identity, we get

$$(4.5.10) \quad [z, x, y, x]^{3^{e-1}} = [y, z, x, x]^{3^{e-1}}.$$

Next we replace x by zx in (4.5.9) and get

$$(4.5.11) \quad 1 = [x, y, z, z]^{3^{e-1}} [z, y, z, x]^{3^{e-1}} [x, z, y, z]^{3^{e-1}}.$$

This $[z, y, z, x]^{3^{e-1}} = 1$, therefore also $[z, y, y, x]^{3^{e-1}} = 1$. Replacing x by zx in (4.5.10), we get $[z, x, y, x]^{3^{e-1}} = [x, y, z, x]^{3^{e-1}}$. Now we replace x by xy and z by yz in $[x, y, z, x]^{3^{e-1}} [x, y, z, y]^{3^{e-1}} [x, y, z, z]^{3^{e-1}} = [x, xyz, z, xyz]^{3^{e-1}}$. After expansion we obtain $[x, y, z, x]^{3^{e-1}} [x, y, z, z]^{3^{e-1}} = 1$. If we replace z by yz once again, we obtain $[x, y, z, y]^{3^{e-1}} = 1$. From here it follows $\mathcal{U}_{e-1}(\gamma_4(K)) = 1$.

Let $\omega \in H'$. By the Hall-Petrescu formula [15, Satz III.9.4] we get

$$([x, y]\omega)^{3^e} = \omega^{3^e} [\omega, [x, y], [x, y]]^{\binom{3^e}{3}} [\omega, [x, y], \omega]^{-\binom{3^e}{3}} = \omega^{3^e},$$

hence $\mathcal{U}_e(H') = 1$.

4-Engel 2-groups. Assume now that $p = 2$. Then Lemma 4.3 implies that every two-generator subgroup of H is nilpotent of class ≤ 7 , and every three-generator subgroup has class ≤ 9 and the ninth term of the lower central series of exponent 2. It is clear that the result holds true if $e = 1$, and when $e = 2$, Theorem 4.1 holds true even without assuming that G is 4-Engel [22]. Thus we may assume that $e > 2$.

Let $x, y \in H$. At first note that, as above, $\mathcal{U}_e(\gamma_5(\langle x, y \rangle)) = 1$. We claim that $\mathcal{U}_{e-1}(\gamma_5(\langle x, y \rangle)) = 1$. Because of the class restriction we may assume that $\mathcal{U}_{e-1}(\gamma_6(\langle x, y \rangle)) = 1$. As $[x, y, x]$ commutes with $[x, y, x, x]$, expansion of $[x^{2^e}, y, x] = 1$ using Lemma 4.4 yields

$$(4.5.12) \quad [x, y, x, x]^{\binom{2^e}{2}} [x, y, x, x]^{\binom{2^e}{3}} = 1.$$

Commuting (4.5.12) with x and y , respectively, we get

$$(4.5.13) \quad [x, y, x, x]^{2^{e-1}} = [x, y, x, x]^{2^{e-1}} = 1.$$

Similarly, expanding $[x^{2^e}, y, y] = 1$, we get $[x, y, x, y]^{2^{e-1}} = 1$, and this, together with (4.5.13), clearly implies that $\mathcal{U}_{e-1}(\gamma_5(\langle x, y \rangle)) = 1$. Moreover, from (4.5.12) we now conclude that $[x, y, x, x]^{2^{e-1}} = 1$. As $\gamma_4(\langle x, y \rangle)$ is abelian, we therefore have that $\mathcal{U}_{e-1}(\gamma_4(\langle x, y \rangle)) = 1$. Applying Lemma 4.4 for $[x^{2^{e+1}}, y] = 1$, we get

$$(4.5.14) \quad [x, y]^{2^{e+1}} [x, y, x]^{\binom{2^{e+1}}{2}} = 1.$$

Replacing x by yx in the equation (4.5.14), we see that $[x, y]^{2^{e+1}} = [x, y, x]^{2^e} = 1$. From here it is not difficult to conclude that $\mathcal{U}_e(\gamma_3(\langle x, y \rangle)) = 1$.

Now let $x, y \in H$ and $\omega \in H'$. Expanding $([x, y]\omega)^{2^{e+1}}$ using the Hall-Petrescu formula [15] and above relations, we get

$$(4.5.15) \quad ([x, y]\omega)^{2^{e+1}} = \omega^{2^{e+1}} [\omega, [x, y]]^{\binom{2^{e+1}}{2}}.$$

We prove that for any $x, y, z \in H$, $[x, y, z]$ has order dividing 2^e . At first we show that $\mathcal{U}_{e-1}(\gamma_5(\langle x, y, z \rangle)) = 1$. To do this we need the following lemma.

Lemma 4.6. *Let G a 4-Engel 2-group of exponent 2^e . Then $\mathcal{U}_{e-1}(\gamma_4(\langle a, b, c \rangle)) = 1$ for all $a, b, c \in G$.*

Proof. Let $a, b, c \in G$ and put $K = \langle a, b, c \rangle$. By Lemma 4.3 we have $\mathcal{U}_1(\gamma_8(K)) = \gamma_9(K) = 1$. For $e = 1$ the conclusion is clear, for $e = 2$ it follows directly from the polycyclic presentation of $B(3, 4)$. So we may assume that $e > 2$. Because of the class restriction we may assume without loss of generality that $\mathcal{U}_{e-1}(\gamma_5(K)) = 1$. Since $\mathcal{U}_{e-1}(\gamma_4(\langle ab, c \rangle)) = 1$ (see above), we have

$$\begin{aligned} 1 &= [ab, c, ab, c]^{2^{e-1}} \\ &= [a, c, b, c]^{2^{e-1}} [b, c, a, c]^{2^{e-1}} \\ &= [a, b, c, c]^{2^{e-1}}, \end{aligned}$$

Replacing c by ac in this relation, we get

$$(4.6.1) \quad [a, b, a, c]^{2^{e-1}} [a, b, c, a]^{2^{e-1}} = 1.$$

This equation is equivalent to

$$(4.6.2) \quad [a, b, b, c]^{2^{e-1}} [a, b, c, b]^{2^{e-1}} = 1.$$

By a result of Traustason [27], the group $\langle a, [a, b] \rangle = \langle a, a^b \rangle$ has class ≤ 4 and is thus metabelian. Expanding $b^{-1}a^{2^e}b = (a[a, b])^{2^e}$ as in the proof of Lemma 4.4 and using the fact that G is 4-Engel, we get

$$\begin{aligned} 1 &= [a^{2^e}, b] \\ &= [a, [a, b]]^{\binom{2^e}{2}} [a, [a, b], [a, b]]^{\binom{2^e}{3}} [a, [a, b], a]^{\binom{2^e}{3}} \\ &= [a, [a, b]]^{\binom{2^e}{2}}, \end{aligned}$$

hence $[a, b, a]^{2^{e-1}} = [a, b, b]^{2^{e-1}} = 1$. Now we have

$$1 = [[a, b, a]^{2^{e-1}}, c] = [a, b, a, c]^{2^{e-1}} [a, b, a, c, [a, b, a]]^{\binom{2^{e-1}}{2}}.$$

From the polycyclic presentation of E_3 we see that $[a, b, a, c, [a, b, a]] = 1$, therefore also $[a, b, a, c]^{2^{e-1}} = 1$. Similarly we get that $[a, b, b, c]^{2^{e-1}} = 1$. The equations (4.6.1) and (4.6.2) now imply that $[a, b, c, a]^{2^{e-1}} = [a, b, c, b]^{2^{e-1}} = 1$. This concludes the proof. \square

This result in particular implies that $\mathcal{U}_{e-1}(\gamma_4(\langle x, y, z \rangle)) \leq Z(\langle x, y, z \rangle)$. For any x_1, x_2, x_3, x_4 and x_5 in $\langle x, y, z \rangle$ we have

$$\begin{aligned} 1 &= [[x_1, x_2, x_3, x_4]^{2^{e-1}}, x_5] \\ &= [x_1, x_2, x_3, x_4, x_5]^{2^{e-1}} [x_1, x_2, x_3, x_4, x_5, [x_1, x_2, x_3, x_4]]^{\binom{2^{e-1}}{2}} \\ &= [x_1, x_2, x_3, x_4, x_5]^{2^{e-1}}, \end{aligned}$$

since $e > 2$ and $\mathcal{U}_1(\gamma_9(\langle x, y, z \rangle)) = 1$. It follows that $\mathcal{U}_{e-1}(\gamma_5(\langle x, y, z \rangle)) = 1$, as required.

Expanding $1 = [[x, y]^{2^e}, z]$ using Lemma 4.4, we get

$$(4.6.3) \quad 1 = [x, y, z]^{2^e} [x, y, z, [x, y]]^{\binom{2^e}{2}} [x, y, z, [x, y], [x, y], [x, y]]^{\binom{2^e}{4}}.$$

As H is center-by-4-Engel, we have that $[x, y, z, [x, y], [x, y], [x, y]] = 1$. Furthermore, since $\mathcal{U}_{e-1}(\gamma_5(\langle x, y, z \rangle)) = 1$, the equation (4.6.3) becomes $[x, y, z]^{2^e} = 1$, as required. It follows now from (4.5.15) that $([x, y]\omega)^{2^{e+1}} = \omega^{2^{e+1}}$ which clearly implies that $\mathcal{U}_{e+1}(H') = 1$. This concludes the proof.

5. 3-ENGEL GROUPS

In this section we show that for every 3-Engel group G of finite exponent, the exponent of $H_2(G)$ divides $\exp G$. Standard argument shows that we may assume that G is a finite p -group. Furthermore, Corollary 4.2 shows that we only need to consider the cases when $p = 2$ or $p = 5$. At first we deal with 5-groups.

Theorem 5.1. *Let H be a finite 5-group and let Z be a central subgroup of H such that $G = H/Z$ is a 3-Engel group. Then $\mathcal{U}_e(H') \leq [\mathcal{U}_e(H), H]$.*

Proof. Assume that $[\mathcal{U}_e(H), H] = 1$. Let $x, y \in H$. As every 2-generator 3-Engel 2-torsion-free group is nilpotent of class ≤ 3 [13], the group $\langle x, y \rangle$ is nilpotent of class ≤ 4 . Thus

$$(5.1.1) \quad 1 = [x, y^{5^e}] = [x, y]^{5^e} [x, y, y]^{5^e} [x, y, y, y]^{5^e}.$$

Commuting (5.1.1) with y , we get that $[x, y]^{5^e} = 1$. For $\omega \in H'$ we now get

$$\begin{aligned} (\omega[x, y])^{5^e} &= \omega^{5^e} \prod_{0 < i+j < 5^e} [\omega, {}_i[y, x], {}_j\omega]^{5^e} [y, x]^{5^e} \\ &= \omega^{5^e}. \end{aligned}$$

It follows that $\mathcal{U}_e(H') = 1$, as required. \square

It remains to consider 2-groups. We prove the following result.

Theorem 5.2. *Let H be a finite 2-group and let Z be a central subgroup of H such that $G = H/Z$ is a 3-Engel group. Then $\mathcal{U}_e(H') \leq [\mathcal{U}_e(H), H]$.*

Lemma 5.3. *Let H be as in Theorem 5.2.*

- (a) *Every r -generator subgroup of H is nilpotent of class $\leq r + 3$.*
- (b) *All commutators in H with a triple entry are central.*
- (c) *H satisfies the laws $[x, y, z, [w, t], u, v] = 1$ and $[x, y, z, w, [x, y, z]] = 1$.*

Proof. As $H/Z(H)$ is a 3-Engel 2-group, (a) follows directly from [10, Theorem 4.1 (a)], whereas (b) is a direct consequence of the fact that in every 3-Engel group, all commutators with a triple entry are trivial [10, Corollary 2.5]. The first law of (c) is a consequence of [10, Theorem 4.1 (b)]. To prove the second law, one can use the nilpotent quotient algorithm implemented in GAP to construct the largest class 7 quotient Q of the free 4-generator group satisfying the law $[x, y, y, y, z] = 1$. Then it can be verified that $[x, y, z, w, [x, y, z]]^5 = 1$ is a law in Q . As H is a 2-group, the assertion is proved. \square

Lemma 5.4. *Let G be a 3-Engel group of exponent 2^e and $a, b, c \in G$. Then $[a, b, c]^{2^{e-1}} = 1$.*

Proof. Since the group $\langle c, [a, b] \rangle$ is nilpotent of class ≤ 4 [13], it is metabelian. Then

$$\begin{aligned} 1 &= (c[a, b])^{2^e} \\ &= \prod_{0 < i+j < 2^e} [c, {}_i[b, a], {}_j c]^{2^e} \\ &= [c, [b, a]]^{2^e}, \end{aligned}$$

and the result follows. \square

Proof of Theorem 5.2. Assume that $[\mathcal{U}_e(H), H] = 1$. For $x, y \in H$ we have that $\langle x, y \rangle$ is nilpotent of class ≤ 5 . Since $[x^{2^e}, y] = 1$, we can apply [24, Lemma 9] to obtain

$$(5.4.1) \quad 1 = [x, y]^{2^e} [x, y, x]^{2^e} [x, y, x, x]^{2^e} [x, y, x, [x, y]]^{\alpha(2^e)},$$

where $\alpha(n) = n(n-1)(2n-1)/6$. Commuting (5.4.1) with $z \in H$, we get $[x, y, x, z]^{2^{e-1}} = 1$. Thus the equation (5.4.1) can be rewritten as

$$(5.4.2) \quad 1 = [x, y]^{2^e} [x, y, x]^{2^e}.$$

Replacing x by yx in (5.4.2) and using (5.4.2), we see that $[x, y, y]^{2^{e-1}} = 1$, hence also $[x, y, x]^{2^{e-1}} = 1$ and $[x, y]^{2^e} = 1$. Now let $\omega \in H'$. We can expand $(\omega[x, y])^{2^e}$ using the collection process and above obtained relations. Since the class of $\langle \omega, [x, y] \rangle$ is at most 5, we get (cf. [24, Lemma 9])

$$(5.4.3) \quad (\omega[x, y])^{2^e} = \omega^{2^e} [x, y, \omega]^{2^e} [x, y, \omega, \omega]^{2^e} [x, y, \omega, \omega, [x, y]]^{3 \cdot 2^e}.$$

The commutator $[x, y, \omega, \omega, \omega]$ is a product of conjugates of commutators of the form $[x, y, [x_1, y_1], [x_2, y_2], [x_3, y_3]]^{\pm 1}$, where $x_i, y_i \in H$. By Lemma 5.3, the commutator $[x, y, [x_1, y_1], [x_2, y_2]]$ belongs to $Z_2(H)$, therefore $[x, y, [x_1, y_1], [x_2, y_2], [x_3, y_3]] = 1$. It follows that $[x, y, \omega, \omega, \omega] = 1$. A similar argument also shows that the commutator $[x, y, \omega, \omega, [x, y]]$ is trivial. Let $z, w \in H$. Using Lemma 5.3 and Lemma 5.4, we get

$$\begin{aligned} 1 &= [[x, y, z]^{2^{e-1}}, w] \\ &= [x, y, z, w]^{2^{e-1}} [x, y, z, w, [x, y, z]]^{2^{e-1}} \\ &= [x, y, z, w]^{2^{e-1}}. \end{aligned}$$

By the Hall-Witt identity this implies $[[x, y], [z, w]]^{2^{e-1}} = 1$. As $H''' = 1$ by Lemma 5.3, it follows that $[x, y, w]^{2^{e-1}} = 1$. Hence (5.4.3) implies $(\omega[x, y])^{2^e} = \omega^{2^e}$. It follows from here that $\mathcal{U}_e(H') = 1$, as required. \square

As in Section 4 we get the following result.

Corollary 5.5. *Let G be a 3-Engel group of exponent e . Then $\exp H_2(G)$ divides e .*

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