POWERFUL ACTIONS AND NONABELIAN TENSOR PRODUCTS OF POWERFUL p-GROUPS

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Abstract. We introduce the notion of powerful action of a p-group upon another p-group. This represents a generalization of powerful p-groups introduced by Lubotzky and Mann in 1987. We derive some properties of powerful actions and study faithful powerful actions. We also prove that the nonabelian tensor product of powerful p-groups acting powerfully and compatibly upon each other is again a powerful p-group.

1. Introduction

A finite p-group G is said to be *powerful* if p is odd and G/G^p is abelian, or $p = 2$ and G/G^4 is abelian. The theory of powerful p-groups was systematically developed by Lubotzky and Mann [7], and has proven to be of significant importance in the theory of finite p-groups and pro-p groups [3].

Let M and N be finite p-groups, with N acting on M. We say that N acts *powerfully* on M if p is odd and the induced action of N upon M/M^p is trivial, or $p = 2$ and N acts trivially on M/M^4. Two important examples are the following. If a finite p-group G acts upon itself by conjugation, then this action is powerful if and only if G is powerful. On the other hand, if N is a normal subgroup of a finite p-group G, then G acts powerfully on N by conjugation if and only if N is powerfully embedded in G [7].

The purpose of this paper is twofold. Our first aim is to derive some of the fundamental properties of powerful actions. Most of the results are natural generalizations of the corresponding properties of powerful p-groups. One of the main results shows that whenever a finite p-group N acts on a finite p-group M, there exist large characteristic subgroups K in N and H in M such that K acts powerfully on H and H is powerful. When N acts faithfully and powerfully on M, we derive bounds for the rank of N in terms of the rank of M. These estimates appear to be better than the bounds for general faithful actions obtained by Hall (see Roseblade [11]) in the case when M is abelian, and Segal and Shalev [12] in the nonabelian case.

As an application of powerful actions we study the nonabelian tensor product of powerful p-groups. The nonabelian tensor product of groups was introduced by Brown and Loday [1] in 1987, following the ideas of Dennis [2]. This construction has several applications in homotopy theory, homology theory and K-theory. Nonabelian tensor products of finite p-groups have been extensively studied [4, 6, 8]. It is shown in [8] that the nonabelian tensor square of a powerful p-group is again powerful. This may easily fail to be true when dealing with arbitrary nonabelian tensor products of powerful p-groups. The answer whether such a product is again

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powerful seems to depend upon the choice of actions between given groups. Here we show that when M and N are powerful p-groups acting compatibly and powerfully upon each other, then the resulting nonabelian tensor product $M \otimes N$ is a powerful p-group. Furthermore, it is shown that an isomorphic copy of $M \otimes N$ powerfully embeds into a canonical overgroup $\eta(M,N)$ of M and N. Based on this result, we find estimates for the order, exponent and rank of $M \otimes N$. These bounds, at least in our particular case, vastly improve the previously known estimates obtained by Ellis and McDermott [6].

2. Powerful actions

Let M and N be groups, with N acting upon M. Define

$$D_N(M) = \langle m^{-1}m^n \mid m \in M, n \in N\rangle.$$

As $(m^{-1}m^n)^{n'} = m^{-n'}(m^{n'})^{n^{n'}}$ for all $m \in M$ and $n, n' \in N$, it follows that $D_N(M)$ is an N-invariant subgroup of M. It is now clear that N acts powerfully on M precisely when either p is odd and $D_N(M) \le M^p$, or $p = 2$ and $D_N(M) \le M^4$.

An extension of a powerful p-group by a powerful p-group may fail to be powerful. Our first observation shows that this is however true whenever the induced action is powerful. This follows from the following result.

Proposition 2.1. *Let M and N be finite p-groups, with N acting upon M.*

(a) *M is powerfully embedded in $M \rtimes N$ if and only if M is powerful and N acts powerfully.*
(b) *If both M and N are powerful, and N acts powerfully, then $M \rtimes N$ is also powerful.*

Proof. We only prove the assertion for p odd, the case $p = 2$ is similar and we omit it.

(a) Let M be powerful and suppose that N acts powerfully on M. Let $m_1, m_2 \in M$ and $n \in N$. Then $[m_1, m_2 n] = (m_1^{-1}m_1^n)[m_1, m_2]^n \in M^p$. This shows that $[M, M \rtimes N] \le M^p$. Conversely, suppose that M is powerfully embedded in $M \rtimes N$. Then M is powerful, and we have $m_1^{-1}m_1^n = [m_1, m_2 n][m_1, m_2]^{-n} \in M^p$. This shows that N acts powerfully on M.

(b) follows by a similar argument, using the identity

$$[m_1 n_1, m_2 n_2] = (m_1^{-1}m_1^{n_2})^{n_1}[n_1, n_2][m_1, m_2]^{n_1 n_2}(m_2^{-1}m_2^{n_1})^{-n_2}$$

which holds for all $m_1, m_2 \in M$ and $n_1, n_2 \in N$. □

Next we prove some elementary properties of powerful actions.

Lemma 2.2. *Let M and N be finite p-groups, let N act on M and let K be an N-invariant normal subgroup of M.*

(a) *If N acts powerfully on M, then N also acts powerfully on M/K.*
(b) *If p is odd and $K \le M^p$, or $p = 2$ and $K \le M^4$, and if N acts powerfully on M/K, then N acts powerfully on M.*

Proof. (a) Let N act powerfully on M. If p is odd, then we have that $D_N(M/K) = D_N(M)K/K \le M^pK/K = (M/K)^p$. If $p = 2$, then a similar argument shows that $D_N(M/K) \le (M/K)^4$.

(b) Assume now that N acts powerfully on M/K, and $K \le M^p$, where p is odd. Then $D_N(M/K) \le (M/K)^p = M^p/K$, hence $D_N(M) \le M^p$. If $p = 2$, the proof is similar. □

Lubotzky and Mann [7] showed that whenever N is powerfully embedded in a finite p-group G, then both N^p and $[N,N]$ are powerfully embedded in G. The next lemma generalizes this result.

Lemma 2.3. *Let M and N be finite p-groups, let M be powerful and let N act powerfully on M. Then N acts powerfully on M^p and $[M,M]$.*

Proof. Let p be odd. We want to prove that, under the above assumptions, N acts powerfully on M^p. Without loss of generality we may assume that $(M^p)^p = 1$. Since N acts powerfully on M, it follows that $D_N(M)^p = 1$. As M is powerful, this implies that $[M,M]^p \leq (M^p)^p = 1$. Using [7, Proposition 1.6], we get $[D_N(M), M] \leq [M^p, M] = [M,M]^p = 1$, therefore $D_N(M)$ is central in M. Thus we get $1 = (m^{-1}m^n)^p = m^{-p}(m^p)^n$, therefore $D_N(M^p) = 1$, as required.

Now let $p = 2$. We claim that N acts powerfully on M^2. Taking quotients we may assume that $(M^2)^4 = 1$. We need to show that $D_N(M^2) = 1$. Since M^2 is powerfully embedded in M [7, Theorem 4.1.1], we get that $M^2 \leq Z(M)$. As above we also conclude that $D_N(M) \leq Z(M)$. Let $m \in M$ and $n \in N$. Since $m^{-1}m^n \in M^4$, we get $m^{-2}(m^2)^n = (m^{-1}m^n)^2 = 1$, therefore $D_N(M^2) = 1$.

It remains to prove that N acts powerfully on $[M,M]$. We only show this for p odd, the case $p = 2$ follows along the same lines. Without loss of generality we may assume that $[M,M]^p = 1$, hence M is nilpotent of class ≤ 2. As $[M^p, M] = [M,M]^p = 1$, we conclude that $D_N(M)$ is central in M. Let $m_1, m_2 \in M$ and $n \in N$. We can write $m_1^n = m_1 z_1$ and $m_2^n = m_2 z_2$ for some $z_1, z_2 \in Z(M)$. From here it follows that $[m_1, m_2]^n = [m_1 z_1, m_2 z_2] = [m_1, m_2]$, therefore $D_N([M,M]) = 1$. This proves the assertion. □

Corollary 2.4. *Let a finite p-group N act powerfully on a powerful p-group M. If $\exp M = p^e$, then N acts on M nilpotently of class $\leq c$, where c is equal to either e if p is odd, or $\lfloor (e+1)/2 \rfloor$ if $p = 2$.*

Proof. Define $D_N^1(M) = D_N(M)$ and $D_N^k(M) = D_N(D_N^{k-1}(M))$ for $k > 1$. Then Lemma 2.3, together with induction, shows that $D_N^k(M) \leq M^{p^k}$ if p is odd, or $D_N^k(M) \leq M^{2^{2k}}$ if $p = 2$. It follows from here that $D_N^c(M) = 1$, where c is equal to either e if p is odd, or $\lfloor (e+1)/2 \rfloor$ if $p = 2$. This concludes the proof. □

Corollary 2.5. *Let M be a powerful p-group of exponent p^e, and let N be a p-subgroup of $\operatorname{Aut} M$. Suppose that N acts powerfully on M.*

(a) *If p is odd, then both N and $D_N(M)$ are nilpotent of class $\leq e-1$, and $\exp N$ divides p^{e-1}.*
(b) *If $p = 2$, then both N and $D_N(M)$ are nilpotent of class $\leq \lfloor (e-1)/2 \rfloor$, and $\exp N$ divides $2^{\lfloor (e-1)/2 \rfloor}$.*

Proof. If p is odd then we have a normal series of N-invariant subgroups

$$M \geq M^p \geq M^{p^2} \geq \cdots \geq M^{p^e} = 1.$$

Since N acts powerfully, we conclude that N induces the trivial action on each factor $M^{p^i}/M^{p^{i+1}}$. By Hall-Kalužnin's theorem [3, p. 4] we get the result for p odd. In the case when $p = 2$ we replace the above series by $M \geq M^4 \geq M^{16} \geq \cdots \geq M^{2^{2\lfloor (e+1)/2 \rfloor}} = 1$ and use the same argument as above. □

One of the most important features of powerful p-groups is the fact that whenever G is a finite p-group of rank r, then there exists a characteristic powerful subgroup $H \leq G$ such that $|G : H|$ is r-bounded [7, Theorem 1.14]. Here we generalize this result. Our proof follows the argument from [7]. We need the following auxiliary result.

Lemma 2.6. *Let $S(n,p)$ be a Sylow p-subgroup of $\mathrm{GL}(n,p)$. Let V be any verbal subgroup of a free group of countable rank such that $V(S(n,p)) = 1$. Let M be a powerful p-group with $\mathrm{d}(M) \leq n$, and let N be a finite p-group acting upon M. If p is odd, then $V(N)$ acts powerfully on M. When $p = 2$, $V(N)^2$ acts powerfully on M.*

Proof. First we deal with the case when p is odd. We may assume that $M^p = 1$, hence M is elementary abelian of order at most p^n. The group $N/C_N(M)$ embeds into $\operatorname{Aut} M$, hence it also embeds into $S(n,p)$. It follows that $V(N/C_N(M)) = 1$, therefore $V(N) \leq C_N(M)$. In other words, $D_{V(N)}(M) = 1$, as required.

Assume now that $p = 2$ and $M^4 = 1$. Hence M is abelian and M/M^2 is elementary abelian of order $\leq 2^d$. As above we conclude that $V(N) \leq C_N(M/M^2)$, therefore $D_{V(N)}(M) \leq M^2$. Let $b \in V(N)$ and $m \in M$. We have that $m^b = mm_1^2$ for some $m_1 \in M$. It follows that $m^{b^2} = mm_1^2(m_1^b)^2$. We can write $m_1^b = m_1m_2^2$ for some $m_2 \in M$. Now this gives $m^{b^2} = mm_1^4m_2^4 = m$. Thus $V(N)^2$ acts trivially on M. This finishes the proof. □

Theorem 2.7. *Let M and N be finite p-groups, $\mathrm{sr}(M) = r$, $\mathrm{sr}(N) = s$, and set $n = \lceil \log_2 r \rceil$. Let N act upon M. Then there exist a powerful characteristic subgroup H in M, and a characteristic subgroup K in N such that K acts powerfully on H, and $|M : H| \leq p^{r(n+\epsilon)}$ and $|N : K| \leq p^{s(n+\epsilon)}$, where $\epsilon = 0$ if p is odd, or $\epsilon = 1$ if $p = 2$.*

Proof. Applying [7, Theorem 1.14], we find a powerful characteristic subgroup H in M such that $|M : H| \leq p^{r(n+\epsilon)}$. To find K, we adopt a similar approach as in [7]. Let V be the variety of all p-groups having a normal series of length $\leq n$ with elementary abelian factors. Then $S(n,p)$ belongs to V, hence $K = V(N)^{p^\epsilon}$ acts powerfully upon H by Lemma 2.6. By definition of V we have that $|N : K| \leq p^{s(n+\epsilon)}$. This concludes the proof. □

Hall (see Roseblade [11]) proved that if M is an abelian p-group of rank r, then any p-subgroup N of $\operatorname{Aut} M$ can be generated by $r(5r-1)/2$ elements. Segal and Shalev [12] showed that if M is any p-group of rank r, then any p-subgroup of its automorphism group can be generated by $5r^2$ elements. These estimates can be further improved in the case when N acts powerfully, even if M is nonabelian. We have the following result.

Theorem 2.8. *Let M and N be finite p-groups and let N act faithfully and powerfully upon M. If $\mathrm{sr}(M) = r$, then $\mathrm{sr}(N) \leq 2r^2$. If M is abelian, then $\mathrm{d}(N) \leq r^2$.*

Proof. At first we prove that if M is abelian, then $\mathrm{d}(N) \leq r^2$. This part of the proof is an adaptation of Roseblade's argument [11]. For each nontrivial $n \in N$ let $t(n)$ be the largest integer t such that n acts trivially on M/p^tM. Let $n_1, \ldots, n_d$ be a minimal generating set of N. We can choose these generators such that $t(n_1) \leq \cdots \leq t(n_d)$ and $\sum_{i=1}^d t(n_i)$ is the largest possible. Let $m_1, \ldots, m_r$ be a basis for M. Then we can write

$$m_i^{n_j} = m_i + p^{t(n_j)} \sum_{k=1}^{r} \alpha_{ik}(j) m_k$$

for some $\alpha_{ik}(j) \in \mathbb{Z}$. Thus we can represent the action of n_j by the matrix $1 + p^{t(n_j)}[\alpha_{ik}(j)]_{i,k}$. Suppose there exists an integer u such that

$$[\alpha_{ik}(u)]_{i,k} \equiv \sum_{j=1}^{u-1} \beta_j [\alpha_{ik}(j)]_{i,k} \mod p$$

for some $\beta_j \in \mathbb{Z}$. Define

$$\tilde{n} = \prod_{j=1}^{u-1} n_j^{\beta_j p^{t(n_u)-t(n_j)}}.$$

Then the matrix corresponding to the action of $\tilde{n}$ on $M/p^{t(n_u)+1}M$ with respect to the basis $m_1, \ldots, m_r$ is precisely

$$\prod_{j=1}^{u-1} (1 + p^{t(n_j)}[\alpha_{ik}(j)]_{i,k})^{\beta_j p^{t(n_u)-t(n_j)}}.$$

By our assumption, for every $n \in N \backslash \{1\}$ we have that $t(n) \geq 1$ if p is odd, and $t(n) \geq 2$ in the case when $p = 2$. Modulo $p^{t(n_u)+1}$, this is therefore equivalent to

$$1 + p^{t(n_u)} \sum_{j=1}^{u-1} \beta_j [\alpha_{ik}(j)]_{i,k}.$$

It follows that $\tilde{n}$ induces the same action on $M/p^{t(n_u)+1}M$ as n_u, therefore we conclude that $t(\tilde{n}^{-1}n_u) \geq t(n_u) + 1$. Now replace the generating set $\{n_1, \ldots, n_d\}$ of N by another generating set $\{n_1, \ldots, n_{u-1}, \tilde{n}^{-1}n_u, n_{u+1}, \ldots, n_d\}$. We have that

$$t(\tilde{n}^{-1}n_u) + \sum_{i \neq u} t(n_i) > \sum_{i=1}^{d} t(n_i),$$

contradicting the assumption. It follows that the images of matrices $[\alpha_{ik}(j)]_{i,k}$ in $\mathrm{GL}(r,p)$ are linearly independent, therefore $d \leq r^2$.

To prove the theorem in the case when M is nonabelian, let $P = M \rtimes N$. Let K be a maximal normal abelian subgroup of P contained in M. As $N/(N \cap C_P(K))$ acts faithfully and powerfully on the abelian group K, we have that $\mathrm{d}(N/(N \cap C_P(K))) \leq r^2$ by the above. Therefore it suffices to show that $\mathrm{d}(N \cap C_P(K)) \leq r^2$. At first we note that $C_M(K) = K$ by the maximality of K. It follows from here that $[M, C_P(K)] \leq K$, hence $C_P(K)/C_{C_P(K)}(M)$ can be embedded in $\mathrm{Der}(M/K, K)$. Now we use a similar approach as in [12, Lemma 2.1] to conclude that $\mathrm{sr}(\mathrm{Der}(M/K, K)) \leq r^2$. On the other hand, we clearly have that $N \cap C_{C_P(K)}(M)$ is trivial, hence $N \cap C_P(K)$ embeds into $\mathrm{Der}(M/K, K)$. This concludes the proof. $\square$

As every automorphism of M that acts trivially upon M/M^p also acts trivially on $M/\Phi(M)$, we get the following.

Corollary 2.9. *Let M be a finite p-group and suppose that N is any subgroup of* $\mathrm{Aut}\, M$ *acting trivially either on M/M^p if p is odd, or upon M/M^4 if $p = 2$. If* $\mathrm{sr}(M) = r$*, then* $\mathrm{sr}(N) \leq 2r^2$*. If M is abelian, then* $\mathrm{sr}(N) \leq r^2$.

3. Nonabelian tensor products of powerful p-groups

Let M and N be groups acting upon each other in a compatible way, that is,

$$\tilde{m}^{n^m} = ((\tilde{m}^{m^{-1}})^n)^m \quad \text{and} \quad \tilde{n}^{m^n} = ((\tilde{n}^{n^{-1}})^m)^n,$$

for $m, \tilde{m} \in M$ and $n, \tilde{n} \in N$, and acting upon themselves by conjugation. The *nonabelian tensor product* $M \otimes N$ of M and N is a group generated by the symbols $m \otimes n$, where $m \in M$ and $n \in N$, with defining relations

$$\tilde{m}m \otimes n = (\tilde{m}^m \otimes n^m)(m \otimes n) \quad \text{and} \quad m \otimes \tilde{n}n = (m \otimes n)(m^n \otimes \tilde{n}^n),$$

where $m, \tilde{m} \in M$ and $n, \tilde{n} \in N$. When $M = N$ and all actions are conjugations, then $M \otimes M$ is called the *nonabelian tensor square* of M. Note that when M and

N act trivially upon each other, then $M \otimes N$ is isomorphic to the 'usual' tensor product $M^{\mathrm{ab}} \otimes N^{\mathrm{ab}}$.

When the groups M and N act compatibly on each other, we have

$$(m^{-1}m^n)^{m'} = m^{-m'}(m^{m'})^{n^{m'}}$$

for all $m, m' \in M$ and $n \in N$. This shows that $D_N(M)$ is a normal subgroup in M. Our first result in this section shows that, under certain natural conditions, $D_N(M)$ is powerfully embedded in M. To be more precise, we have the following.

Proposition 3.1. *Let M and N be powerful p-groups acting upon each other compatibly and powerfully. Then $D_N(M)$ is powerfully embedded in M, and $D_M(N)$ is powerfully embedded in N.*

Proof. Let p be odd. We may assume that $D_N(M)^p = [D_N(M), M, M] = 1$. Since $D_N(M) \leq Z_2(M)$ and N acts powerfully on M, we get $[D_N(M), D_N(M)] \leq [D_N(M), M^p] = [D_N(M)^p, M] = 1$, therefore $D_N(M)$ is abelian. Let $m \in M$. From the above assumptions it follows that $\langle D_N(M), m\rangle$ is nilpotent of class ≤ 2, hence regular. For $a \in D_N(M)$ and $m \in M$ we thus get $(am)^p = m^p$ by our assumption. Letting $a = m^{-1}m^n$, we get $(m^p)^n = m^p$, thus N acts trivially on M^p, i.e., $D_N(M^p) = 1$. In particular, this implies that $D_N([M, M]) = D_N(D_N(M)) = 1$. For $m \in M$ and $n_1, n_2 \in N$ we therefore get that $m^{-n_1}m^{n_2n_1} = m^{-1}m^{n_2}$. This implies that the map $\phi_m : N \to D_N(M)$ defined by $n^{\phi_m} = m^{-1}m^n$ is a homomorphism for all $m \in M$. Now we conclude that

$$m^{-1}m^{n^p} = (n^p)^{\phi_m} = (n^{\phi_m})^p = 1,$$

hence N^p acts trivially on M. Let $m, m_1 \in M$ and $n \in N$. As M and N act compatibly upon each other, we get $m_1^{m^{-1}m^n} = m_1^{n^{-m}n}$. Observing that $n^{-m}n \in D_M(N) \leq N^p$, we conclude that $m_1^{m^{-1}m^n} = m_1$, therefore $[D_N(M), M] = 1$. This shows that $D_N(M)$ is powerfully embedded in M.

When $p = 2$, the argument above needs to be slightly modified. We may assume without loss of generality that $D_N(M)^4 = [D_N(M), M, M] = 1$. We want to show that these assumptions yield $[D_N(M), M] = 1$. Therefore we may assume that $[D_N(M), M]^2 = 1$. As above we get that $D_N(M)$ is abelian and $\langle D_N(M), m\rangle$ is nilpotent of class ≤ 2. This gives $(am)^4 = m^4$ for all $a \in D_N(M)$ and $m \in M$. The rest of the proof now follows along the above lines. □

As mentioned in the introduction, the nonabelian tensor product of powerful p-groups M and N may fail to be powerful. For instance, if M and N are powerful normal subgroups of a finite p-group G, then $[M, N]$ is not necessarily powerful, see the example below. If $M \otimes N$ were powerful in this particular situation, then $[M, N]$, being an image of $M \otimes N$, would have been powerful, which is not the case.

Example 3.2. Let p be an odd prime and $n \geq 5$ and $m \geq 5$. For $1 \leq i < j \leq n$ let $t_{ij}(\alpha)$ be the matrix in $\mathrm{GL}(n, p^m)$ with 1's on the diagonal, α in the (i, j)-entry, and zeros elsewhere. It is straightforward to see that $t_{ij}(\alpha)^k = t_{ij}(k\alpha)$, and that $[t_{ij}(\alpha), t_{jk}(\beta)] = t_{ik}(\alpha\beta)$ for $i < j < k$, and $[t_{ij}(\alpha), t_{k\ell}(\beta)] = 1$ for $i < j$, $k < \ell$ and $k \neq j$. Consider now the group

$$G = \langle t_{ij}(p) \mid 1 \leq i < j \leq n\rangle.$$

Then it can be verified that G is a powerful p-group. Let

$$N = \langle t_{1i}(p^{i-1}),\, t_{3j}(p^{j-3}) \mid i > 1,\, j > 3\rangle.$$

N is a powerful normal subgroup of G. As $[[t_{12}(p), t_{23}(p)], [t_{34}(p), t_{45}(p)]] = t_{15}(p^4)$, the group $[N, G]$ is not powerful.

In order to determine when the nonabelian tensor products of powerful p-groups are again powerful, we recall here a construction introduced by Ellis and Leonard [5] and Rocco [10], see also Nakaoka [9]. Let M and N be groups acting compatibly upon each other and let $\varphi : N \to N^\varphi$ be an isomorphism of N onto N^φ. Define

$$\eta(M,N) = \langle M, N^\varphi \mid [m,n^\varphi]^{m_1} = [m^{m_1},(n^{m_1})^\varphi],\ [m,n^\varphi]^{n_1^\varphi} = [m^{n_1},(n^{n_1})^\varphi] \\ \text{for all } m,m_1 \in M, n,n_1 \in N\rangle.$$

By [5], the groups M and N are embedded in $\eta(M,N)$. It is well known [5, 9] that there is a natural isomorphism between $M \otimes N$ and the subgroup $[M,N^\varphi]$ of $\eta(M,N)$ induced by $m \otimes n \mapsto [m,n^\varphi]$. In [8] we prove that if G is a powerful p-group, then the groups $[G,G^\varphi]$ and $\gamma_2(\eta(G,G))$ are powerfully embedded in $\eta(G,G)$. The following theorem may be seen as a generalization of this result.

Theorem 3.3. *Let M and N be powerful p-groups acting upon each other compatibly and powerfully. Then both $[M,N^\varphi]$ and $\gamma_2(\eta(M,N))$ are powerfully embedded in $\eta(M,N)$.*

Proof. We only prove that $\gamma_2(\eta(M,N))$ is powerfully embedded in $\eta(M,N)$. The rest of the proof is similar and we omit it.

First let p be odd. Without loss of generality we may assume that $\gamma_4(\eta(M,N)) = \gamma_2(\eta(M,N))^p = 1$. We need to show that it follows from here that $\gamma_3(\eta(M,N)) = 1$. As M is powerful, we have that $[M^p,M] = [M,M]^p = 1$, and similarly also $[(N^\varphi)^p,N^\varphi] = 1$. It follows that both M and N are nilpotent of class ≤ 2. Expansion of the commutator $[m,(n^\varphi)^p]$ in $\eta(M,N)$ gives $[m,(n^\varphi)^p] = [m,n^\varphi]^p[m,n^\varphi,n^\varphi]^{p(p-1)/2} = 1$, hence $[M,(N^\varphi)^p] = 1$. By a similar argument we also conclude that $[M^p,N^\varphi] = 1$. In particular, $[M,M,N^\varphi] = [N^\varphi,N^\varphi,M] = 1$. In order to finish the proof, we need to show that the commutators of the form $[m_1,n^\varphi,m_2]$ and $[m,n_1^\varphi,n_2^\varphi]$, where $m,m_1,m_2 \in M$ and $n,n_1,n_2 \in N$, are trivial in $\eta(M,N)$. By symmetry it suffices to verify this only for the first commutator. To this end, note there exists $\bar{n} \in N$ such that $n^{m_2} = n\bar{n}^p$. Thus

$$\begin{aligned} [m_1,n^\varphi,m_2] &= [m_1,n^\varphi]^{-1}[m_1,n^\varphi]^{m_2} \\ &= [m_1,n^\varphi]^{-1}[m_1^{m_2},(n^{m_2})^\varphi] \\ &= [m_1,n^\varphi]^{-1}[m_1[m_1,m_2],n^\varphi(\bar{n}^\varphi)^p] \\ &= 1, \end{aligned}$$

as required.

Now assume that $p = 2$ and $\gamma_4(\eta(M,N)) = \gamma_2(\eta(M,N))^4 = 1$. We are trying to prove that $\gamma_3(\eta(M,N)) = 1$, therefore we may assume that $\gamma_3(\eta(M,N))^2 = 1$. With these assumptions we can now repeat the above proof by replacing p with 4 throughout. $\square$

Corollary 3.4. *Let M and N be powerful p-groups acting upon each other compatibly and powerfully. Then $M \otimes N$ is a powerful p-group.*

In particular, when M and N be normal subgroups of a finite p-group G, and if they are both powerfully embedded in G, then they act upon each other compatibly and powerfully by conjugation. This implies the following.

Corollary 3.5. *Let M and N be normal subgroups of a finite p-group G. If both M and N are powerfully embedded in G, then $M \otimes N$ is a powerful p-group.*

Theorem 3.3 provides sharp estimates for the order, exponent and rank of the nonabelian tensor product of powerful p-groups when the mutual actions are powerful. We have the following result.

Proposition 3.6. *Let M and N be powerful p-groups acting upon each other compatibly and powerfully. Suppose that* $\mathrm{d}(M) = d_1$, $\mathrm{d}(N) = d_2$, $\exp M = p^{e_1}$, $\exp \gamma_2(M) = p^{f_1}$, $\exp N = p^{e_2}$, *and* $\exp \gamma_2(N) = p^{f_2}$. *Denote* $e = \min\{e_1, e_2\}$. *Then we have the following:*

(a) $\mathrm{d}(M \otimes N) \leq d_1 d_2$ *and* $\mathrm{d}(\gamma_2(\eta(M,N))) \leq \binom{d_1}{2} + \binom{d_2}{2} + d_1 d_2$.
(b) $\exp(M \otimes N) \mid p^e$ *and* $\exp \gamma_2(\eta(M,N)) \mid p^{\max\{f_1,f_2,e\}}$.
(c) $|M \otimes N| \leq p^{d_1 d_2 e}$ *and* $|\gamma_2(\eta(M,N))| \leq p^{(\binom{d_1}{2}+\binom{d_2}{2}+d_1 d_2)\max\{f_1,f_2,e\}}$.

Proof. Let $\{m_1, \ldots, m_{d_1}\}$ be a generating set of M, and $\{n_1, \ldots, n_{d_2}\}$ a generating set of N. Then $\gamma_2(\eta(M,N))$ is the normal closure in $\eta(M,N)$ of the set

$$\begin{aligned}\mathcal{S} = \{[m_i, m_j],\, [n_k^\varphi, n_\ell^\varphi],\, [m_u, n_v^\varphi] \mid & 1 \leq j < i \leq d_1,\, 1 \leq \ell < k \leq d_2, \\ & 1 \leq u \leq d_1,\, 1 \leq v \leq d_2\}.\end{aligned}$$

As $\gamma_2(\eta(M,N))$ is powerfully embedded in $\eta(M,N)$, it is actually generated by the set $\mathcal{S}$, see [7, Proposition 1.10]. A similar argument shows that $[M, N^\varphi]$ is generated by the set

$$\mathcal{T} = \{[m_u, n_v^\varphi] \mid 1 \leq u \leq d_1,\, 1 \leq v \leq d_2\}.$$

This proves (a). To prove (b), one can adapt the argument from the proof of [8, Theorem 2.4] to show that $[m, n^\varphi]^{p^{e_1}} = [m^{p^{e_1}}, n^\varphi] = 1$ and $[m, n^\varphi]^{p^{e_2}} = [m, (n^\varphi)^{p^{e_2}}] = 1$ for all $m \in M$ and $n \in N$, therefore the order of $[m, n^\varphi]$ divides p^e. It follows that $[M, N^\varphi]$ is generated by elements of order p^e. As $[M, N^\varphi]$ is powerful, [7, Corollary 1.9] implies that the exponent of $[M, N^\varphi]$ divides p^e. Similarly, the elements of the set $\mathcal{S}$ have order dividing $p^{\max\{f_1,f_2,e\}}$ and as $\gamma_2(\eta(M,N))$ is powerful, (b) is proved. The assertion (c) is now a direct consequence of (a), (b), and [7, Proposition 2.5]. □

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