ON PRO-p GROUPS WITH POTENT FILTRATIONS

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Abstract. In this note we prove that if G is PF-group of finite exponent, then the exponent of the second homology group $H_2(G,M)$ divides the exponent of G for every profinite trivial $[\![\hat{\mathbb{Z}}G]\!]$-module M. We introduce the notion of the exponential rank of a pro-p group, and find a bound for the exponential rank of a PF-group.

1. Introduction

In 1987, Lubotzky and Mann [10, 11] introduced the notion of powerful p-groups and powerful pro-p groups. These groups had been implicitly studied before by Lazard [8] and Arganbright [2]. Powerful groups have a particularly nice power-commutator structure, and have had an important role in the theory of finite p-groups and pro-p groups. In their paper [10], Lubotzky and Mann obtained some properties of the Schur multiplier $H_2(G,\mathbb{Z})$ of a powerful p-group G. In particular, they showed that if G is a powerful p-group, then the exponent of $H_2(G,\mathbb{Z})$ divides the exponent of G. The question whether $\exp H_2(G,\mathbb{Z})$ divides $\exp G$ for every finite group seems to have been a longstanding open problem, probably going all the way back to Schur. It is now known that the answer is negative in general, see, for example, [13]. On the other hand, the counterexamples seem to be quite rare. It is still not known whether or not there exists a finite group G of odd order such that $\exp H_2(G,\mathbb{Z})$ does not divide $\exp G$.

Recently, Fernández-Alcober, González-Sanchez, and Jaikin-Zapirain [4] defined a new family of pro-p groups, the so called PF-groups. These groups generalize the concepts of powerful pro-p groups and potent pro-p groups [6]. They have been used successfully in studying the power structure of pro-p groups [4]. Furthermore, González-Sanchez [5] proved that a torsion-free pro-p group is a PF-group if and only if it is p-saturable (in the sense of Lazard). The purpose of this paper is to study the power structure of central extensions of PF-groups. As a consequence we generalize the above mentioned result of Lubotzky and Mann by proving that if G is a PF-group of finite exponent, then $\exp H_2(G,M)$ divides $\exp G$ for every profinite trivial $[\![\hat{\mathbb{Z}}G]\!]$-module M. This also generalizes a result of Ellis [3]. In the second part of the paper we follow the approach from [13] and define the exponential rank $\operatorname{exprank}(G)$ of a center-by-finite-exponent pro-p group G. We first examine the relationship between the exponential rank of a pro-p group and exponential rank of its finite quotients. Then we prove that if G is a PF-group, then $\operatorname{exprank}(G)\le 1$. We show by an example that this estimate is best possible. When G is potent, then this result can be further refined. We namely show that potent pro-p groups have zero exponential rank if p is odd. When $p=2$, the exponential rank is precisely 1 unless the group in question is abelian.

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A word about the notations. If G is a pro-p group, then all the subgroups will be considered in a topological sense, i.e., as topological closures of corresponding abstract subgroups. For other unexplained notations we refer to the book of Ribes and Zaleskii [14], and [4].

2. Central extensions of PF-groups and homology

Let G be a pro-p group. Following [4], we say that a descending chain $(N_i)_{i\in\mathbb{N}}$ of closed subgroups of G is a *potent filtration* of G if its intersection $\cap_{i\in\mathbb{N}}N_i$ is trivial, and $[N_i,G]\le N_{i+1}$ and $[N_i,{}_{p-1}G]\le N_{i+1}^p$ for all $i\in\mathbb{N}$. A subgroup N of G is said to be *PF-embedded* in G if there is a potent filtration of G starting at N. We also say that G is a *PF-group* if it is PF-embedded in itself. The notion of PF-groups is a generalization of that of potent pro-p groups [6], and powerful pro-p groups [10].

The main result of this section is the following.

Theorem 2.1. *Let G be a PF pro-p group and let H be a pro-p group with $Z\le Z(H)$ such that $H/Z\cong G$. Then $[H^{p^i},H]=[H,H]^{p^i}$ for all nonnegative integers i.*

Before proving this theorem, we mention the following two auxiliary results proved in [4].

Lemma 2.2 ([4]). *Let G be a pro-p group and let M and N be closed normal subgroups of G. Then*

$$[N^{p^k},M]\equiv[N,M]^{p^k}\mod\prod_{i=1}^{k}[M,{}_{p^i}N]^{p^{k-i}}$$

for all nonnegative integers k.

Lemma 2.3 ([4]). *Let G be a pro-p group, and N a PF-embedded subgroup of G. Then we have the following:*

(a) *N/K is PF-embedded in G/K for every closed normal subgroup K of G.*
(b) *Both N^p and $[N,G]$ are PF-embedded in G.*
(c) *$[N^{p^i},G^{p^j}]=[N,G]^{p^{i+j}}$ for all $i,j\ge 0$.*
(d) *$(N^{p^i})^{p^j}=N^{p^{i+j}}$ for all $i,j\ge 0$.*

Proof of Theorem 2.1. Let G, H, and Z be as above. Let $\mathfrak{U}$ be the collection of all open normal subgroups of H. Let $U\in\mathfrak{U}$. Then ZU/U is a central subgroup of H/U, and $(H/U)/(ZU/U)\cong H/ZU$ is a PF pro-p group. If we prove that the conclusion of the theorem holds true for all H/U, where $U\in\mathfrak{U}$, then $[H^{p^i},H]U=[H,H]^{p^i}U$ for all $U\in\mathfrak{U}$, and therefore $[H^{p^i},H]=[H,H]^{p^i}$. Thus in order to prove that $[H^{p^i},H]=[H,H]^{p^i}$, it suffices to show this for every finite quotient of H, therefore we may assume without loss of generality that H is a finite p-group. Let $G=N_1\ge N_2\ge\cdots\ge N_k=1$ be a potent filtration of G. Taking preimages in H, we obtain a descending chain $H=M_1\ge M_2\ge\cdots\ge M_k=Z$ of closed subgroups of H such that $[M_i,H]\le M_{i+1}$, and $[M_i,{}_{p-1}H]\le M_{i+1}^pZ$ for all $i=1,\dots,k$. The last condition implies that

$$[M_i,{}_pH]\le[M_{i+1}^p,H] \tag{2.3.1}$$

for all $i=1,\dots,k$. We claim that $([M_i,H])_{i\in\mathbb{N}}$ is a potent filtration for H. The only nontrivial thing to be verified is that $[M_i,{}_pH]\le[M_{i+1},H]^p$. Using Lemma 2.2, we get $[M_{i+1}^p,H]\le[M_{i+1},H]^p[M_{i+1},{}_pH]\le[M_{i+1},H]^p[M_{i+2}^p,H]$. By induction, $[M_{i+1}^p,H]\le[M_{i+1},H]^p[M_{i+j}^p,H]$ for all $j\ge 1$. As $M_k=Z$, we conclude that $[M_{i+1}^p,H]\le[M_{i+1},H]^p$, hence also $[M_i,{}_pH]\le[M_{i+1},H]^p$, as required.

We now claim that $[M_i^{p^j}, H] = [M_i, H]^{p^j}$ for all positive integers i and j. We prove this by induction on j. The above argument implies that $[M_i^p, H] \leq [M_i, H]^p$. On the other hand, Lemma 2.2 gives $[M_i, H]^p \leq [M_i^p, H][M_i, {}_p H] \leq [M_i^p, H][M_{i+1}^p, H] = [M_i^p, H]$, therefore $[M_i^p, H] = [M_i, H]^p$. Suppose now that $[M_i^{p^r}, H] = [M_i, H]^{p^r}$ for all positive integers i and $r < j$, where $j > 1$. We have that

$$[M_i^{p^j}, H] \equiv [M_i, H]^{p^j} \mod \prod_{\ell=1}^{j} [H, {}_{p^\ell} M_i]^{p^{j-\ell}},$$

by Lemma 2.2. As $[M_i, H]$ are PF-embedded in H, induction argument gives that $[M_i, {}_{t(p-1)+1} H] \leq [M_i, H]^{p^t}$ for all $t \geq 0$. As $p^\ell \geq \ell(p-1)+1$ for all $\ell \geq 1$, we therefore conclude that $[H, {}_{p^\ell} M_i]^{p^{j-\ell}} \leq [M_i, {}_{p^\ell} H]^{p^{j-\ell}} \leq ([M_i, H]^{p^\ell})^{p^{j-\ell}} = [M_i, H]^{p^j}$. This shows that $[M_i^{p^j}, H] \leq [M_i, H]^{p^j}$. To prove the reverse inclusion, note first that $p^\ell \geq \ell(p-1)+2$ for all $\ell \geq 2$, therefore $[H, {}_{p^\ell} M_i]^{p^{j-\ell}} \leq [M_i, {}_{p^\ell} H]^{p^{j-\ell}} \leq [[M_i, {}_{\ell(p-1)+1} H], H]^{p^{j-\ell}} \leq [[M_i, H]^{p^\ell}, H]^{p^{j-\ell}} = [[M_i, H]^{p^j}, H] \leq [M_i^{p^j}, H]$ for all $\ell \geq 2$. It remains to consider $[H, {}_p M_i]^{p^{j-1}}$. We clearly have that $[H, {}_p M_i]^{p^{j-1}} \leq [M_i, {}_p H]^{p^{j-1}} \leq [M_{i+1}^p, H]^{p^{j-1}} \leq [M_i^p, H]^{p^{j-1}}$ by (2.3.1). Let us prove that the equation (2.3.1) still holds when M_i are replaced by M_i^p throughout. We prove this by reverse induction on i. Using Lemma 2.2 and induction assumption, we get $[M_i^p, {}_p H] = [M_i, {}_p H]^p \leq [M_{i+1}^p, H]^p \leq [(M_{i+1}^p)^p, H][H, {}_p M_{i+1}^p] \leq [(M_{i+1}^p)^p, H][M_{i+1}^p, {}_p H] \leq [(M_{i+1}^p)^p, H][(M_{i+2}^p)^p, H] = [(M_{i+1}^p)^p, H]$, as required. Thus can apply the induction assumption on j to conclude that $[H, {}_p M_i]^{p^{j-1}} \leq [M_i^p, H]^{p^{j-1}} = [(M_i^p)^{p^{j-1}}, H]$. By Lemma 2.3 we have that the equality $(N_i^p)^{p^{j-1}} = N_i^{p^j}$ holds, hence $(M_i^p)^{p^{j-1}} Z = M_i^{p^j} Z$. Commuting with H, we get $[(M_i^p)^{p^{j-1}}, H] = [M_i^{p^j}, H]$. This concludes the proof. □

The above result has the following consequence for the homology of PF-groups.

Corollary 2.4. *Let G be a PF pro-p group of finite exponent and let M be a profinite trivial $[\![\hat{\mathbb{Z}}G]\!]$-module. Then $\exp H_2(G, M)$ divides $\exp G$.*

Proof. First assume that G is finite. Applying Theorem 2.1 to a covering group of G, we get that $\exp H_2(G, \mathbb{Z})$ divides $\exp G$. Let M be a trivial $\mathbb{Z}G$-module. Then the Universal Coefficient Theorem implies that $H_2(G, M) \cong (H_2(G, \mathbb{Z}) \otimes M) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(G^{\mathrm{ab}}, M)$, hence $\exp H_2(G, M)$ divides $\exp G$. This proves the theorem in the finite case. As for the pro-p case, let $\mathcal{U}$ be the collection of open normal subgroups of G, and M a profinite trivial $[\![\hat{\mathbb{Z}}G]\!]$-module. Then we have [14, Corollary 6.5.8] that

$$H_2(G, M) = \varprojlim_{U \in \mathcal{U}} H_2(G/U, M_U),$$

hence the result follows from the above conclusion. □

Corollary 2.4 also holds for potent pro-p groups, i.e, pro-p groups satisfying $\gamma_{p-1}(G) \leq G^p$ if p is odd, or $\gamma_2(G) \leq G^4$ when $p = 2$ [6]. For, it is straightforward to see that every potent pro-p group is a PF-group. Another related class of groups was considered by Ellis [3]. He introduced the class $\mathcal{C}_p$ consisting of finite p-groups G satisfying $[G^{p^{i-1}}, G, G] \leq G^{p^i}$ for all $1 \leq i \leq e$, where $\exp G = p^e$. Ellis proved that if G is a finite p-group belonging to $\mathcal{C}_p$, then $\exp H_2(G, \mathbb{Z})$ divides $\exp G$. Extending this notion, we define $\hat{\mathcal{C}}_p$ to be the class of all pro-p groups G satisfying $[G^{p^{i-1}}, G, G] \leq G^{p^i}$ for all $i \in \mathbb{N}$. It is now clear that if $p > 3$, then every $\hat{\mathcal{C}}_p$-group is potent. Thus Corollary 2.4 also applies to pro-p groups belonging to $\hat{\mathcal{C}}_p$, where $p > 3$.

3. Exponential rank

Let n be an integer. A group G is said to be *n-abelian* if it satisfies the law $(xy)^n = x^n y^n$. The study of n-abelian groups was initiated by Levi in [9]. Alperin [1] showed that if G is n-abelian for some $n \neq 0, 1$, then both $\exp G/Z(G)$ and $\exp \gamma_2(G)$ divide $n(n-1)$. Kappe [7] considered the sets $\mathcal{E}(G) = \{n \in \mathbb{Z} \mid G \text{ is } n\text{-abelian}\}$. She found arithmetic characterizations of these sets. In the case of finite p-groups these were further refined in [13].

Let G be a pro-p group and suppose that $\exp G/Z(G) = p^e$. Then $G/Z(G)$ is locally finite by a result of Zelmanov [15]. Using a result of Mann [12], we conclude that $\exp G'$ is (p, e)-bounded (Mann's result holds true for abstract groups, but can be extended to the topological setting, since taking powers is continuous). It follows that there exists $n = n(p, e) > 1$ such that G is n-abelian. Adapting the argument from [13], we have that there exists a nonnegative integer r such that $\mathcal{E}(G) = p^{e+r}\mathbb{Z} \cup (p^{e+r}\mathbb{Z} + 1)$. As in [13] we say that r is the *exponential rank* of G, and we write $r = \operatorname{exprank}(G)$. Our first result shows that there is a relationship between $\operatorname{exprank}(G)$ and the exponential rank of finite quotients of G.

Proposition 3.1. *Let G be a pro-p group with $\exp G/Z(G) = p^e$. Then*

$$s = \sup\{\operatorname{exprank}(G/U) \mid U \text{ an open normal subgroup of } G\}$$

is finite, and $\operatorname{exprank}(G) \leq s$.

Proof. Let $r = \operatorname{exprank}(G)$ and let Q be any finite quotient of G. Let $\exp Q/Z(Q) = p^f$ and $\operatorname{exprank} Q = t$. Then $f \leq e$. As G is p^{e+r}-abelian, so is Q. This implies that $t \leq r + e - f$, therefore $s < \infty$. To prove the second part, note that, by definition, Q is p^{f+t}-abelian, hence it is also p^{e+s}-abelian. Since this is true for any finite quotient of G, it follows that G is p^{e+s}-abelian. From here we conclude that $r \leq s$, as required. □

The next lemma is an elementary consequence of Hall's collection process.

Lemma 3.2. *Let G be a group and let $x, y \in G$. Then*

$$(xy)^{p^k} \equiv x^{p^k} y^{p^k} \mod \gamma_2(\langle x, y\rangle)^{p^k} \prod_{i=1}^{k} \gamma_{p^i}(\langle x, y\rangle)^{p^{k-i}}$$

for all nonnegative integers k.

Theorem 3.3. *Let G be a center-by-finite-exponent PF pro-p group. Then its exponential rank is at most 1.*

Proof. By Proposition 3.1 we may assume that G is a finite PF p-group. Let $\exp G/Z(G) = p^e$. Lemma 2.3 implies that $\exp \gamma_2(G) = p^e$. Let $G = N_1 \geq \cdots \geq N_k = 1$ be a potent filtration for G. By induction on i we can prove that $\gamma_{i(p-1)+1}(G) \leq N_{i+1}^{p^i}$ for all $i \geq 1$. Now, Lemma 3.2 gives

$$(xy)^{p^{e+1}} \equiv x^{p^{e+1}} y^{p^{e+1}} \mod \gamma_2(\langle x, y\rangle)^{p^{e+1}} \prod_{i=1}^{e+1} \gamma_{p^i}(\langle x, y\rangle)^{p^{e+1-i}} \tag{3.3.1}$$

for all $x, y \in G$. Clearly, $\gamma_2(\langle x, y\rangle)^{p^{e+1}} = 1$. Furthermore, we have that $p^i > (i-1)(p-1) + 1$ for all $i \geq 1$, hence $\gamma_{p^i}(G)^{p^{e+1-i}} \leq [\gamma_{(i-1)(p-1)+1}(G)^{p^{e+1-i}}, G] \leq$

$[(N_i^{p^{i-1}})^{p^{e+1-i}}, G] = [N_i^{p^e}, G] = [N_i, G]^{p^e} = 1$. Thus the equation (3.3.1) can be rewritten as $(xy)^{p^{e+1}} = x^{p^{e+1}} y^{p^{e+1}}$, hence $\mathrm{exprank}(G) \le 1$. □

The following example, taken from [4], shows that for each prime p there exists a finite PF p-group G with $\mathrm{exprank}(G) = 1$.

Example 3.4. Let p be a prime and n a positive integer. Let $H = \langle x_1 \rangle \times \cdots \times \langle x_p \rangle$, where $|x_1| = \cdots = |x_{p-1}| = p^n$ and $|x_p| = p^{n+1}$. Form $G = H \rtimes \langle \alpha \rangle$, where α is an automorphism of H of order p^n acting on H in the following way: $x_i^\alpha = x_i x_{i+1}$ for $1 \le i \le p-2$, $x_{p-1}^\alpha = x_{p-1} x_p^p$, and $x_p^\alpha = x_p$. Then it can be verified [4] that G is a PF-group. As $[x_p^{p^n}, \alpha] = 1$, we conclude that $\exp G/Z(G) = p^n$. Short calculation shows that $(\alpha x_1)^{p^n} \neq 1$, whereas $\alpha^{p^n} x_1^{p^n} = 1$. Thus $\mathrm{exprank}(G) = 1$.

Theorem 3.5. *Let G be a center-by-finite-exponent potent pro-p group. If p is odd, then* $\mathrm{exprank}(G) = 0$. *If $p = 2$ and G is nonabelian, then* $\mathrm{exprank}(G) = 1$.

Proof. If $p = 2$, then G is powerful and the conclusion follows from [13]. Thus we assume from here on that p is odd. Let $\exp G/Z(G) = p^e$. Then $\exp \gamma_2(G) = p^e$. We have that

$$(xy)^{p^e} \equiv x^{p^e} y^{p^e} \mod \gamma_2(\langle x, y \rangle)^{p^e} \prod_{i=1}^{e} \gamma_{p^i}(\langle x, y \rangle)^{p^{e-i}}.$$

We prove by induction on i that $\gamma_{p^i}(G)^{p^{e-i}} = 1$ for all $i \ge 1$. The case $i = 1$ follows from $\gamma_p(G)^{p^{e-1}} \le [G^p, G]^{p^{e-1}} = \gamma_2(G)^{p^e} = 1$. For the induction step observe that $\gamma_{p^{i+1}}(G)^{p^{e-i-1}} = [\gamma_{p-1}(G), {}_{p^{i+1}-p+1} G]^{p^{e-i-1}} \le [G^p, {}_{p^{i+1}-p+1} G]^{p^{e-i-1}} = \gamma_{p^{i+1}-p+2}(G)^{p^{e-i}} \le \gamma_{p^i}(G)^{p^{e-i}} = 1$. This shows that G is p^e-abelian. □

References

[1] J. L. Alperin, *A classification of n-abelian groups*, Canad. J. Math. **21** (1969), 1238–1244.
[2] D. E. Arganbright, *The power-commutator structure of finite p-groups*, Pacific J. Math. **29** (1969), 11–17.
[3] G. Ellis, *On the relation between upper central quotients and lower central series of a group*, Trans. Amer. Math. Soc. **353** (2001), no. 10, 4219–4234.
[4] G. A. Fernández-Alcober, J. González-Sanchez, and A. Jaikin-Zapirain, *Omega subgroups of pro-p groups*, Israel J. Math. **166** (2008), 393–412.
[5] J. González-Sanchez, *On p-saturable groups*, J. Algebra **315** (2007), 809–823.
[6] J. González-Sanchez, and A. Jaikin-Zapirain, *On the structure of normal subgroups of potent p-groups*, J. Algebra **276** (2004), 193–209.
[7] L.-C. Kappe, *On n-Levi groups*, Arch. Math. **47** (1986), 198–210.
[8] M. Lazard, *Groupes analytiques p-adiques*, IHES Publ. Math. **26** (1965), 389–603.
[9] F. W. Levi, *Notes on group theory I*, J. Indian Math. Soc. **8** (1944), 1–7.
[10] A. Lubotzky, and A. Mann, *Powerful p-groups. I. Finite groups*, J. Algebra **105** (1987), 484–505.
[11] A. Lubotzky, and A. Mann, *Powerful p-groups. II. p-adic analytic groups*, J. Algebra **105** (1987), 506–515.
[12] A. Mann, *The exponents of central factor and commutator groups*, J. Group Theory **10** (2007), no. 4, 435–436.
[13] P. Moravec, *Schur multipliers and power endomorphisms of groups*, J. Algebra **308** (2007), no. 1, 12–25.
[14] L. Ribes, and P. Zaleskii, *Profinite groups*, Springer, 2000.
[15] E. I. Zelmanov, *On periodic compact groups*, Israel J. Math. **77** (1992), no. 1-2, 83–95.

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