

POWER CENTRALIZED SEMIGROUPS

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ABSTRACT. A semigroup is said to be power centralized if for every pair of elements x and y there exists a power of x commuting with y . The structure of power centralized groups and semigroups is investigated. In particular, we characterize 0-simple power centralized semigroups and describe subdirectly irreducible power centralized semigroups. Connections between periodic semigroups with central idempotents and periodic power commutative semigroups are discussed.

1. INTRODUCTION

A semigroup S is said to be *power commutative* when for any $x, y \in S$ there exists a positive integer $n = n(x, y)$ such that $(xy)^n = (yx)^n$. The study of power commutative semigroups was initiated by Davenport [4]. It was shown there that these semigroups are precisely semilattices of unipotent semigroups (i.e., semigroups each having exactly one idempotent). Periodic power commutative semigroups were characterized in a similar way by Galbiati in [5]. When n can be chosen independently of x and y , the semigroup is said to be a PC_n -semigroup. Another class of semigroups which is closely related to PC_n -semigroups are the so-called n -central semigroups. A semigroup S is said to be n -central [10] when $x^n y = y x^n$ for all $x, y \in S$. In the group-theoretical setting, a group is n -central if and only if it is a PC_n -group. n -central groups have been studied by many authors, see the papers of Adjan [2], Gupta and Rhemtulla [7], Morse and Kappe [9] and the author [11].

Close connection between n -central semigroups and PC_n -semigroups leads to the following generalization of n -centrality. A semigroup S is said to be *power centralized* if for any $x, y \in S$ there exists a positive integer $n = n(x, y)$ such that x^n commutes with y . This class of semigroups is the main object of investigation in this paper. Our interest in these semigroups has its origins in certain problems on center-by-periodic groups; see [11]. Note however that the class of power centralized groups is much richer, since it contains examples of nonabelian torsion-free simple groups (Example 2.1). It is easy to see that a group is power centralized precisely when it is power commutative. We generalize results of [7] and [10] by proving that a finitely generated soluble-by-finite group with uniformly power centralized normal closures is embeddable into the direct product of a finite m -central group and a free abelian group of finite rank. Moreover, every such group G is nearly exponential, i.e., for any $x, y \in G$ there exists a positive integer $n = n(x, y) > 1$ such that $(xy)^n = x^n y^n$. As a consequence we prove that every torsion-free locally soluble-by-finite group with uniformly power centralized normal closures is abelian. Note that this is not true in general since there exist nonabelian torsion-free center-by-periodic groups [2].

The rest of the paper is devoted to power centralized semigroups. Note that this class of semigroups includes for instance periodic semigroups with central idempotents and power joined semigroups (a semigroup S is said to be *power joined* if

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for every $x, y \in S$ there exist positive integers m and n such that $x^m = y^n$). It is proved that every power centralized semigroup is a semilattice of archimedean power centralized semigroups. This result enables characterizations of 0-simple power centralized semigroups and power centralized semigroups that are nil extensions of regular semigroups. We also describe the structure of archimedean power centralized semigroups. In all these cases power centralized semigroups are closely related to power centralized groups.

In Section 4 we deal with subdirectly irreducible power centralized semigroups. In particular, we describe subdirectly irreducible power centralized semigroups with the globally idempotent core and obtain structural information about certain subdirectly irreducible power centralized semigroups with trivial annihilator.

The last section of this paper deals with periodic power centralized semigroups. A periodic semigroup is power centralized if and only if its idempotents are central. We prove that every periodic power centralized semigroup is power commutative and nearly exponential. The converse does not hold true in general. Thus we derive some sufficient conditions for a periodic power commutative semigroup to be power centralized.

The reader is referred to Petrich [13] for results and definitions not mentioned here. For group-theoretic results and notations we mainly refer to Robinson [15].

2. POWER CENTRALIZED GROUPS

Since every center-by-periodic group is power centralized, one can expect that there is not much to be said about the structure of power centralized groups in general. To demonstrate this further, we show that there exists a nonabelian torsion-free simple group which is power centralized, and all whose proper noncyclic subgroups are center-by-periodic.

Example 2.1. According to Traustason [17], a group G is said to be a *group with the congruence intersection property* (or *CIP group*) if $H^G \cap K^G = (H \cap K)^G$ for every $H, K \leq G$. It is easy to see that every CIP group is power centralized. To show this, let $x, y \in G$. If x and y commute, then we are done. Otherwise $1 \neq [x, y] \in \langle x \rangle^G \cap \langle y \rangle^G = (\langle x \rangle \cap \langle y \rangle)^G$, hence $\langle x \rangle \cap \langle y \rangle \neq 1$. Thus there exists an integer n such that $x^n \in \langle y \rangle$, whence x^n commutes with y . Traustason showed that every CIP group is either a Dedekind group (i.e., every subgroup is normal) or G has a simple factor which is an NSIP group; here a group is said to be an *NSIP group* if it is torsion-free and the intersection of any of its nontrivial subgroups is nontrivial. Obraztsov [12] showed, using geometric methods, that simple NSIP groups actually exist. More precisely, there exists a simple NSIP group $G = \langle a_1, a_2, \dots \rangle$ such that every noncyclic subgroup of G is a conjugate in G of some $G_k = \langle a_1, a_2, \dots, a_k \rangle$, $k \geq 2$. Furthermore, if p is a sufficiently large prime, then G may be chosen to be such that every G_k is p^{k-1} -central. Note that G is power centralized.

The above example shows that we need to restrict ourselves to some special classes of power centralized groups in order to get some structural results. At first we mention an elementary lemma.

Lemma 2.2. *A group G is power centralized if and only if it is power commutative.*

Proof. Let G be a group and let $x, y \in G$. Suppose first that G is power centralized. Then there exists a positive integer n such that $(xy)^n y = y(xy)^n$, hence $(xy)^n = (yx)^n$. Conversely, suppose G is power commutative. Then there exists a positive integer n such that $x^n = (xy \cdot y^{-1})^n = (y^{-1}xy)^n = y^{-1}x^n y$, thus x^n commutes with y . This concludes the proof. \square

At this point we recall [10] that if n is a positive integer, then there exists a positive integer $m = f(n) > 1$ such that every locally (soluble-by-finite) n -central

group is m -abelian; here a group G is said to be m -abelian when $(xy)^m = x^m y^m$ for every $x, y \in G$. The proof of this fact relies on the solution of the Restricted Burnside Problem [18]. When we want to extend this result to power centralized groups, the main difficulty arising here is the fact that the smallest positive integer n for which x^n commutes with y depends on x and y and may be unbounded when x and y run over the group. This leads to the following definition.

Definition 2.3. Let S be a semigroup and X a nonempty subset of S . We say that X is *uniformly power centralized* in S when for every $s \in S$ there exist a positive integer $n = n(s)$ such that s^n commutes with every $x \in X$.

For instance, every finite nonempty subset of a power centralized semigroup S is uniformly power centralized in S . Moreover, every finitely generated subsemigroup of a power centralized semigroup S is uniformly power centralized in S .

For a semigroup S we say that it is *nearly exponential* when for every $x, y \in S$ there exists a positive integer $n = n(x, y) > 1$ such that $(xy)^n = x^n y^n$. The following result is now an analogue of the above mentioned group-theoretical result.

Theorem 2.4. *Let G be a finitely generated soluble-by-finite group. The following assertions are equivalent.*

- (i) *For every $x \in G$ the normal closure $\langle x \rangle^G$ of x in G is uniformly power centralized in G .*
- (ii) *There exists a positive integer m such that G is isomorphic to a subgroup of the direct product of a finite m -central group and a free abelian group of finite rank.*

Furthermore, if G is a locally soluble-by-finite group satisfying the property (i), then G is nearly exponential.

Proof. Let G be a finitely generated soluble-by-finite group satisfying (i). Let x be an arbitrary element of G . Since $\langle x \rangle^G$ is uniformly power centralized, $G/C_G(\langle x \rangle^G)$ is a finitely generated torsion group. Since this group is also soluble-by-finite, it follows that $G/C_G(\langle x \rangle^G)$ is a finite group for every $x \in G$. Let $\{t_1, \dots, t_r\}$ be a transversal to $C_G(\langle x \rangle^G)$ in G and put $K = \cap_{i=1}^r C_G(\langle t_i \rangle^G)$. Then $|G : K| < \infty$, hence there exists a positive integer k such that $x^k \in K$. Since x^k centralizes $\{t_1, \dots, t_r\}$ and $C_G(x)$, we obtain $x^k \in Z(G)$, hence $G/Z(G)$ is a torsion group; since it is finitely generated and soluble-by-finite, we conclude that G is center-by-finite. It follows from Schur's theorem [15, Theorem 4.12] that G' is finite. Let T be the set consisting of all elements of finite order in G . Since G' is finite, we have $G' \subseteq T$, hence T is a characteristic subgroup of G . The factor group G/T is finitely generated, torsion-free and abelian. Since $G/Z(G)$ is finite, there exists a maximal torsion-free abelian subgroup A of $Z(G)$, such that G/A is a torsion group. The group G/A is finitely generated, hence it is finite. Therefore there exists a positive integer m such that G/A is m -central. Beside that we observe that $T \cap A = 1$, so G naturally embeds into $(G/A) \times (G/T)$. This proves (i) \Rightarrow (ii), whereas the converse implication is trivial.

Now let G be a locally soluble-by-finite group such that the normal closure of every element of G is uniformly power centralized in G . Let $a, b \in G$ and put $H = \langle a, b \rangle$. Then H is a finitely generated soluble-by-finite group, hence the above argument implies that H' is a finite subgroup of H . Let $e = e(a, b)$ be the exponent of H' and let $n = n(a, b) > 1$ be a positive integer such that a^n and b^n centralize all elements of H' . Note that such n exists since H is power centralized and $|H'| < \infty$. We have $(ab)^n = a^n b^n c$ for some $c \in H'$, hence $(ab)^{ne} = a^{ne} b^{ne} c^e = a^{ne} b^{ne}$. This concludes the proof that G is nearly exponential. \square

It is proved in [7] that every torsion-free locally soluble n -central group is abelian. Note that the proof of Theorem 2.4 implies a generalization of this result.

Corollary 2.5. *Let G be a torsion-free locally soluble-by-finite group with uniformly power centralized normal closures. Then G is abelian.*

3. SEMILATTICE DECOMPOSITION OF POWER CENTRALIZED SEMIGROUPS

Recall that a semigroup S is said to be *archimedean* if for every $a, b \in S$ there are positive integers m and n such that $a^m \in S^1 b S^1$ and $b^n \in S^1 a S^1$. The following lemma is a starting point of investigation of power centralized semigroups.

Lemma 3.1. *Every power centralized semigroup is a semilattice of archimedean power centralized semigroups.*

Proof. Let S be a power centralized semigroup, let $x, y \in S$ and suppose that $x \in S^1 y S^1$. Then $x = ayb$ for some $a, b \in S^1$. Since S is power centralized, there exists a positive integer n such that $(yba)^n$ commutes with y . This implies $x^{n+1} = (ayb)^{n+1} = a(yba)^n yb = ay(yba)^n b \in S^1 y^2 S^1$. By a result of Putcha [14], S is a semilattice of archimedean semigroups which are clearly power centralized. \square

This gives a characterization of 0-simple power centralized semigroups.

Theorem 3.2. *A semigroup is 0-simple and power centralized if and only if it is a power centralized group with a zero adjoined.*

Proof. Let S be a 0-simple power centralized semigroup. By Lemma 3.1, S is a semilattice of archimedean power centralized semigroups. Since $S^1 a S^1 = S$ for every $a \in S \setminus \{0\}$, the nonzero elements of S are in the same semilattice component C of S . If $0 \in C$, then S is a nil semigroup which contradicts the fact that S is 0-simple. Thus we have $S = C^0$, where C is a simple semigroup. Since C is a subsemigroup of S , we also conclude that C is power centralized.

Suppose now that C is not completely simple. By a result of Jones [8], there exists a subsemigroup of C such that the bicyclic semigroup $C(p, q) = \langle p, q \mid pq = 1 \rangle$ is its homomorphic image. This implies that there exists a positive integer n such that p commutes with q^n . On the other hand, we observe that $q^n p = pq^n = q^{n-1}$ which is clearly impossible. Thus C is completely simple. By a theorem of Rees and Šuškevič [13], C is isomorphic to a Rees matrix semigroup $\mathcal{M}[G; I, J; P]$ over a group G with the $J \times I$ sandwich matrix P . Additionally, we may assume that the matrix P is normalized, i.e., $p_{j, i_0} = p_{j_0, i} = e$ for some $i_0 \in I$, $j_0 \in J$ and all $i \in I$, $j \in J$; here e denotes the identity of G . Let g and h be any elements of G and put $x = (i_0, g, j_0)$ and $y = (i, h, j)$. Since there exists a positive integer n such that x^n commutes with y , we obtain, after a short calculation, $(i_0, g^n h, j) = (i, h g^n, j_0)$. This implies $|I| = |J| = 1$, therefore C is a group which is also power centralized.

The converse statement is obvious. \square

Since every power centralized semigroup is a semilattice of archimedean power centralized semigroups, we would like to have an insight into the structure of these semilattice components. First we describe those which contain idempotents.

Proposition 3.3. *S is an archimedean power centralized semigroup containing an idempotent if and only if S is an ideal extension of a power centralized group by a nil semigroup.*

Proof. Suppose first that S is an archimedean power centralized semigroup containing an idempotent. By a result of Chrislock [3], S is an ideal extension of a simple semigroup C by a nil semigroup. Since S is power centralized, the same is true for C . Beside that, C is a group by Theorem 3.2. Conversely, let S be an ideal extension of a power centralized group G by a nil semigroup N . Using [3] once again, we conclude that S is archimedean and contains an idempotent. To see that S is power centralized, observe first that the above mentioned extension is a retract extension. Namely, the map $\rho : S \rightarrow G$ defined by $s\rho = se$, where e is the identity element of G , is clearly a

retraction homomorphism. Pick $x, y \in S$ and $n \in \mathbb{N}$. Since G is power centralized, there exists an integer $k \geq n$ such that $(x\rho)^k$ commutes with $y\rho$. In case when $x \in G$ or $y \in G$ we obtain $x^k y = (x^k y)\rho = (x\rho)^k(y\rho) = (y\rho)(x\rho)^k = (yx^k)\rho = yx^k$. Therefore we may assume that $x, y \in N \setminus \{0\}$. Let $I = \{i \in \mathbb{N} : x^i y = yx^i \text{ in } N\}$. This is clearly a nonempty set. If $x^i y \neq 0$ in N for some $i \in I$, $i \geq n$, then $x^i y = yx^i$ in S . So assume that $x^i y = 0$ in N for all $i \in I$. There exists a positive integer l such that $l \in I$, $l \geq k \geq n$ and $(x\rho)^l$ commutes with $y\rho$. Since $x^l y = yx^l = 0$ in N , we obtain $x^l y = (x^l y)\rho = (x\rho)^l(y\rho) = (y\rho)(x\rho)^l = (yx^l)\rho = yx^l$ in S , hence S is power centralized. \square

Proposition 3.4. *A semigroup is regular and power centralized if and only if it is a semilattice of power centralized groups.*

Proof. This is clear since the idempotents of a power centralized semigroup are central. \square

Theorem 3.5. *Let S be an ideal extension of a regular semigroup K by a nil semigroup. Then S is power centralized if and only if K is a power centralized Clifford semigroup.*

Proof. Let S be an ideal extension of a regular semigroup K by a nil semigroup N . If S is power centralized, then so is K . By Proposition 3.4, K is a power centralized Clifford semigroup which proves the ‘if’ part of the theorem. Conversely, assume S is an ideal extension of a power centralized Clifford semigroup K by a nil semigroup N . Let K be a semilattice of groups $\{G_\alpha : \alpha \in \Omega\}$. Let $a \in S$ and let k be the least positive integer such that $a^k \in G_\alpha$. Let e_α be the identity element of G_α . Since $(ae_\alpha)^k$ belongs to G_α , we obtain $ae_\alpha \in G_\alpha$, hence $a^l \in G_\alpha$ for $l \geq k$. Define the map $\rho : S \rightarrow K$ by the rule $a\rho = ae_\alpha$. Clearly ρ leaves the elements of K fixed. Let $x, y \in S$ and suppose that $x^i \in G_\alpha$, $y^j \in G_\beta$, $(xy)^k \in G_\gamma$ for some $i, j, k \in \mathbb{N}$, $k \geq i$, $k \geq j$. As $(xy)^k = e_\gamma(xy)^k$ and $e_\gamma x^k y^k$ belong to $G_{\alpha\beta\gamma}$, we get $\alpha\beta\gamma = \gamma$, hence $e_\alpha e_\beta e_\gamma = e_\gamma$. Similarly we obtain $\alpha\beta\gamma = \alpha\beta$, since $e_\alpha e_\beta (xy)^k$ and $(e_\alpha x)^k (e_\beta y)^k$ belong to the same semilattice component of K . This implies $e_\alpha e_\beta = e_\gamma$, hence $(xy)\rho = xye_\gamma = xye_\alpha e_\beta = x(ye_\beta)e_\alpha = (xe_\alpha)(ye_\beta) = (a\rho)(b\rho)$. Thus ρ is a retract homomorphism. The rest of the proof now follows the lines of the proof of Proposition 3.3. \square

Proposition 3.3 describes archimedean power centralized semigroups containing idempotents. When there are no idempotents present, we obtain the following result.

Theorem 3.6. *Let S be a semigroup and suppose I is an ideal in S which is uniformly power centralized in S . Suppose that I is archimedean without idempotents. Then I has a nontrivial group homomorphic image.*

Proof. Let a be any element of I . Define

$$\rho_a = \{(x, y) \in I \times I : xa^i = ya^j \text{ for some positive integers } i, j\}.$$

Clearly ρ_a is an equivalence relation on I . Let $(x, y) \in \rho_a$ and $z \in I$. Then there exist $i, j \in \mathbb{N}$ such that $xa^i = ya^j$. We may assume without loss of generality that a^i and a^j commute with z . Then $zxa^i = zya^j$ and $xza^i = xa^i z = ya^j z = yz^j$, hence ρ_a is a congruence relation on I . Consider the semigroup I/ρ_a . Since $(xa, x) \in \rho_a$ for every $x \in I$, we have that the ρ_a -class containing a is a right identity of I/ρ_a . Let $x \in I$ be arbitrary. As I is uniformly power centralized in S , there exists $n \in \mathbb{N}$ such that x^n commutes with every element of I . Since I is archimedean, there exist $u, v \in I$ and $m \in \mathbb{N}$ such that $a^m = ux^n v = x(x^{n-1}uv)$. It follows that $(x(x^{n-1}uv), a) \in \rho_a$, whence I/ρ_a is right simple. This shows that I/ρ_a is a group. Note that since I does not contain idempotents, $(a, a^2) \notin \rho_a$, hence I/ρ_a is a nontrivial group. \square

Corollary 3.7. *Let S be an n -central archimedean semigroup without idempotents. Then S has a nontrivial group homomorphic image.*

4. SUBDIRECTLY IRREDUCIBLE POWER CENTRALIZED SEMIGROUPS

A semigroup S is said to be *subdirectly irreducible* if whenever S is a subdirect product of a family of semigroups $(S_\lambda)_{\lambda \in \Lambda}$, then there exists $\lambda_0 \in \Lambda$ such that the projection homomorphism $\pi_{\lambda_0} : \prod_{\lambda \in \Lambda} S_\lambda \rightarrow S_{\lambda_0}$ maps S isomorphically onto S_{λ_0} . Subdirectly irreducible semigroups are building blocks of semigroups, since every semigroup is a subdirect product of subdirectly irreducible semigroups by Birkhoff's theorem.

General theory of subdirectly irreducible semigroups was developed by Schein in [16]. We briefly mention some of the results which will be needed later. For instance, it is known that S is subdirectly irreducible if and only if S^0 is subdirectly irreducible. Every subdirectly irreducible semigroup contains the *core*, the least nontrivial ideal of S . If K is the core of S , then we have either $K^2 = K$ or $K^2 = \{0\}$. In the first case we say that K is *globally idempotent*, and in the second case K is called *nilpotent*.

A semigroup is said to be a *homogroup* if it contains the least nonempty ideal (the *kernel*) which is a group. Subdirectly irreducible homogroups without zero have a nice structure.

Lemma 4.1 ([16]). *Every subdirectly irreducible homogroup without zero is a group.*

We are now in a position to present a characterization of subdirectly irreducible power centralized semigroups with the globally idempotent core.

Theorem 4.2. *S is a subdirectly irreducible power centralized semigroup with the globally idempotent core if and only if it is isomorphic either to G or G^0 , where G is a subdirectly irreducible power centralized group.*

Proof. Let K be the globally nilpotent core of S . If $0 \notin S$, then K is a simple power centralized semigroup. By Theorem 3.2, K is a power centralized group. Therefore S is a homogroup, implying that S is a power centralized group by Lemma 4.1. Hence we may assume that S contains a zero element. Let S^* be the set of all nonzero elements of S . We will prove that S^* is a subsemigroup of S . Suppose that there exist $a, b \in S$, $a, b \neq 0$ such that $ab = 0$. By Lemma 3.1, S is a semilattice of archimedean power centralized semigroups. Let S_0 be the semilattice component containing zero of S . This is clearly an ideal of S . By Theorem 3.2, K is a group with zero adjoined, hence 0 is the only element of S contained in S_0 . This implies that $I = \{x \in S : ax = 0\}$ is an ideal of S . As I is nontrivial, we have $K \subseteq I$, hence $aK = \{0\}$. This implies that $J = \{y \in S : yK = \{0\}\}$ is a nontrivial ideal of S , hence $K \subseteq J$ and therefore $K^2 = \{0\}$. But this is a contradiction since K is globally idempotent, hence S^* is a subsemigroup of S .

Now, if $|S^*| = 1$, then S is a two-element semilattice. If $|S^*| > 1$, then S^* contains no zero element. As S^* is subdirectly irreducible power centralized semigroup with the globally idempotent core, it is isomorphic to a subdirectly irreducible power centralized group G , hence $S \cong G^0$.

Since the converse is trivial, we have the result. \square

If S is a semigroup with zero, let $\text{Ann}(S)$ be the *annihilator* of S , that is, $\text{Ann}(S) = \{a \in S : as = sa = 0 \text{ for every } s \in S\}$. This is an ideal of S . More generally, for a nonempty subset X of S we define $\text{Ann}_S^{(l)}(X) = \{a \in S : aX = \{0\}\}$ and $\text{Ann}_S^{(r)}(X) = \{a \in S : Xa = \{0\}\}$ to be the *left (resp. right) annihilator* of X in S . It is obvious that $\text{Ann}_S^{(l)}(X)$ is a left ideal of S , $\text{Ann}_S^{(r)}(X)$ is a right ideal of S . We also use the notation $\text{Ann}_S(X) = \text{Ann}_S^{(r)}(X) \cap \text{Ann}_S^{(l)}(X)$.

According to [16] we say that $s \in S$ is a *disjunctive element* of S if the congruence $C_{\{s\}} = \{(a, b) \in S \times S : (\forall x, y \in S^1) xay = s \iff xby = s\}$ is the equality relation on S . The next result is now a direct consequence of [16].

Proposition 4.3. *A power centralized semigroup with zero element and with non-trivial annihilator is subdirectly irreducible if and only if it has a non-zero disjunctive element.*

Further consideration also gives some information in the case when the annihilator is trivial.

Theorem 4.4. *Let S be a subdirectly irreducible power centralized semigroup with zero and with a trivial annihilator. If S has the nilpotent core which is uniformly power centralized in S , then S is a monoid.*

Proof. Let K be the nilpotent core of S and assume that K is uniformly power centralized in S . We shall prove that $\text{Ann}_S^{(l)}(K) = \text{Ann}_S^{(r)}(K)$. Suppose that there is $x \in \text{Ann}_S^{(r)}(K) \setminus \text{Ann}_S^{(l)}(K)$. Then xK is a nontrivial ideal of S , hence $K \subseteq xK$ which implies $K = xK$. Since S is power centralized and K is uniformly power centralized in S , there exists a positive integer n such that x^n commutes with every element of K . This gives $K = x^n K = Kx^n = \{0\}$, which is clearly a contradiction, hence $\text{Ann}_S^{(r)}(K) \subseteq \text{Ann}_S^{(l)}(K)$. Similarly we prove $\text{Ann}_S^{(l)}(K) \subseteq \text{Ann}_S^{(r)}(K)$, therefore $\text{Ann}_S^{(r)}(K) = \text{Ann}_S^{(l)}(K) = \text{Ann}_S(K)$. Since the annihilator of S is trivial, the set $N = S \setminus \text{Ann}_S(K)$ is not empty. We want to prove that N is a subsemigroup of S . Suppose that there exist $x, y \in N$ such that $xy \in \text{Ann}_S(K)$. Then $Ky \cup yK = K$ and $xK = xKy$. If m is such that y^m commutes with all elements of K , then $xK = xKy^m = xy^m K = \{0\}$, which is in contradiction with $x \in N$. Therefore N is a subsemigroup of S .

Now pick $k \in K \setminus \{0\}$. Since K is the core of S , we have $K = Nk \cup kN \cup \{0\}$. If $k = ke_1$ for some $e_1 \in N$, then $k = ke_1^n = e_1^n k$. If $k = e_2 ke_3$ for some $e_2, e_3 \in N$, then $k = e_2^n ke_3^n = e_2^n e_3^n k$. Therefore we may assume that $k = ek$ for some $e \in N$. Let $I = \{x \in S : e^m x = x \text{ for some positive integer } m\}$. I is a nonempty set since $k \in I$. Let $x \in I$ and $s \in S$. Then there is a positive integer m such that $e^m x = x$. Let n be an integer such that e^n commutes with s . Then $xs = e^m(xs)$ and $sx = se^{nm}x = e^{nm}(sx)$, hence I is an ideal of S . Since K is the core of S , we get $K \subseteq I$. Let

$$\rho = \{(x, y) \in S \times S : e^m x = e^n y \text{ for some positive integers } m, n\}.$$

From the proof of Theorem 3.6 it follows that ρ is a congruence relation on S . The next step in the proof is to show that ρ is the equality relation on S . Since S is subdirectly irreducible, it suffices to show that ρ is the equality relation on the core K . Let $(k_1, k_2) \in \rho$ for $k_1, k_2 \in K$. There exist positive integers m and n such that $e^m k_1 = e^n k_2$. Since $K \subseteq I$, there exist positive integers m_1 and m_2 such that $e^{m_1} k_1 = k_1$ and $e^{m_2} k_2 = k_2$. But then we have $k_1 = e^{m_1 m_2 m} k_1 = e^{m_1 m_2 n} k_2 = k_2$, hence ρ is indeed the equality relation.

Now observe that $s\rho se$ for any $s \in S$, hence e is a right identity of S . Let n be a positive integer such that e^n commutes with s . Then we have $s = se = se^n = e^n s$. But $(e, e^2) \in \rho$, thus e is an idempotent. This implies $s = es$, hence S is a monoid, as required. \square

5. PERIODIC SEMIGROUPS WITH CENTRAL IDEMPOTENTS

As every element of a periodic semigroup S has some power in $E(S)$, we conclude that a periodic semigroup S is power centralized if and only if the set of idempotents $E(S)$ is in the center of S . Examples of these include periodic groups, nil semigroups and nil extensions of periodic groups. In the finite case these are

essentially all possibilities, since every finite semigroup with central idempotents subdirectly embeds into the direct product $N \times M_1 \times \cdots \times M_n \times G$, where N is a nil semigroup, M_1, \dots, M_n are monoids and nil extensions of groups, and G is a group, see Almeida and Weil [1]. Note also that the proof of this result in [1] can be amended to prove a similar result for periodic semigroups in which idempotents are central and every strictly descending chain of idempotents is finite. On the other hand, power centralized semigroups are closely related to power commutative groups. In fact, we have the following.

Proposition 5.1. *Every periodic power centralized semigroup is power commutative and nearly exponential.*

Proof. Let S be a periodic power centralized semigroup. Then every idempotent of S belongs to the center of S . By Theorem 4.3 in [5], S is power commutative. Now let $x, y \in S$. By [5, Theorem 4.3], S is a semilattice Ω of unipotent semigroups S_α , $\alpha \in \Omega$. Suppose $x \in S_\alpha$ and $y \in S_\beta$ for some $\alpha, \beta \in \Omega$. Since S is periodic, there exists a positive integer $l = l(x, y)$ such that $(xy)^l$, x^l in y^l are idempotents. Without loss of generality we may assume $l > 1$. But $(xy)^l$ and $x^l y^l$ are idempotents belonging to the same semilattice component $S_{\alpha\beta}$, hence $(xy)^l = x^l y^l$. \square

This raises a question when the converse of Proposition 5.1 holds. First note that there are finite power commutative semigroups which are not power centralized. Consider for instance the semigroup S on three elements a , b and c , where a is a zero element, $b^2 = cb = a$, $bc = b$ and c is an idempotent. It can be easily verified that S is a power commutative semigroup, but $c^n b \neq bc^n$ for any $n \geq 1$. This leads to the following definition, suggested by M. Petrich.

Definition 5.2. Let \mathfrak{X} be the class of all semigroups satisfying the following condition for all $x \in S$: If x commutes with an idempotent $e \in S$, then x also commutes with every idempotent f of S with $f \geq e$.

Before stating the main results of this section, recall that a semigroup S is *quasi regular* if for every $a \in S$ there exists a positive integer n such that a^n is regular. A quasi regular semigroup S is *quasi strongly regular* if each regular element of S is completely regular. In a quasi regular semigroup S define

$$a\mathcal{J}^*b \iff S^1 a^m S^1 = S^1 b^n S^1,$$

where m and n are the smallest positive integers such that a^m and b^n are regular. We have the following result.

Lemma 5.3 (Galbiati and Veronesi [6]). *Let S be a quasi strongly regular semigroup. Then the relation \mathcal{J}^* on S is a semilattice congruence and every \mathcal{J}^* -class is a completely archimedean semigroup.*

Here a semigroup is said to be *completely archimedean* if it is archimedean and contains a primitive idempotent.

In the rest of the paper power commutative periodic semigroups play a major role. We first show that there are some nice characterizations of these groups within the above defined class \mathfrak{X} .

Proposition 5.4. *For a periodic semigroup S in \mathfrak{X} the following assertions are equivalent.*

- (i) \mathcal{J}^* is an idempotent-separating congruence on S .
- (ii) S is a semilattice of nil extensions of groups.
- (iii) S is power commutative.

Proof. (i) \Rightarrow (ii). Let $e \in E(S)$ and let J_e^* be the \mathcal{J}^* -class containing the idempotent e . Since the congruence \mathcal{J}^* is idempotent separating, J_e^* is a unipotent semigroup.

Additionally, e commutes with every element of J_e^* . To show that the semigroup J_e^* is archimedean, let G_e be the maximal subgroup of J_e^* . We claim that $G_e = eJ_e^*$. Namely, it is easy to see that $G_e = \{a \in J_e^* : a \in eJ_e^*, e \in aJ_e^* \cap J_e^*a\}$. This clearly gives $G_e \subseteq eJ_e^*$. For the converse inclusion note that since J_e^* is periodic, for every $x \in J_e^*$ there exists a positive integer r such that $x^r = e$. This implies $e \in aJ_e^* \cap J_e^*a$ for every $a \in eJ_e^*$, thus $eJ_e^* \subseteq G_e$, as required. It is now straightforward to show that G_e is a minimal ideal in J_e^* , therefore J_e^* has the kernel K . Since J_e^* contains exactly one idempotent, K is completely simple. It is clear that for every $a \in J_e^*$ there exists a positive integer m such that $a^m \in K$, hence J_e^* is a nil extension of a completely simple semigroup, therefore it is archimedean. Since S is periodic, every \mathcal{J}^* -class contains at least one idempotent. It follows that S is a disjoint union of J_e^* where $e \in E(S)$. Next we want to show that $E(S)$ is a semilattice. Let $e, f \in E(S)$. Let m and n be the smallest positive integers such that $(ef)^m$ and $(fe)^n$ are regular elements of S . There exist idempotents $g, h \in E(S)$ such that $(ef)^m \in G_g$ and $(fe)^n \in G_h$ (here G_g denotes the maximal subgroup of S containing the idempotent g). Using Munn's lemma, we observe that $(ef)^{m+k} \in G_g$ and $(fe)^{n+k} \in G_h$ for every $k \geq 0$. Since S is periodic, we clearly have $G_g \subseteq J_g^*$ and $G_h \subseteq J_h^*$, which implies $(ef)^m \mathcal{J}^*(ef)^{m+k}$ and $(fe)^n \mathcal{J}^*(fe)^{n+k}$ for every $k \geq 0$. Pick any l greater than m and n . We have

$$S^1(ef)^m S^1 = S^1(ef)^{l+1} S^1 = S^1 e (fe)^l f S^1 \subseteq S^1 (fe)^l S^1 \subseteq S^1 (fe)^n S^1.$$

Similarly, $S^1(fe)^n S^1 \subseteq S^1(ef)^m S^1$, hence $(ef, fe) \in \mathcal{J}^*$. It follows from here that $((ef)^p, (fe)^p) \in \mathcal{J}^*$ for every $p \geq 0$. As S is periodic, there exists such p that $(ef)^p, (fe)^p$ are idempotents. But \mathcal{J}^* is idempotent-separating, thus $(ef)^p = (fe)^p \in E(S)$. Then we have $(ef)^{p+1} = e(fe)^p f = e(ef)^p f = (ef)^p$ and similarly $(fe)^{p+1} = (fe)^p$, hence $(ef)^{p+1} = (fe)^{p+1}$. This shows that e commutes with $(fe)^p f$. But $((fe)^p f)^2 = (fe)^p f (fe)^p f = (fe)^{2p} f = (fe)^p f$, hence $(fe)^p f$ is an idempotent. Note that $(fe)^p f \leq f$. As $S \in \mathfrak{X}$, it follows in particular that e commutes with f , hence $E(S)$ is a semilattice. Finally, let $e, f \in E(S)$ and let $x \in J_e^*, y \in J_f^*$. Then $(xy, ef) \in \mathcal{J}^*$, hence $J_e^* J_f^* \subseteq J_{ef}^*$. This shows that S is a semilattice of unipotent archimedean semigroups $J_e^*, e \in E(S)$. By [3] and Proposition 3.3, every class J_e^* is an ideal extension of a group by a nil semigroup.

(ii) \Rightarrow (iii). Assume now S is a periodic semigroup belonging to \mathfrak{X} , and that S is a semilattice Ω of nil extensions S_α of groups G_α , where $\alpha \in \Omega$. Since S is periodic, so are G_α 's. It follows that the semigroups S_α are power-joined and unipotent, hence the conclusion follows from [5, Theorem 4.3].

(iii) \Rightarrow (i). Let \mathcal{L}^* and \mathcal{R}^* be equivalence relations on S defined by $a\mathcal{L}^*b \iff S^1 a^m = S^1 b^n$ and $a\mathcal{R}^*b \iff a^m S^1 = b^n S^1$, where m and n are the smallest positive integers such that a^m and b^n are regular. Since S is periodic, we have that $\mathcal{J}^* = \mathcal{L}^* \vee \mathcal{R}^*$ [5]. Thus it follows from [5, Theorem 4.3] that \mathcal{J}^* is an idempotent-separating congruence on S . \square

Clearly every periodic semigroup with central idempotents is in \mathfrak{X} . Our final result shows that the converse also holds if we assume that the semigroup is power commutative.

Theorem 5.5. *A periodic semigroup S is power centralized if and only if it is a power commutative semigroup belonging to \mathfrak{X} .*

Proof. Let S be a power commutative semigroup and let $S \in \mathfrak{X}$. By [5, Theorem 4.3], S is a semilattice of unipotent semigroups S_α , where $\alpha \in \Omega$. For $e, f \in E(S)$ there exists a positive integer n such that $(ef)^n = (fe)^n \in E(S)$. A similar argument as in the proof of Proposition 5.4 shows that e commutes with f , hence $E(S)$ is a semilattice; moreover, it is straightforward to see that $\Omega \cong E(S)$. Now let $e \in E(S)$ and let x be an arbitrary element of S . We want to show that x commutes

with e . We have $e \in S_\alpha$ and $x \in S_\beta$ for some $\alpha, \beta \in \Omega$. In case when $\alpha = \beta$ we obtain $x^n = e$ for some positive integer n , hence $xe = ex$. Therefore we may assume that $\alpha \neq \beta$. Let f be the idempotent contained in S_β . Then x commutes with f . We have $xe, ex \in S_{\alpha\beta}$, hence $xef = xef = efex$ and $efx = efex = exef$. This gives $xef = exfe = efx$. It follows from here that x commutes with ef . Since $ef \leq e$ and $S \in \mathfrak{X}$, we obtain $ex = xe$. \square

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